Malaya Journal of Matematik, Vol. 7, No. 1, 96-103, 2019,
https://doi.org/10.26637/MJM0701/0019

Existence results for fractional delay integro-differential equations with multi-point boundary conditions

P. Duraisamy[1] and T. Nandha Gopal [2*]

Abstract
This paper addresses the issue of existence and uniqueness of solutions to the fractional delay integro-differential equations with multi-point boundary conditions. The existence results are proved by applying Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative whereas uniqueness result is proved by the contraction mapping principle. Examples are provided to illustrate the main results.

Keywords
Fractional differential equations, Delay Integro-differential equations, Existence, Fixed Point.

AMS Subject Classification
34A08, 34A12.

[1] *Department of Mathematics, Gobi Arts and Science College, Gobichettipalayam, Tamilnadu, India.*
[2] *Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India.*
*__Corresponding author__: nandhu792002@yahoo.co.in;
Article History: Received **24** August **2018**; Accepted **11** December **2018**

Contents

1. Introduction

The increasing interest of fractional differential equations is motivated by their applications in various fields of science such as physics, fluid mechanics, chemistry, biology, control theory, signal processing, heat conduction in materials with memory [6],[15],[22] and the references therein. The main advantage of using fractional differential equations is related to the fact that we can describe the dynamics of complex non-local systems with memory. Fixed-point theory has wide applications in several areas such as economics, dynamic systems, the theory of differential and integral equations and so on. There have been some papers dealing with the existence of solutions of nonlinear fractional differential equations by using fixed point technique [13],[17],[19].

In [4], the authors discussed the existence of solution for a Riemann-Liouville fractional differential equation with multi-point boundary conditions whereas the existence of solution for multi-point BVPs of Caputo fractional differential equation is discussed in [23]. Area of fractional differential equations with multi-point boundary conditions have attracted many researchers [7],[8],[12],[20] and the references therein. Shri Akiladevi and Balachandran [16] discussed the existence and uniqueness of solution to the fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions. The fractional differential equations with delay has drawn the attention of researchers in the recent years, for detail we refer [1],[2],[3],[9],[18],[21].

Motivated by this consideration, in this paper, we shall discuss the existence and uniqueness of solutions for the fractional delay integrodifferential equations with multi-point boundary conditions of the form by using appropriate fixed point theorems:

$$\left.\begin{aligned} &D^{\xi}u(t) = g\Big(t,u(t),u(\mu(t)),\int_0^t h(t,s,u(s),u(\theta(s)))ds\Big),\\ &2<\xi\leq 3, t\in J=[0,1],\\ &u(0)=0,\ \ u(\zeta)=0,\ \ u(1)=0,\ \ 0<\zeta<1, \end{aligned}\right\} \quad (1.1)$$

where $g: J\times X^3 \to X, h:\Omega\times X^2\to X, \mu,\theta: J\to J$ are continuous functions with $0\leq \mu(t),\theta(t)\leq t, t\in J$. Here $\Omega = \{(t,s): 0\leq s\leq t\leq 1\}$.

Here we use the notation $Hu(t) = \int_0^t h(t,s,u(s),u(\theta(s)))ds$. With this context in the mind, the outline of this paper is as follows. In Section 2, we give some definitions and lemmas which are required to our main results. In Section 3, we use Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative to prove the existence results whereas the uniqueness result by using the contraction mapping principle. Finally, in section 4, we shall some numerical examples, which shall explicate the applicability of our results.

2. Preliminaries

Let us recall some basic definitions and Lemmas which will be used in our main results [5],[10],[11],[14].

Definition 2.1: The Riemann-Liouville fractional integral of a function $g \in L^1(R^+)$ of order $\xi > 0$ is defined by

$$I^{\xi} g(t) = \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} g(s)ds$$

provided the integral exists.

Definition 2.2: The Riemann-Liouville fractional derivative of order $\xi > 0, n-1 < \xi \le n, n \in N$ is defined as

$$D^{\xi}_{0+} g(t) = \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\xi-1} g(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Lemma 2.3 The equality $D^{\xi} I^{\xi} g(t) = g(t), \xi > 0$ holds for $g \in L(0,1)$.

Lemma 2.4 Let $\xi > 0$. Then the differential equation $D^{\xi} u = 0$ has a unique solution $u(t) = c_1 t^{\xi-1} + c_2 t^{\xi-2} + \ldots + c_n t^{\xi-n}, c_i \in R, i = 1,2,\ldots,n$, where $n-1 < \xi \le n$.

