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A note on strong zero-divisor graphs of near-rings

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Abstract
For a near-ring N, the strong zero-divisor graph $\Gamma_s(N)$ is a graph with vertices $V^*(N)$, the set of all non-zero left N-subset having non-zero annihilators and two vertices I and J are adjacent if and only if $IJ = 0$. In this paper, we study diameter and girth of the graph $\Gamma_s(N)$ wherein the nilpotent and invariant vertices are playing a significant role. We show that if $diam(\Gamma_s(N)) > 3$, then N is necessarily a strongly semi-prime near-ring. Also we find the $\chi(\Gamma_s(N))$ and investigate some characterizations of cliques and maximal cliques in $\Gamma_s(N)$.

Keywords
Near-ring; essential ideal; diameter; girth; chromatic number.

AMS Subject Classification
16Y30, 13A15.

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Contents

1. Introduction

Let N be a zero symmetric (right) near-ring and $V(N)$ be the set of all left N-subsets with non-zero left annihilators. The strong zero-divisor graph denoted $\Gamma_S(N)$ is a directed simple graph with the set of vertices $V^*(N) = V(N)\{0\}$ such that any two distinct I and $J \in V^*(N)$ are adjacent if and only if $IJ = 0$.

The concept of zero-divisor graph of a commutative ring was first introduced by Beck in [4]. Beck [4] has mainly investigated coloring of the ring. He has conjectured that $\chi(\Gamma(R)) = clique(\Gamma(R))$. Anderson et all redefined the notion of zero-divisor graphs in [2] and proved that such a graph is always connected and its diameter is less than or equal to 3. Anderson and Mulay in [3] studied diameter and girth of zero-divisor graph of a commutative ring. The notion of zero-divisor graph was extended to a non-commutative ring [1] and various properties of diameter and girth were established. Behboodhi [5] studied annihilator ideal graphs dealing with the annihilators of ideals of a commutative ring. Redmond[8] has generalised the notion of zero-divisor graph. For an ideal I of a commutative ring R, Redmond [8] defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$.

In this paper, we study some graph theoretic aspect of a near-ring N. For basic definitions and results related to near-ring, we would like to mention Pilz [7]. A subset I of N is left(right)N-subset of N if $NI \subseteq I(IN \subseteq I)$ and I is invariant if it is both left as well as right N-subset of N. If I is a left N-subset of N, then $l(I) = \{x \in N \mid xI = 0\}$ is the left annihilator of I. For any N-subset I, $l(I)$ is also a left N-subset of N. If I and J be two left N-subsets, then so is $I \cap J$. A left N-subset I of N is nilpotent with index $n(n \in Z_+)$ if $I^n = 0$ and $I^m \neq 0$ for $m < n$. The near-ring N is strongly semi-prime if it has no non-zero nilpotent invariant subsets. A left N-subset(ideal) I of N is essential in N if for any non-zero left N-subset(ideal) A of N, $I \cap A \neq 0$.

Recall that a graph G is connected if there is a path between any two distinct vertices. The graph G is complete if every two vertices are adjacent. The distance between two distinct vertices x and y of G is the length of the shortest path from x to y denoted $d(x,y)$. If no such path exists, then $d(x,y) = \infty$. The diameter of the graph G is $diam(G) = sup\{d(x,y)|x$ and y are distinct vertices of $G\}$. The girth of G is the length of distance of the shortest cycle in G, denoted $gr(G)$. If there is no such cycle, then $gr(G) = \infty$. The minimal numbers of colors so that no two adjacent elements of the graph G have same color is the chromatic number of G denoted $\chi(G)$.

In this paper, we study diameter and girth of the strong zero-divisor graphs of near-rings wherein the nilpotency and invariant character of vertices playing a significant role. We prove that any path joining vertex I to an invariant vertex J is contained in a cycle provided $l(I+J)\neq 0$. We show that a strongly semi-prime near-ring contains no non-zero nilpotent invariant subset if $\Gamma_s(N)>3$. Moreover, we show that in this case $\Gamma_s(N)$ contains not more than two invariant subsets I_1 and I_2 so that $l(I_1)$ and $l(I_2)$ are essential. In addition to the above, we investigate the coloring of $\Gamma_s(N)$ and some characterisations of cliques and maximal cliques of $\Gamma_s(N)$.

