On certain associated graphs of set-valued signed graphs

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Abstract

Let X be a non-empty set and let Σ be a signed graph, with corresponding underlying graph G and the signature σ. An injective function $f : V(\Sigma) \to \mathscr{P}(X)$ is said to be a set-labeling of Σ if f is a set-labeling of the underlying graph G and the signature of Σ is defined by $\sigma(uv) = (-1)^{|f(u) \oplus f(v)|}$. A signed graph Σ together with a set-labeling f is known as a set-labeled signed graph and is denoted by Σ_f. In this paper, we discuss the characteristics of certain signed graphs associated with given set-valued signed graphs.

Keywords

Signed graphs, balanced signed graphs, set-valuations of signed graphs.

Mathematics Subject Classification

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [5, 9, 18] and and for the topics in signed graphs we refer to [19, 20]. Unless mentioned otherwise, all graphs considered here are simple, finite, connected and have no isolated vertices.

Let X be a non-empty set and $\mathscr{P}(X)$ be its power set. A *set-labeling* (or a *set-valuation*) of a graph G is an injective function $f : V(G) \to \mathscr{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathscr{P}(X)$ is defined by $f^{\oplus}(uv) = f(u) \oplus f(v) \ \forall \ uv \in E(G)$, where $\oplus$ is the symmetric difference of two sets. A graph G which admits a set-labeling is called an *set-labeled graph* (or a *set-valued graph*)(see [1]). A *set-indexer* of a graph G is an injective function $f : V(G) \to \mathscr{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathscr{P}(X)$ is also injective. A graph G which admits a set-indexer is called a *set-indexed graph* (see [1]).

An edge of a graph G having only one end vertex is known as a *half edge* of G and an edge of G without end vertices is called *loose edge* of G.

A *signed graph* (see [19, 20]), denoted by $\Sigma(G, \sigma)$, is a graph $G(V, E)$ together with a function $\sigma : E(G) \to \{+, -\}$ that assigns a sign, either $+$ or $-$, to each ordinary edge in G. The function σ is called the *signature* or *sign function* of Σ, which is defined on all edges except half edges and is required to be positive on free loops. An edge e of a signed graph Σ is said to be a *positive edge* if $\sigma(e) = +$ and an edge $\sigma(e)$ of a signed graph Σ is said to be a *negative edge* if $\sigma(e) = -$. The set E^+ denotes the set of all positive edges in Σ and the set E^- denotes the set of negative edges in Σ. A simple cycle (or path) of a signed graph Σ is said to be *balanced* (see [3, 10]) if the product of signs of its edges is $+$. A signed graph Σ is said to be a *balanced signed graph* if it contains no half edges and all of its simple cycles are balanced. It is to be noted that the number of all negative signed graph is balanced if and only if it is bipartite.

Balance or imbalance is the basic and the most important property of a signed graph. The following theorem, popularly known as *Harary's Balance Theorem*, establishes a criteria for balance in a signed graph.

Theorem 1.1. [10] *The following statements about a signed graph are equivalent.*

   (i) *A signed graph Σ is balanced.*

   (ii) *Σ has no half edges and there is a partition (V_1, V_2) of $V(\Sigma)$ such that $E^- = E(V_1, V_2)$.*

  (iii) *Σ has no half edges and any two paths with the same end points have the same sign.*

A signed graph Σ is said to be *clusterable* or *partitionable* (see [19, 20]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. If $V(\Sigma)$ can be partitioned in to k subsets with the above mentioned conditions, then the signed graph Σ is said to be *k-clusterable*. In this paper, we study the 2-clusterability of signed graphs only.

Note that 2-clusterability always implies balance in a signed graph Σ. But, the converse need not be true. If all edges in Σ are positive edges, then Σ is balanced but not 2-clusterable.

we define the notion of a set-labeling of a signed graph as follows.

Definition 1.2. [4] Let X be a non-empty set and let Σ be a signed graph, with corresponding underlying graph G and the signature σ. An injective function $f : V(\Sigma) \to \mathscr{P}(X)$ is said to be a *set-labeling* (or *set-valuation*) of Σ if f is a set-labeling of the underlying graph G and the signature of Σ is defined by $\sigma(uv) = (-1)^{|f(u) \oplus f(v)|}$. A signed graph Σ together with a set-labeling f is known as a *set-labeled signed graph* (or set valued signed graph) and is denoted by Σ_f.