Lemma 2.5 Let $\xi > 0$. Then the following equality holds for $u \in L(0,1)$;
$I^{\xi} D^{\xi} u(t) = u(t) + c_1 t^{\xi-1} + c_2 t^{\xi-2} + \ldots + c_n t^{\xi-n}, c_i \in R$,
$i = 1,2,\ldots,n$, there $n-1 < \xi \le n$.

Lemma 2.6 For $k(t) \in C(J)$, the following BVP

$$D^{\xi} u(t) = k(t),\ t \in J,\ 2 < \xi \le 3, \tag{2.1}$$

$$u(0) = 0, u(\zeta) = 0, u(1) = 0, 0 < \zeta < 1$$

has a unique solution given by

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \\ &+ \frac{t^{\xi-1}}{\zeta^{\xi-1} - \zeta^{\xi-2}} \left\{ \zeta^{\xi-2} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right. \\ &\left. - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right\} \\ &+ \frac{t^{\xi-2}}{\zeta^{\xi-2} - \zeta^{\xi-1}} \left\{ \zeta^{\xi-1} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right. \\ &\left. - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right\} \end{aligned}$$

Proof:
For some vector constants $c_1, c_2, c_3 \in X$ the general solution of (2.1) can be written as

$$u(t) = I^{\xi} k(t) + c_1 t^{\xi-1} + c_2 t^{\xi-2} + c_3 t^{\xi-3} \tag{2.2}$$

Using the boundary condition $u(0) = 0$, we get $c_3 = 0$.
By the boundary conditions $u(\zeta) = 0$ and $u(1) = 0$, we get

$$I^{\xi} k(\zeta) + c_1 \zeta^{\xi-1} + c_2 \zeta^{\xi-2} = 0 \tag{2.3}$$

and

$$I^{\xi} k(1) + c_1 + c_2 = 0 \tag{2.4}$$

Solving equations (2.3) and (2.4), we get

$$c_1 = \frac{1}{\zeta^{\xi-1} - \zeta^{\xi-2}} (\zeta^{\xi-2} I^{\xi} k(1) - I^{\xi} k(\zeta))$$

$$c_2 = \frac{1}{\zeta^{\xi-2} - \zeta^{\xi-1}} (\zeta^{\xi-1} I^{\xi} k(1) - I^{\xi} k(\zeta))$$

Substituting the values of c_1, c_2 & c_3 in (2.2), we get

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \\ &+ \frac{t^{\xi-1}}{\zeta^{\xi-1} - \zeta^{\xi-2}} \left\{ \zeta^{\xi-2} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right. \\ &\left. - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right\} \\ &+ \frac{t^{\xi-2}}{\zeta^{\xi-2} - \zeta^{\xi-1}} \left\{ \zeta^{\xi-1} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right. \\ &\left. - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} k(s)ds \right\} \end{aligned}$$

This completes the proof.

3. Main Results

Let $Z = C(J,X)$ denote the Banach Space of all continuous functions from $J \to R$ endowed with the norm defined by $||u|| = sup\{|u(t)|, t \in J\}$.

Define the operator $F : Z \to Z$ by

$$\begin{aligned} (Fu)(t) &= \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),H(u(s)))ds \\ &+ \frac{t^{\xi-1}}{\zeta^{\xi-1} - \zeta^{\xi-2}} \left\{ \zeta^{\xi-2} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \right. \\ &\times g(s,u(s),u(\mu(s)),H(u(s)))ds \end{aligned}$$

$$
\begin{aligned}
&- \int_0^{\zeta} \frac{(n-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),H(u(s)))ds \Big\} \\
&+ \frac{t^{\xi-2}}{\zeta^{\xi-2}-\zeta^{\xi-1}} \Big\{ \zeta^{\xi-1} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times g(s,u(s),u(\mu(s)),H(u(s)))ds \\
&- \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),H(u(s)))ds \Big\}
\end{aligned}
$$

for $t \in J$. Note that the problem (1.1) has solutions, if the operator F has fixed points.

For the forthcoming analysis, we need the following assumptions:

(A1) $\exists$ positive constants L_g and $L_h \ni$
(i) $\|g(t,u_1,v_1,w_1) - g(t,u_2,v_2,w_2)\| \leq L_g(\|u_1-u_2\| + \|v_1-v_2\| + \|w_1-w_2\|)$,
. $t \in J, u_1,u_2,v_1,v_2,w_1,w_2 \in X$.
(ii) $\|h(t,s,u_1,v_1) - h(t,s,u_2,v_2)\| \leq L_h(\|u_1-u_2\| + \|v_1-v_2\|), t,s \in J, u_1,u_2,v_1,v_2 \in X$.