Below we discuss some examples of strong zero-divisor graphs $\Gamma_s(N)$ in contrast to zero-divisor graph $\Gamma(N)$

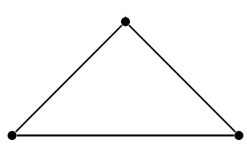
Example 1.1. *If the graph $\Gamma_s(N)$ contains a point only, then $N\cong Z_4$ or $\frac{Z_2[x]}{<x^2>}$. In this case $gr(\Gamma_s(N))=\infty$. Also $\Gamma_s(Z_4)\cong\Gamma(Z_4)$*

$\Gamma_s(Z_4)$ or $\Gamma(Z_4)$

Example 1.2. *If the graph $\Gamma_s(N)$ contains two points, then $\Gamma_s(N)$ contains no cycle and $gr(\Gamma_s(N)=\infty$. In this case $N\cong Z_2\times Z_2$ or Z_6. Here $Z_2\times Z_2\ncong Z_6$, however their strong zero divisor graphs $\Gamma_s(Z_2\times Z_2)$ and $\Gamma_s(Z_6)$ are isomorphic i.e. $\Gamma_s(Z_2\times Z_2)\cong\Gamma(Z_2\times Z_2)\cong\Gamma_s(Z_6)$ $(Z_2\times Z_2\ncong Z_6)$*

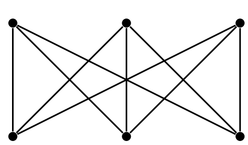
$\Gamma_s(Z_2\times Z_2)$, $\Gamma(Z_2\times Z_2)$ or $\Gamma_s(Z_6)$

Example 1.3. *The graph $\Gamma_s(\frac{Z_3[x]}{\langle x^2\rangle})$ is graph with finite girth 3. Here $I=\{\langle x^2\rangle, x+\langle x^2\rangle, 2x+\langle x^2\rangle\}$ is a nilpotent ideal as $I^2=0$.*
$\Gamma_s(\frac{Z_3[x]}{\langle x^2\rangle})\cong\Gamma_s(\frac{Z_2[x,y]}{\langle x^2,y^2,xy\rangle})$ but $\frac{Z_3[x]}{x^2}\ncong\frac{Z_2[x,y]}{\langle x^2,y^2,xy\rangle}$.



$\Gamma_s(\frac{Z_3[x]}{\langle x^2\rangle})$ or $\Gamma_s(\frac{Z_2[x,y]}{\langle x^2,y^2,xy\rangle})$

Example 1.4. *The graph below is a complete bipartite graph with girth 4. $\Gamma_s(\frac{Z_4[x]}{\langle x^2\rangle})\ncong\Gamma_s(\frac{Z_2[x,y]}{\langle x^2,xy,y^2\rangle})$ $(\frac{Z_4[x]}{\langle x^2\rangle}\cong\frac{Z_2[x,y]}{\langle x^2,xy,y^2\rangle})$*



$\Gamma_s(\frac{Z_4[x]}{\langle x^2\rangle})$

2. Diameter and girth

In this section, we present some of characteristic of paths, diameter and girth of $\Gamma_s(N)$. We note that the vertex 0 is adjacent to every other vertices which we exclude here for obvious reason.

A vertex $I\in\Gamma_s(N)$ is an invariant vertex if it is an invariant N subset of the near-ring N. The right annihilator $r(I)=\{x\in N\mid Ix=0\}$ of a left N-subset I of N is a right N-subset of N not necessarily coincide to $l(I)$. However in a strongly semi-prime near-ring N, in case of an invariant subset I, $Il(I)=0$ as $(Il(I))^2=I(l(I)I)l(I)=0$ giving thereby $l(I)\subseteq r(I)$. Similarly $r(I)\subseteq l(I)$. Thus we state the following lemma.

Lemma 2.1. *[6] Let N be a strongly semi-prime near-ring. Then for an invariant subset I of N, $l(I)=r(I)$.*

Let I be a left N-subset with $l(I)\neq 0$ and let $x(\neq 0)\in l(I)$. If $J\subseteq l(I)$ be a non-zero nilpotent N-subset. Then there exists a positive integer m such that $xJ^m=0$ but $xJ^{m-1}=0$. It is clear that $l(I+J)\subseteq l(I)\cap l(J)$.