Definition 1.3. [4] A set-labeling f of a signed graph Σ is said to be a *set-indexer* of Σ if f is a set-indexer of the underlying graph G.

If the context is clear, we can represent a set-valued signed graph or a set-indexed signed graph simply by Σ itself. In this section, we discuss the 2-clusterability and balance of set-valued signed graphs.

The following are some of the important and relevant results proved in [4].

Theorem 1.4. [4] *An edge e of a set-labeled signed graph is a positive edge if and only if the set-labels of its end vertices are of the same parity.*

Theorem 1.5. [4] *A set-valued signed graph is 2-clusterable if and only if at least two adjacent vertices in Σ have opposite parity set-labels.*

2. Associated graphs of set-valued signed graphs

We say that two vertices (or edges) of a set-valued graph (or signed graph) are of the same parity if their set-labels are of

even cardinality or odd cardinality simultaneously. Otherwise, we say that the vertices are of different parity. Using this terminology, in this section, we discuss the admissibility of induced set-labeling by certain signed graphs associated with the set-labeled signed graphs.

If some new vertices of degree 2 are added to some of the edges of a graph G, the resulting graph H is called a *subdivision* or *expansion* of G (see [17]). In the following theorem, we discuss whether the subdivision of a set-labeled signed graph admits an induced set-labeling.

Theorem 2.1. *A subdivision Σ' of a set-labeled signed graph Σ is a balanced signed graph under induced set-labeling if and only if Σ is balanced.*

Proof. Let Σ be a balanced set-labeled signed graph and C be a cycle in Σ. Here, note that the number negative edges in C is even. Take any edge $e = uv$ in C. Subdivide the edge uv by introducing a new vertex w to it. Then, let Σ' be the revised signed graph and $C' = (C - \{uv\}) \cup \{uw, vw\}$ be the cycle in Σ' corresponding to C in Σ. Here, the vertex w in Σ' gets the same set-label of the edge uv in Σ. Now, we have to consider the following cases.

Case-1: Let u and v are same parity vertices. Then, by Theorem 1.4, the edge uv is a positive edge. Then, the edge uv will be removed and two edges uw and vw are created instead in the reduced graph. Here, we need to consider the following two cases.

Subcase-1.1: If w is of the same parity to u and v, then the new edges uw and vw are also positive edges. This will not affect the number of negative edges in the modified cycle C' in Σ'.

Subcase-1.2: If w is of the different parity to u and v, then the new edges uw and vw are negative edges. Therefore, note that each such subdivision generates two negative edges in place of a positive edges which keeps the number of negative edges even in the modified cycle C' in Σ'.

Case-2: Let u and v are different parity vertices. Then, by Theorem 1.4, the edge uv is a negative edge. Then, in C', the vertex w will be of same parity with exactly one of the vertices u and v. Then, exactly one of uw and vw is a positive edge (and the other is a negative edge). In all such cases, one negative edge and one positive edge are created in the revised cycle C'. This clearly ensures that the number of negative edges in C' is also even always.

In all the above cases, we can find that the number of negative edges either remain unchanged or increase by an even number, indicating that the number of negative edges in C' is always even if the number of edges in C is even. Therefore, Σ' is balanced.

Conversely, assume that the subdivision Σ' of the set-labeled signed graph Σ is balanced with respect to an induced set-labeling of the set-labeling of Σ. Take an edge uv of a cycle C in Σ such that uv is subdivided by a vertex w in Σ'. Clearly, the set-label of uv in Σ and the set-label of w in Σ' are the same. Now, Consider the following cases.

Case-1: If the edges uw and vw in C' are positive, then we have u, v and w are of the same parity and hence the edge uv in C is also positive. Then, the number of negative edges in C' and C are the same even number.

Case-2: If the edges uw and vw in C' are negative, then we have u and v are of same parity which is different from the parity of w. In this case, the edge uv in C is positive. Then, the number of negative edges in C is decreased by 2 from that of C'. Hence, the number of negative edges in C is also an even number.

Case-3: If exactly one of the edges uw and vw in C' is negative, then we have u and v are of different parity, where exactly one of them have the same parity that of w. In this case, the edge uv in C is negative. Then, the number of negative edges in C is decreased by 2 from that of C'. Then, the number of negative edges in C' and C are the same even number.