(A2) $\|g(t,u,v,w)\| \leq l(t)\phi(\|u\|), \quad (t,u,v,w) \in J \times X^3$, where $l \in L^1(J,R^+)$ and
$\phi : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(A3) Let

$$
\Lambda_1 = \frac{1}{\Gamma(\xi+1)} \Big\{ 1 + \frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} [|\zeta^{\xi-1}| + |\zeta^{\xi-2}| + 2\zeta^{\xi}] \Big\}
$$

(A4) $\exists$ a continuous nondecreasing function $\chi : [0,\infty) \to (0,\infty)$ and the functions $\vartheta_1, \vartheta_2 \in L^1(J,R^+) \ni$ for each $(t,u,v,w) \in J \times X^3$,

$$
\|g(t,u,v,w)\| \leq \vartheta_1(t)\chi(\|u\|) + \vartheta_2(t).
$$

(A5) $\exists$ a constant $M > 0 \ni M\Omega^{-1} > 1$,
where $\Omega = (\chi(M)\|\vartheta_1\|_{L^1} + \|\vartheta_2\|_{L^1})\Lambda_1$.

3.1 Existence Result via Krasnoselskii's Fixed Point Theorem

Theorem 3.1: Suppose that the assumptions (A1)-(A3) hold with

$$
\Lambda = \frac{2L_g}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} \frac{1}{\Gamma(\xi+1)} \Big\{ [|\zeta^{\xi-2}| + |\zeta^{\xi-1}| + 2|\zeta^{\xi}|] + \frac{L_h}{\xi+1} [|\zeta^{\xi-2}| + |\zeta^{\xi-1}| + 2|\zeta^{\xi+1}|] \Big\} < 1.
$$

Then the BVP (1.1) has at least one solution on J.

Proof:

We can fix $r \geq \|l\|_{L^1}\phi(r)\Lambda_1$ and consider $B_r = \{u \in Z : \|u\| \leq r\}$. We define the operators P and Q on B_r as

$$
\begin{aligned}
(Pu)(t) &= \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds \\
(Qu)(t) &= \frac{t^{\xi-1}}{\zeta^{\xi-1}-\zeta^{\xi-2}} \Big\{ \zeta^{\xi-2} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times g(s,u(s),u(\mu(s)),Hu(s))ds \\
&\quad - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds \Big\} \\
&\quad + \frac{t^{\xi-2}}{\zeta^{\xi-2}-\zeta^{\xi-1}} \Big\{ \zeta^{\xi-1} \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times g(s,u(s),u(\mu(s)),Hu(s))ds \\
&\quad - \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds \Big\}
\end{aligned}
$$

For $u,v \in B_r$, we find that

$$
\begin{aligned}
\|Pu+Qv\| &\leq \|l\|_{L^1}\phi(r) \frac{1}{\Gamma(\xi+1)} \Big\{ 1 + \frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} \\
&\quad \times \Big[|\zeta^{\xi-1}| + |\zeta^{\xi-2}| + 2\zeta^{\xi} \Big] \Big\} \\
&= \|l\|_{L^1}\phi(r)\Lambda_1 \leq r
\end{aligned}
$$

Thus $Pu+Qv \in B_r$. Now, we prove that Q is a contraction.

$$
\begin{aligned}
&\|(Qu)(t) - (Qv)(t)\| \\
&\leq \frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} \Big\{ |\zeta^{\xi-2}| \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \|g(s,u(s),u(\mu(s)),Hu(s)) \\
&\quad - g(s,v(s),v(\mu(s)),Hv(s))\|ds \\
&\quad + |\zeta^{\xi-1}| \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \|g(s,u(s),u(\mu(s)),Hu(s)) \\
&\quad - g(s,v(s),v(\mu(s)),Hv(s))\|ds \\
&\quad + 2\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} \|g(s,u(s),u(\mu(s)),Hu(s)) \\
&\quad - g(s,v(s),v(\mu(s)),Hv(s))\|ds \Big\} \\
&\leq \frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} \Big\{ (|\zeta^{\xi-2}| + |\zeta^{\xi-1}|) \\
&\quad \times \int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} L_f(2\|u-v\| \\
&\quad + \| \int_0^s h(s,\tau,u(\tau),u(\tau))d\tau \\
&\quad - \int_0^s h(s,\tau,v(\tau),v(\tau))d\tau\|)ds \\
&\quad + 2\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} L_f(2\|u-v\| \\
&\quad + \| \int_0^s h(s,\tau,u(\tau),u(\tau))d\tau
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^s h(s,\tau,v(\tau),v(\tau))d\tau\|)ds\Big\} \\
\leq\ & 2L_g\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big[\frac{1}{\Gamma(\xi+1)}\{|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|\} \\
&+\frac{L_h}{\Gamma(\xi+2)}\{|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|\}\Big]\|u-v\| \\
=\ & \Lambda\|u-v\|
\end{aligned}
$$