Lemma 2.2. *[6] Let N be a near-ring such that the left annihilators are distributively generated. If I be a left N-subset with $l(I)\neq 0$ and $J\subseteq l(I)$ is a nilpotent left N-subset of N, then $l(I+J)\neq 0$.*

Proof. Let $x(\neq 0)\in l(I)$ such that $xJ^m=0$ and $xJ^{m-1}\neq 0$ for some positive integer m. Now $xJ^{m-1}J=xJ^m=0$ and $xJ^{m-1}I=xJ^{m-2}JI=0$. Thus $xJ^{m-1}(I+J)=0$ giving thereby $xJ^{m-1}\subseteq l(I+J)$. Thus $l(I+J)\neq 0$. □

Thus in this lemma, we see that the nilpotency of $J\subseteq l(I)$ leads us to $l(I+J)\neq 0$.

Throughout the paper, by a near-ring N we mean a strongly semi-prime near-ring unless otherwise specified.

Theorem 2.3. *Let N be a near-ring and J be an invariant N-subset such that $l(I+J)\neq 0$ for some $I\in V(\Gamma_S(N))$. Then any path joining I and J is contained in a cycle of $\Gamma_S(N)$.*

Proof. Let $P: I\longrightarrow K_1\longrightarrow K_2\longrightarrow .....\longrightarrow K_n\longrightarrow J$ be any path. Now $l(I+J)\subseteq l(I)\cap l(J)$ implies $l(I)\cap l(J)\neq 0$ as $l(I+J)\neq 0$. Let $M=l(I)\cap l(J)$ which is a non-zero left N-subset of N. Then $I\longrightarrow K_1\longrightarrow K_2\longrightarrow .....\longrightarrow K_n\longrightarrow J\longrightarrow M\longrightarrow I$ is a cycle containing the path P since $JM=J(l(I)\cap l(J))=J(l(I)\cap r(J))=0$ and $MI=(l(I)\cap l(J))I=0$. □

Theorem 2.4. *Let N be a near-ring such that girth$(\Gamma_S(N))>3$. Then N has no non-zero nilpotent invariant subset*

Proof. Let $I(\neq 0)$ be a nilpotent invariant subset of N and n be the least positive integer such that $I^n=0$. Now $I.I^{n-1}=0$ gives that $I^{n-1}\subseteq r(I)=l(I)$[Lemma 2.2]. Thus $l(I)$ is a non-zero left N-subset of N so that $I^{n-1}l(I)=I^{n-2}.Il(I)=I^{n-2}Ir(I)=0$. Thus $l(I)\longrightarrow I\longrightarrow I^{n-1}\longrightarrow l(I)$ is a circuit, which is a contradiction. □

In the example 1.3 $gr(\Gamma_s(\frac{Z_4[x]}{\langle x^2\rangle})) = 3$. Here $\frac{Z_4[x]}{\langle x^2\rangle}$ is not strongly semi-prime. The non-zero ideal $I = \{\langle x^2\rangle, x+\langle x^2\rangle, 2x+\langle x^2\rangle, 3x+\langle x^2\rangle\}$ is nilpotent as $I^2 = 0$.

A left N-subset I of N is said to be simple if there exists no non-zero left N- subset J such that $J \subseteq I$

Theorem 2.5. *Let I be an essential simple N-subset of N, then I is adjacent to every nilpotent $J \in V(\Gamma_S(N))$.*

Proof. Let $J \in V(\Gamma_S(N))$ be nilpotent and let m be the least positive integer such that$J^m = 0$. Now $I \cap J \neq 0$ as I is essential. Also $I \cap J = I$ since I is a simple N-subset giving thereby $J \subseteq I$. If $IJ = 0$, then we are done. For otherwise $IJ = I \cap J = I$ which gives that $IJ^2 = IJ.J = IJ = I$. Similarly $IJ^3 = IJ^2J = IJ = I$. Continuing in this way we get that $IJ^m = I$ which gives that $I = 0$, a contradiction. □

Theorem 2.6. *Let N be a near-ring such that the left annihilators are distributively generated. Let I be an invariant simple N-subset and J be a nilpotent left N-subset such that $l(I+J) = 0$. Then for some $q \in l(I)$, $d(I+qJ, J) = 3$.*

Proof. We give the proof in two steps such as

(i) Step 1: We show that for any two non-adjacent vertices I and J where I is an invariant N-subset so that $l(I+J) \neq 0, d(I,J) = 3$. Since I is an invariant N-subset there for $Il(I) = Ir(I) = 0$[Lemma 2.1]. We claim $l(I)l(J) = 0$. For let $xy(\neq 0) \in l(I)l(J)$ gives that $Ixy \subseteq Il(I)l(J) = Ir(I)l(J) = 0$, giving thereby $Ixy = 0$. Thus $xy \in r(I) = l(I)$. Also $y \in l(J)$ implies $xy \in l(J)$ giving thereby $xy \in l(I+J)$. Thus $l(I+J) \neq 0$, a contradiction. Thus $l(I)l(J) = 0$. Thus $I \longrightarrow r(I) = l(I) \longrightarrow l(J) \longrightarrow J$ is a directed path. Thus $d(I,J) = 3$.