Form all the above cases, we note that the number of negative edges in any cycle C of Σ is even, provided the number of negative edges in the corresponding cycle C' in Σ' is even. Hence, Σ is balanced, completing the proof. $\square$

A signed graph Σ' is said to be homeomorphic to another signed graph Σ if Σ' is obtained by removing a vertex v with $d(v) = 2$ and is not a vertex of any triangle in Σ, and joining the two pendant vertices thus formed by a new edge. This operation is said to be an *elementary transformation* on Σ. The following theorem discusses the balance of a signed graph that is homeomorphic to a given balanced set-valued signed graph Σ.

Theorem 2.2. *Let Σ' be signed graph which is homeomorphic to a balanced set-valued signed graph Σ. Then, Σ' is balanced with respect to induced set-labeling if and only Σ is balanced.*

If Σ' is a signed graph obtained from the signed graph by applying finite number of elementary transformations, then Σ can be considered as a subdivision of Σ'. Hence, the necessary condition in Theorem 2.1 is the sufficient condition for Theorem 2.2 and the sufficient condition in Theorem 2.1 is the necessary condition for 2.2.

The *line graph* of an undirected graph G is a graph, denoted by $L(G)$, which represents the adjacencies between edges of G (see [9]). That is, given a graph G, the line graph $L(G)$ of G is the graph such that $V(L(G)) = E(G)$ and $E(L(G)) = \{(e, e') : e, e' \in E(G)$ have a common endpoint in $G\}$.

Verification of the balance of the signed line graph of a set-valued graph Σ, under an induced set-labeling is very complex. But we discuss the condition for the line graphs of some special graph classes in the following results.

An *edge contraction* of a graph G is an operation which removes an edge from G and merge its two end vertices preserving all adjacency of the merged vertices. The graph obtained by contracting an edge e of a given graph G is denoted by $G \circ e$. It is customary that the new vertex obtained by merging the end vertices of the contracted edge is labeled by the same set-label of the contracted edge in the given graph.

The following theorem establishes the balance of a signed graph obtained by contracting some edges of a set-valued signed graph.

Theorem 2.3. *Let Σ be a set-valued signed graph. A signed graph $\Sigma' = \Sigma \circ uv$ is balanced under induced set-labeling if and only if Σ is a balanced.*

Proof. First assume that Σ is balanced. Note that if we contract an edge which is not contained in any cycle of Σ, it will not affect the balance of the reduced graph. Hence, let C be a cycle in Σ. Then, C has even number of negative edges. Now choose an edge $e = uv$ in C. Let u' and v' be the vertices adjacent to u and v in respectively and w be the new vertex obtained after contraction. Here, we have to consider the following cases.

Case-1: Assume u and v are of the same parity. Then, uv is a positive edge. In the reduced graph, w is of even parity. Then, there are the following possibilities.

Case-1.1: If u' and v' are of the same parity with u and v, then all the edges $u'u, uv, vv'$ are positive edges and after contraction, the edges $u'w$ and $v'w$ are either positive edges or negative edges. In this case, the number of negative edges in the cycle C and in the reduced cycle C' are the same or C' has 2 edges more than that of C.

Case-1.2: If u' and v' are of the different parity with u and v, then the edges $u'u$ and vv' are negative edges and after contraction, the edges $u'w$ and $v'w$ are either positive edges or negative edges. In this case, the number of negative edges in the cycle C and in the reduced cycle C' are the same or C' has 2 edges less than that of C.

Case-1.3: If u' and u are of the same parity and v and v' are of the different parity, then the edges $u'u$ is positive and vv' is negative edge and after contraction, the one of edges $u'w$ and $v'w$ is positive edge and the other is a negative edge. In this case, the number of negative edges in the cycle C and in the reduced cycle C' are the same.

Case-2: Assume u and v are of the same parity. Then, uv is a negative edge. In the reduced graph w is of odd parity. Then, there are the following possibilities.

Case-2.1: If u' and v' are of the same parity with u and v respectively, then the edges $u'u$ and vv' are positive edges and after contraction, one of the edges $u'w$ and $v'w$ is positive edge and the other is a negative edge. In this case, the number of negative edges in the cycle C and in the reduced cycle C' are the same.

Case-2.2: If u' and v' are of the different parity with u and v respectively, then all the edges $u'u$ and vv' are negative 0edges and after contraction, one of the edges $u'w$ and $v'w$ is positive edge and the other is a negative edge. In this case, the number of negative edges in the cycle C' is 2 less than that of the cycle C.