Since $\Lambda<1$, we say that Q is a contraction mapping. Continuity of g and h implies that the operator P is continuous. Also

$$
\begin{aligned}
\|(Pu)(t)\| &\leq \int_0^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\leq \frac{\|l\|_{L^1}\phi(r)}{\Gamma(\xi+1)}.
\end{aligned}
$$

Therefore P is uniformly bounded on B_r.

Next to prove that the compactness of the operator P, it is enough to show that P is equicontinuous. Now, for any $t_1,t_2\in J$ with $t_1<t_2$ and $u\in B_r$, we have

$$
\begin{aligned}
&|(Pu)(t_2)-(Pu)(t_1)| \\
&= \Big|\int_0^{t_2}\frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)}g(s,u(s),u(\mu(s)),Hu(s))ds \\
&\quad -\int_0^{t_1}\frac{(t_1-s)^{\xi-1}}{\Gamma(\xi)}g(s,u(s),u(\mu(s)),Hu(s))ds\Big| \\
&\leq \int_0^{t_1}\frac{[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]}{\Gamma(\xi)}l(s)\phi(\|x\|)ds \\
&\quad +\int_{t_1}^{t_2}\frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)}l(s)\phi(\|x\|)ds \\
&\leq \phi(r)\Big[\int_0^{t_1}\frac{[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]}{\Gamma(\xi)}l(s)ds \\
&\quad +\int_{t_1}^{t_2}\frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)}l(s)ds\Big]
\end{aligned}
$$

which is independent of u and tends to zero as $t_2\to t_1$. Thus, P is relatively compact on B_r. Hence, by Arzela-Ascoli Theorem, we have P is compact on B_r. Therefore by the Krasnoselskii's Fixed Point Theorem, the problem (1.1) has at least one solution on J.

3.2 Existence Result via Leray-Schauder Nonlinear Alternative

Theorem 3.2:

Assume that the hypotheses (A3)-(A5) holds. Then the BVP (1.1) has at least one solution on J.

Proof:

The operator $F:Z\to Z$ is continuous. Now, we show that F maps bounded sets into bounded sets in Z. Fix $B_r=\{u\in Z:\|u\|\leq r\}$ in Z.

For $u\in B_r$, we have

$$
\begin{aligned}
&\|(Fu)(t)\| \\
&\leq \int_0^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta}\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\quad +\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-1}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta}\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\leq (\|\vartheta_1\|_{L^1}\chi(r)+\|\vartheta_2\|_{L^1})\frac{1}{\Gamma(\xi+1)} \\
&\quad \times\Big\{1+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}(|\zeta^{\xi-1}|+|\zeta^{\xi-2}|+2\zeta^{\xi})\Big\} \\
&= (\|\vartheta_1\|_{L^1}\chi(r)+\|\vartheta_2\|_{L^1})\Lambda_1.
\end{aligned}
$$

Next we show that F maps bounded sets into equicontinuous sets in B_r. Let $t_1,t_2\in J$ with $t_1<t_2$ and $u\in B_r$. Then