(ii) Since $I(\neq 0)$ is simple invariant and J is nilpotent, therefore $IJ = 0$[Theorem 2.5]. Thus $(I+J)^2 = I^2 + J^2 = 0$. Also $l(I+J)^2 = 0$, as $x \in l(I+J)^2$ gives $x(I+J) \subseteq l(I+J)$. Thus $x(I+J) = 0$ giving thereby $x \in l(I+J) = 0$ as $IJ = 0 = JI$. Since J is nilpotent, therefore qJ is also so for some $q(\neq 0) \in l(I) = r(I)$. Also $qJ^2 \neq 0$, for otherwise $q(I^2+J^2) = qI^2 + qJ^2 = qI.I + qJ^2 = 0$. Thus $q(\neq 0) \in l(I^2+J^2) = l(I+J)^2$, a contradiction. Again $I+qJ \neq J$, for otherwise $I \subseteq J$ implies $I+J = J$. Thus $l(I+J) = l(J)(\neq 0)$, a contradiction. Hence $I+qJ, J$ are distinct and $I+J = I+qJ+J$ which gives $l(I+qJ+J) = 0$ and $(I+qJ)J = IJ + qJ^2 = qJ^2(\neq 0)$. Hence $d(I+qJ, J) = 3$[caseI].

□

Theorem 2.7. *Let N be a near-ring such that $\Gamma_s(N) > 3$, then N does not contain more than two invariant vertices I_1 and I_2 such that $l(I_1)$ and $l(I_2)$ are essential.*

Proof. Let I_1, I_2 and I_3 be three invariant N-subsets such that $l(I_1), l(I_2)$ and $l(I_3)$ are essential. Let $J_1 = l(I_1) \cap I_2 \neq 0, J_2 = l(I_2) \cap I_3 \neq 0$ and $J_3 = l(I_3) \cap I_1 \neq 0$. Clearly each $J_i, i = 1,2,3$ are left N-subsets. Also $l(I_1) \cap I_2 \subseteq I_2$ which gives that $l(I_2) \subseteq l(l(I_1) \cap I_2)$. Thus $l(l(I_1) \cap I_2) = l(J_1) \neq 0$. Similarly $l(J_2) \neq 0$ and $l(J_3) \neq 0$. Also J_1, J_2 and J_3 are distinct. For otherwise $J_1^2 = J_2J_1 = (l(I_2) \cap I_3)(l(I_1) \cap I_2) \subseteq l(I_2)I_2 = 0$ implies that $J_1 = 0$, a contradiction. Thus J_1, J_2 are J_3 are distinct such that $J_1J_2 = 0$, $J_2J_3 = 0$ and $J_3J_1 = 0$. Thus $J_1 \to J_2 \to J_3 \to J_1$ is a cycle, a contradiction. □

Theorem 2.8. *[6] Let N be a such that $\Gamma_s(N)$ contains a cycle with an invariant vertex in it. Then $gr(\Gamma_s(N)) \leq 4$.*

3. Coloring of $\Gamma_s(N)$

In this section we present some characterization of cliques as well as of maximal cliques in $\Gamma_s(N)$. Also we establish some bounds for chromatic no of the graph.

Theorem 3.1. *Let N be a near-ring and $I_1, I_2, ..., I_k$ be ideals of N such that $l(I_i)$'s are maximal as annihilator. Then the following are equivalent.*

(i) $P_i = l(I_i)'s$ are prime ideals so that $P_i \cap P_j = 0$ for $i \neq j$.

(ii) $\{I_1, I_2, ..., I_k\}$ is a clique.