Case-2.3: If u' and u are of the same parity and v and v' are of the different parity, then the edge $u'u$ is a positive edge and vv' is a negative edge and after contraction, edges $u'w$ and $v'w$ will be positive edges. In this case, the number of negative edges in the reduced cycle C' is 2 less than that of C.

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In all the above mentioned cases, we note that the number of negative edges in C' is even. Hence, $\Sigma \circ uv$ is balanced.

In a similar and reverse manner of arguments, the converse part of the theorem can also be verified. □

3. 2-Clusterability of associated graphs

A signed graph Σ is said to be *clusterable* or *partitionable* (see [19, 20]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters.

If $V(\Sigma)$ can be partitioned in to k subsets with the above mentioned conditions, then the signed graph Σ is said to be *k-clusterable*. In this paper, we study the 2-clusterability of signed graphs only.

Note that 2-clusterability always implies balance in a signed graph Σ. But, the converse need not be true. If all edges in Σ are positive edges, then Σ is balanced but not 2-clusterable.

In this section, we discuss the 2-clusterability of the above mentioned associated graphs of a set-valued signed graph. The following theorem establishes the 2-clusterability of a subdivision of a set-valued 2-clusterable signed graph.

Theorem 3.1. *A subdivision Σ' of a set-valued signed graph Σ is 2-clusterable if and only if Σ is 2-clusterable.*

Proof. Let Σ be 2-clusterable with the 2-cluster (X_1, X_2). Choose an arbitrary edge $e = uv$ of Σ for subdividing to get the reduced graph Σ'. If $e \in X_1$, then uv is a positive edge and hence the corresponding new vertex w in Σ' will be an even parity vertex.

If u and v are of odd parity, then $w \in X_2$, which will not be adjacent to any other vertex in X_2. If u and v are of even parity, then the edges uw and vw are positive edges and hence u, v, w are in X_1 itself. In both cases, the 2-clusterability of Σ is preserved in Σ' also. A similar argument can be established for the case when $e \in X_2$ also.

Next, if we subdivide a negative edge $e = uv$, where $u \in X_1$ and $v \in X_2$, then we can see that the new vertex w will be of the same parity with exactly one of u and v. If u and w are of same parity, then uv is a positive edge and vw is a negative edge. Hence, $w \in X_1$. Here also, the 2-clusterability is not affected. Therefore, Σ' is 2-clusterable if Σ is 2-clusterable.

The converse part also can be proved in a similar way. □

The above theorem may not hold for homeomorphic graphs which can be illustrated in Figure 1 as follows.

If uw and vw are the only negative edges in Σ, then elementary transformation applied on the vertex w may remove the negative edges in the revised signed graph, which remove the 2-clusterability of Σ'.

Remark 3.2. *The 2-clusterability of a signed graph $\Sigma' = \Sigma \circ uv$ under induced set-labeling will be preserved if the contracted edge uv is not the only negative edge in the 2-clusterable set-valued signed graph Σ.*
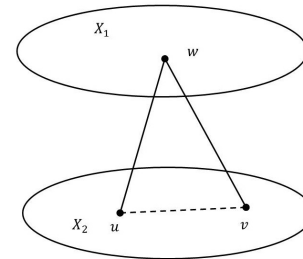

Figure 1

Hence, trivially, we have the following theorem.

Theorem 3.3. *Let $\Sigma' = \Sigma \circ e$, where Σ is a 2-clusterable set-valued signed graph. Then , Σ' is 2-clusterable if and only if Σ 2-clusterable with at least two negative edges.*

4. Conclusion

In this paper, we have discussed the admissibility of induced set-labelings by certain signed graphs associated with given set-labeled signed graphs. There are more open problems in this area. Some of the open problems, we have found during this study, are the following.

Problem 4.1. An edge of a graph G is said to be *contracted* if that edge is removed from G and its end vertices are identified to form a new single vertex such that the new vertex preserves the adjacency of the contracted vertices. Defining suitable induced set-labels for the signed graphs obtained by finite number of edge contractions and verifying the balance and clusterability of these reduced signed graphs, under induced set-labelings.

Problem 4.2. Verify the balance and clusterability of line graphs of different classes of set-labeled signed graphunder induced set-labelings.

Problem 4.3. Verify the balance and clusterability of total graphs of different classes of set-labeled signed graph, under induced set-labelings.

Problem 4.4. Verify the balance and clusterability of a signed digraph corresponding to a set-valuation defined on it.

Further studies on other characteristics of signed graphs and their various associated graphs corresponding to different set-labeled graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

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