$$
\begin{aligned}
&\|(Fu)(t_2)-(Fu)(t_1)\| \\
&\leq \int_0^{t_1}\frac{[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]}{\Gamma(\xi)} \\
&\quad \times\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_{t_1}^{t_2}\frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\frac{(t_2^{\xi-1}-t_1^{\xi-1})}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta}\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\quad +\frac{(t_2^{\xi-2}-t_1^{\xi-2})}{|\zeta^{\xi-2}-\zeta^{\xi-1}|}\Big\{|\zeta^{\xi-1}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta}\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\leq \chi(r)\Big[\int_0^{t_1}\frac{[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]}{\Gamma(\xi)}\vartheta_1(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2} \frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)} \vartheta_1(s)ds \\
&+\frac{(t_2^{\xi-1}-t_1^{\xi-1})}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_1(s)ds \\
&+\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_1(s)ds\Big\} \\
&+\frac{(t_2^{\xi-2}-t_1^{\xi-2})}{|\zeta^{\xi-2}-\zeta^{\xi-1}|}\Big\{|\zeta^{\xi-1}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_1(s)ds \\
&+\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_1(s)ds\Big\}\Big] \\
&+\Big[\int_0^{t_1} \frac{[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]}{\Gamma(\xi)}\vartheta_2(s)ds \\
&+\int_{t_1}^{t_2} \frac{(t_2-s)^{\xi-1}}{\Gamma(\xi)} \vartheta_2(s)ds \\
&+\frac{(t_2^{\xi-1}-t_1^{\xi-1})}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_2(s)ds \\
&+\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_2(s)ds\Big\} \\
&+\frac{(t_2^{\xi-2}-t_1^{\xi-2})}{|\zeta^{\xi-2}-\zeta^{\xi-1}|}\Big\{|\zeta^{\xi-1}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_2(s)ds \\
&+\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\vartheta_2(s)ds\Big\}\Big]
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above equation tends to zero which is independent of $u \in B_r$. Thus F maps bounded sets into equicontinuous sets in B_r. By Arzela-Ascoli's theorem, F is completely continuous.

Let $u = \lambda_0 Fu$, where $\lambda_0 \in (0,1)$. Then for $t \in J$, we have

$$
\begin{aligned}
u(t) &= \lambda_0 \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds \\
&+ \frac{\lambda_0}{(\zeta^{\xi-1}-\zeta^{\xi-2})}\Big\{\zeta^{\xi-2}\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times g(s,u(s),u(\mu(s)),Hu(s))ds \\
&- \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds\Big\} \\
&+ \frac{\lambda_0}{(\zeta^{\xi-2}-\zeta^{\xi-1})}\Big\{\zeta^{\xi-1}\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \times g(s,u(s),u(\mu(s)),Hu(s))ds \\
&- \int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)} g(s,u(s),u(\mu(s)),Hu(s))ds\Big\}
\end{aligned}
$$

Then $\|u(t)\| \le (\chi(\|u\|)\|\vartheta_1\|_{L^1} + \|\vartheta_2\|_{L^1})\Lambda_1$ and which can be written as

$$\frac{\|u\|}{(\chi(\|u\|)\|\vartheta_1\|_{L^1} + \|\vartheta_2\|_{L^1})\Lambda_1} \le 1.$$

By the assumption of (A5), $\exists\, M \ni \|u\| \neq M$. Set

$$U = \{u \in Z : \|u\| < M\}.$$

The operator $F : U \to Z$ is completely continuous. From the above choice of U, there is no $u \in \partial U \ni u = \lambda_0 Fu$, for $\lambda_0 \in (0,1)$. By the Leray-Schauder nonlinear alternative, we conclude that F has a fixed point $u \in U$ which is a solution to the problem (1.1).

3.3 Uniqueness Result via Banach's Fixed Point Theorem

Theorem 3.3:

Assume that (A1)-(A3) hold with

$$
\begin{aligned}
\Lambda_2 &= 2L_g(1+L_h)\frac{1}{\Gamma(\xi+1)}\Big\{1+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|} \\
&\quad [|\zeta^{\xi-1}|+|\zeta^{\xi-2}|+2\zeta^{\xi}]\Big\} < 1.
\end{aligned}
$$

Then the multipoint BVP (1.1) has a unique solution on J.

Proof:

Let $M_1 = \sup_{t\in J}\|g(t,0,0,0)\|$ and $M_2 = \sup_{t\in J}\|h(t,s,0,0)\|$. Consider $B_r = \{u \in Z : \|u\| \le r\}$, where $r \ge \frac{\Delta_2}{1-\Delta_1}$ with

$$
\begin{aligned}
\Delta_1 &= 2L_g\Big[\frac{1}{\Gamma(\xi+1)}\{1+|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|\} \\
&\quad +L_h\{\frac{1}{\Gamma(\xi+2)}(|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|)\}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_2 &= \frac{L_g M_2}{\Gamma(\xi+2)}(|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|) \\
&\quad +\frac{M_1}{\Gamma(\xi+1)}(1+|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|)
\end{aligned}
$$

To show that $FB_r \subset B_r$, where $F : Z \to Z$.