Proof. Assume that $P_i = l(I_i)'s$ are prime ideals so that $P_i \cap P_j = 0$ for $i \neq j$. Let $I_iI_j \neq 0$, $(1 \leq i,j \leq k)$ gives that $I_i \nsubseteq l(I_j) = P_j$. Now $l(I_i)I_i = 0$ implies $l(I_i)I_i \in P_j$ which gives that $l(I_i) \subseteq P_j$ or $I_i \subseteq P_j$. Thus $l(I_i) = P_i \subseteq P_j$ giving thereby $P_i = P_i \cap P_j = 0$, a contradiction. Thus $I_iI_j = 0$. Conversely, assume that $\{I_1, I_2, ..., I_k\}$ is a clique. Suppose that I and J be two ideals such that $IJ \subseteq P_i$ so that $I \nsubseteq P_i$ and $J \nsubseteq P_i$. Now $IJ \subseteq P_i = l(I_i)$ implies $IJI_i = 0$ giving thereby $I \subseteq l(JI_i)$. Again I_i is an invariant N-subset being an ideal. Thus $x \in l(I_i) = r(I_i)$[Lemma 2.2] implies $I_ix = 0$. Thus $JI_ix = 0$ giving thereby $x \in l(JI_i)$. Hence $l(I_i) \subseteq l(IJ_i)$ giving thereby $l(I) = l(IJ_i) = 0$. Thus $I^2 = 0$, a contradiction. □

Theorem 3.2. *Let N be a near-ring and $I_i, i = 1,2,...,k$ be the ideals of such that $l(I_i)'s$ are pairwise disjoint and maximal as annihilator. Then the following are equivalent.*

(i) $\{I_1, I_2, ..., I_k\}$ is a maximal clique.

(ii) $l(I_i)$'s are only annihilator prime ideals.

Proof. Assume that $\{I_1, I_2, ..., I_k\}$ is a maximal clique. Let I be another ideal distinct from I_i, $1 \leq i \leq k$ where $l(I)$ is maximal with $l(I) \cap l(I_i) = 0$. If $l(I)$ is prime, then we show that $I_iI = 0$. Suppose that $I_iI \neq 0$ which implies $I_i \nsubseteq l(I)$. Now $l(I_i)I_i) \subseteq l(I)$ gives that either $l(I_i) \subseteq l(I)$ or $I_i \subseteq l(I)$. Thus $l(I_i) = l(I_i) \cap l(I) = 0$, a contradiction. Hence $I_iI = 0$ for $i = 1,2,....,k$. Thus $\{I, I_1, I_2, ..., I_k\}$ is a clique which contains $\{I_1, I_2, ..., I_k\}$, a contradiction. Thus $I = I_i$ for some $i, 1 \leq i \leq k$. Thus $l(I_i)$'s are only prime ideals of this type. Conversely, let $C^/$ be a clique such that $C = \{I_1, I_2, ..., I_k\} \subset C^/$. Let $I \in C^/$ such that $I \notin C$. Now $I_iI = 0$ for all i. We claim that $l(I)$ is a prime ideal. Suppose A, B be two ideals such that $AB \subseteq l(I)$ and $A \nsubseteq l(I)$, $B \nsubseteq l(I)$. Now $ABI = 0$ gives $A \subseteq l(BI) = l(I)$, a contradiction. Thus $l(I)$ is a prime ideal and $l(I) = l(I_i)$ for some $1 \le i \le k$. Now $II_i(=0) \subseteq l(I_i)$ gives that either $I \subseteq l(I_i)$ or $I_i \subseteq l(I_i)$. Thus $l(I)$ is a prime ideal, a contradiction. Thus $\{I_1, I_2, ..., I_k\}$ is a maximal clique. □

Example 3.3. *Consider Z_6, the integer modulo 6. Here $I_1 = \{0,3\}$, $I_2 = \{0,2,4\}$ are only ideals. Clearly $\{0, I_1, I_2\}$ is a clique in $\Gamma_s(Z_6)$. Also $AnnI_1 = I_2$, $AnnI_2 = I_1$ which are prime ideals of Z_6.*

Theorem 3.4. *Let N be a near ring and $I_1, I_2, ..., I_k$ be the only ideals such that $P_i = l(I_i), 1 \le i \le k$ are pairwise disjoint and maximal as annihilator. Then $\chi(\Gamma_s(N)) \le k+1$.*