For $u \in B_r$, we have

$$
\begin{aligned}
&\|(Fu)(t)\| \\
&\le \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\quad +\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-1}|\int_0^1 \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \\
&\quad \|g(s,u(s),u(\mu(s)),Hu(s))\|ds \\
&\quad +\int_0^{\zeta} \frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\|ds\Big\} \\
&\le \int_0^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)}[\|g(s,u(s),u(\mu(s)),Hu(s)) \\
&\quad -g(s,0,0,0)\|+\|g(s,0,0,0)\|]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-2}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\\
&\times[\|g(s,u(s),u(\mu(s)),Hu(s))-g(s,0,0,0)\|\\
&+\|g(s,0,0,0)\|]ds\\
&+\int_0^\zeta\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}[\|g(s,u(s),u(\mu(s)),Hu(s))\\
&-g(s,0,0,0)\|+\|g(s,0,0,0)\|]ds\Big\}\\
&+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{|\zeta^{\xi-1}|\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\\
&[\|g(s,u(s),u(\mu(s)),Hu(s))-g(s,0,0,0)\|\\
&+\|g(s,0,0,0)\|]ds\\
&+\int_0^\zeta\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}[\|g(s,u(s),u(\mu(s)),Hu(s))\\
&-g(s,0,0,0)\|+\|g(s,0,0,0)\|]ds\Big\}\\
\leq&\int_0^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}[L_g(\|u(s)\|+\|u(\mu(s))\|+\|Hu(s)\|)+M_1]ds\\
&+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\{(|\zeta^{\xi-1}|+|\zeta^{\xi-2}|)\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\\
&[L_g(\|u(s)\|+\|u(\mu(s))\|+\|Hu(s)\|)+M_1]ds\\
&+2\int_0^\zeta\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}[L_g(\|u(s)\|+\|u(\mu(s))\|\\
&+\|Hu(s)\|)+M_1]ds\\
\leq\;& 2rL_g\Big[\frac{1}{\Gamma(\xi+1)}\{1+|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|\}\\
&+L_h\{\frac{1}{\Gamma(\xi+2)}(|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|)\}\Big]\\
&+\frac{L_gM_2}{\Gamma(\xi+2)}(|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|)\\
&+\frac{M_1}{\Gamma(\xi+1)}(1+|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|)\\
=\;&\Delta_1 r+\Delta_2\leq r.
\end{aligned}
$$

Thus $FB_r\subset B_r$. Now for $u,v\in Z$ and $t\in J$, we have

$$
\begin{aligned}
&\|(Fu)(t)-(Fv)(t)\|\\
\leq\;&\int_0^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\\
&-g(s,v(s),v(\mu(s)),Hv(s))\|ds\\
&+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{(|\zeta^{\xi-1}|+|\zeta^{\xi-2}|)\\
&\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\\
&-g(s,v(s),v(\mu(s)),Hv(s))\|ds\\
&+2\int_0^\zeta\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}\|g(s,u(s),u(\mu(s)),Hu(s))\\
&-g(s,v(s),v(\mu(s)),Hv(s))\|ds\Big\}\\
\leq\;&\int_0^t\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}[L_g(\|u-v\|+\|u-v\|\\
&+\|\int_0^s h(s,\tau,u(\tau),u(\mu(\tau)))d\tau\\
&-\int_0^s h(s,\tau,v(\tau),v(\mu(\tau)))d\tau\|)]ds\\
&+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\Big\{(|\zeta^{\xi-1}|+|\zeta^{\xi-2}|)\\
&\int_0^1\frac{(1-s)^{\xi-1}}{\Gamma(\xi)}[L_g(\|u-v\|\\
&+\|u-v\|+\|\int_0^s h(s,\tau,u(\tau),u(\mu(\tau)))d\tau\\
&-\int_0^s h(s,\tau,v(\tau),v(\mu(\tau)))d\tau\|)]ds\\
&+2\int_0^\zeta\frac{(\zeta-s)^{\xi-1}}{\Gamma(\xi)}[L_g(\|u-v\|+\|u-v\|\\
&+\|\int_0^s h(s,\tau,u(\tau),u(\mu(\tau)))d\tau\\
&-\int_0^s h(s,\tau,v(\tau),v(\mu(\tau)))d\tau\|)]ds\Big\}\\
\leq\;&2L_g(1+L_h)\frac{1}{\Gamma(\xi+1)}\Big\{1+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\\
&[|\zeta^{\xi-1}|+|\zeta^{\xi-2}|+2\zeta^{\xi}]\Big\}\|u-v\|\\
=\;&\Lambda_2\|u-v\|
\end{aligned}
$$

Here Λ_2 depends only on the parameters involved in the problem. Since $\Lambda_2<1$, we say that F is a contraction. Hence, by the Contraction mapping principle, the BVP (1.1) has a unique solution on J.

4. Example

Example 4.1
Consider the following BVP:

$$
\begin{aligned}
D^{\frac{5}{2}}u(t) \;=\;& \frac{1}{(t+15)}\frac{|u(t)|}{1+|u(t)|}+\frac{e^{-2t}}{14+e^t}\frac{|u(\frac{2t}{3})|}{1+|u(\frac{2t}{3})|}\\
&+\frac{1}{15}\int_0^t\frac{e^s}{5}\frac{|u(s^3)|}{1+|u(s^3)|}ds \qquad (4.1)
\end{aligned}
$$

with the boundary conditions:

$$u(0)=0,u(\frac{1}{3})=0,u(1)=0.$$

Here $\xi=\frac{5}{2}$, $\zeta=\frac{1}{3}$. From (4.1), we have

$$
\begin{aligned}
g(t,u(t),u(\mu(t)),Hu(t)) \;=\;& \frac{1}{(t+15)}\frac{|u(t)|}{1+|u(t)|}\\
&+\frac{e^{-2t}}{14+e^t}\frac{|u(\frac{2t}{3})|}{1+|u(\frac{2t}{3})|}
\end{aligned}
$$

$$+\frac{1}{15}\int_0^t \frac{e^s}{5}\frac{|u(s^3)|}{1+|u(s^3)|}ds$$

where $Hu(t) = \int_0^t \frac{e^s}{5}\frac{|u(s^3)|}{1+|u(s^3)|}ds, \mu(t) = \frac{2t}{3}, \theta(t) = t^3$.
The assumption (A1) is satisfied with $Lg = \frac{1}{15}$ and $L_h = \frac{1}{5}$. Also

$$\begin{aligned}\Lambda &= \frac{2L_g}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}\frac{1}{\Gamma(\xi+1)}\Big\{[|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi}|] \\ &\quad +\frac{L_h}{\xi+1}[|\zeta^{\xi-2}|+|\zeta^{\xi-1}|+2|\zeta^{\xi+1}|]\Big\} \\ &= 0.0984 < 1.\end{aligned}$$

Thus all the assumptions of the Theorem 3.2 are satisfied. Hence the problem (4.1) has at least one solution on J.

Example 4.2

Consider the BVP:

$$\begin{aligned}D^{\frac{5}{2}}u(t) &= \frac{1}{62+2t}\frac{u(t)}{1+u(t)}+\frac{e^{-t}}{61+e^t}\frac{u(2t^3)}{1+u(2t^3)} \\ &\quad +\frac{1}{62}\int_0^t\frac{e^{-s}}{7}\frac{u(\cos s)}{1+u(\cos s)}ds \qquad (4.2)\end{aligned}$$

with the boundary conditions:

$$u(0) = 0, u(\frac{1}{2}) = 0, u(1) = 0.$$

Here $\xi = \frac{5}{2}$, $\zeta = \frac{1}{2}$. From (4.2), we have

$$\begin{aligned}g(t,u(t),u(\mu(t)),Hu(t)) &= \frac{1}{62+2t}\frac{u(t)}{1+u(t)} \\ &\quad +\frac{e^{-t}}{61+e^t}\frac{u(2t^3)}{1+u(2t^3)} \\ &\quad +\frac{1}{62}\int_0^t\frac{e^{-s}}{7}\frac{u(\cos s)}{1+u(\cos s)}ds\end{aligned}$$

where $Hu(t) = \int_0^t \frac{e^{-s}}{7}\frac{u(coss)}{1+u(coss)}ds, \mu(t) = 2t^3, \theta(t) = cost$.
The assumption (A1) is satisfied with $Lg = \frac{1}{62}$ and $L_h = \frac{1}{7}$. Also

$$\begin{aligned}\Lambda_2 &= 2L_g(1+L_h)\frac{1}{\Gamma(\xi+1)}\Big\{1+\frac{1}{|\zeta^{\xi-1}-\zeta^{\xi-2}|}[|\zeta^{\xi-1}| \\ &\quad +|\zeta^{\xi-2}|+2\zeta^{\xi}]\Big\} = 0.0556 < 1.\end{aligned}$$

Thus all the assumptions of the Theorem 3.5 are satisfied. Hence the problem (4.2) has a unique solution on J.

References

[1] Balachandran K., Kiruthika S., Rivero M. and Trujillo J.J.; *Existence of solutions for fractional delay integrodifferential equations,* Journal of Applied Nonlinear Dynamics, 2012(1), 309-319.

[2] Balachandran K. and Samuel F.P.; *Existence of solutions for quasilinear delay integrodifferential equations with nonlocal conditions,* Electronic Journal of Differential Equations, 2009(6), 1-7.

[3] Benchohra M., Henderson J., Ntouyas S.K. and Ouahab A.; *Existence results for fractional order fractional differential equations with infinite delay,* Journal of Mathematical Analysis and Applications, 2008(338), 1340-1350.

[4] Bingxian Li., Shurong Sun., Yanan Li. and Ping Zhao.; *Multi-point boundary value problems for a class of Riemann-Liouville fractional differential equations,* Advances in Difference Equations, 2014(151), 11 pages.

[5] Diethelm K.; *The Analysis of Fractional Differential Equations,* Springer-Verlag, Berlin, 2010.

[6] Duraisamy P. and Nandha Gopal T.; *Existence and uniqueness of solutions for a coupled system of higher order fractional differential equations with integral boundary conditions,* Discontinuity, Nonlinearity, and Complexity, 2018(7), 1-14.

[7] Hua-Cheng Zhou. and Fu-Dong Ge., Chun-Hai Kou.; *Existence of solutions to fractional differential equations with multi-point boundary conditions at resonance in Hilbert spaces,* Electronic Journal of Differential Equations, 2016(61), 1-16.

[8] John Allison. and Nickolai Kosmatov.; *Multi-point boundary value problems of fractional order,* Communications in Applied Analysis, 2008(12), 451-458.

[9] Khalida Aissani. and Mouffak Benchohra.; *Fractional integro-differential equations with state-dependent delay,* Advances in Dynamical Systems and Applications, 2014(9), 17-30.

[10] Kilbas A.A., Srivastava H.M. and Trujillo J.J.; *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdan, 2006.

[11] Miller K.S. and Ross B.; *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New Yourk, 1993.

[12] Nemat Nyamoradi; *Existence of solutions for multi-point boundary value problems for fractional differential equations,* Arab Journal of Mathematical Sciences, 2012(18), 165-175.

[13] Nigar Ali., Kamal Shah. and Rahmat Ali Khan.; *Existence of positive solution to a class of fractional differential equations with three point boundary conditions,* Mathematical Sciences Letters, 2016(5), 291-296.

[14] Podlubny I.; *Fractional Differential Equations,* Academic Press, New Yourk, 1999.

[15] Sahadevan R. and Prakash P.; *Exact solution of certain time fractional nonlinear partial differential equations,* Nonlinear Dynamics, 2016(12), DOI:10.1007/s 11071-016-2714-4, 15 pages.

[16] Shri Akiladevi K. and Balachandran K.; *On fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions,* Acta Mathematica Universitatis Comenianae, 2017(2), 187-204.

[17] Shri Akiladevi K., Balachandran K. and Kim J.K.; *Existence results for neutral fractional integrodifferential equations with fractional integral boundary conditions,* Nonlinear Functional Analysis and Applications, 2014(19), 251-270.

[18] Sokhanvar E. and Askari-Hemmat A.; *A numerical method for solving delay-fractional differential and integro-differential equations,* Journal of Mahani Mathematical Research Center, 2015(4), 11-24.

[19] Vikram Singh. and Dwijendra N Pandey.; *Existence results for multi-term time-fractional impilsive differential equations with fractional order boundary conditions,* Malaya Journal of Mathematik, 2017(5), 625-635.

[20] Xiping Liu., Mei Jia., Ming Niu. and Xiufen Xiang.; *Multiple positive solutions for multi-point boundary value problem of fractional differential equations,* Journal of Mathematical Research with Applications, 2017(37), 223-232.

[21] Yayun Fu., Hongliang Liu. and Aigno Xiao.; *Convergence of variational iteration method for fractional delay integrodifferential-algebraic equations,* Mathematical Problems in Engineering, 2017(2017), 10 pages.

[22] Yuansheng Tian.; *Multiple positive solutions to m-point boundary value problem of nonlinear fractional differential equations,* Differential Equations and Dynamical Systems, 2015(23), 43-56.

[23] Yunhong Li. and Anbin Qi.; *Existence of positive solutions for multi-point boundary value problems of caputo fractional differential equation,* International Journal of Dynamical Systems and Differential Equations, 2017(7), 169-183.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