Proof. It is clear that $\{I_1, I_2, ..., I_k\}$ forms a clique[Theorem 3.1]. We give each $I_i's$ a distinct color and one extra color to $'0'$. We claim that these $k+1$ colors are sufficient to color the graph $\Gamma_s(N)$. Consider $I(\neq 0)$ be any ideal. Then $I \nsubseteq l(I_i)$ for some i, for if $I \subseteq l(I_i)$ for each i, then $I \subseteq l(I_1) \cap l(I_2) \cap ..... \cap l(I_k) = 0$, a contradiction. Let $k = min\{i | I \nsubseteq l(I_i)\}$. Thus $I \nsubseteq l(I_k)$ which gives that $II_k \neq 0$. Here we give the color of I_k to I. Let J be another ideal which is also coloured with the color of I_k. Thus $JI_k \neq 0$. We claim $IJ \neq 0$. Suppose $IJ = 0$ gives that $IJ \subseteq l(I_k)$ which gives that either $I \subseteq l(I_k)$ or $J \subseteq l(I_k)$ as $l(I_k)$ is a prime ideal, a contradiction. Next, let I, J be two left N-subsets such that I and J are given the color of some I_k. Thus $II_k \neq 0$ and $JI_k \neq 0$. Here $IJ \neq 0$. For if $IJ = 0$, we get $IJI_k = 0$. Thus $I \subseteq l(JI_k) = r(JI_k)$. But for any $x \in l(I_k) = r(I_k)$, $I_k x = 0$ gives $JI_k x = 0$ which implies that $x \in r(JI_k) = l(JI_k)$. Thus $l(I_k) = l(JI_k)$ giving thereby $II_k = 0$, a contradiction. Thus $IJ \neq 0$. □

Example 3.5. *: Consider $Z_6 = \{0,1,2,3,4,5\}$ which is a near-ring with respect to the tables given below. The only left N subsets are $I_1 = \{0,3\}$, $I_2 = \{0,2,4\}$ and $I_3 = \{0,2,3.4\}$ which are invariant also and $l(I_1) = I_2$ and $l(I_2) = I_1$ are two maximal ideals of the annihilator ideal form. Here the chromatic number $\chi(\Gamma_s(Z_6))$ is $2+1 = 3$, i.e., $\chi(\Gamma_s(Z_6))$ is equal to $p+1$, where p is the number of maximal ideals of the form of left annihilator.*

In the results below, we deal with the essentiality of annihilator ideals in a near-ring N to determine the chromatic number of $\Gamma_s(N)$.

Theorem 3.6. *Let N be a near-ring with unity, then the following two are equivalent.*

(i) If for a left N-subset I of N, $l(I)$ is essential, then $I = 0$.

(ii) N is strongly semi-prime.

Example 3.7. *Consider the ring $Z_6 = \{0,1,2,3,4,5\}$ which is strongly semi-prime with unity. Here $I_1 = l(I_2) = \{0,3\}$ and $I_2 = l(I_1) = \{0,2,4\}$ are the only non-zero ideals and $Z_6 = Ann(0)$ is the only essential ideal.*

Example 3.8. *$Z_4 = \{0,1,2,3\}$ is a ring with unity. Here Z_4 is not strongly semi-prime as for $I = \{0,2\}$, $I^2 = 0$ and $l(I)$ is an essential ideal of Z_4*

Theorem 3.9. *:[6]Let N be a near-ring such that $\Gamma(N)$ has no infinite clique, then the near-ring N satisfies the acc on essential left N-subsets.*

Proof. Let $I_1 < I_2 < I_3 < ......$be an ascending chain for essential left N-subsets. Suppose $I_i < I_{i+1}$. Now $I_i \cap l(I_i) \le I_{i+1} \cap l(I_i)$. Here $I_i \cap l(I_i) \neq 0$ as I_i is essential. Thus $I_{i+1} \cap l(I_i) \neq 0$. Also $I_i \cap l(I_i) \neq I_{i+1} \cap l(I_i)$ for otherwise $(I_i \cap l(I_i))^2 = (I_{i+1} \cap l(I_i))(I_i \cap l(I_i)) \subseteq l(I_i)I_i = 0$, a contradiction. Now consider an element $x_n \in I_n \cap l(I_{n-1})$ such that $x_n \notin I_{n-1} \cap l(I_{n-1})$. Here for $i \neq j$ (suppose $i > j$), $x_i x_j \in (I_i \cap l(I_{i-1}))(I_j \cap l(I_{j-1})) \subseteq l(I_{i-1})I_j = 0$. Thus we get an infinite clique in N, a contradiction. □

Theorem 3.10. *:[6]Let N be a near-ring without unity. If $\Gamma_s(N)$ has no infinite clique, then N satisfies the acc on invariant subsets having essential left N-subsets.*

Proof. Let $I_1 < I_2 < I_3....$ be an ascending chain of invariant subsets with essential left annihilators. Suppose $I_i \subsetneq I_{i+1}$. Let $x_{i+1}(\neq 0) \in I_{i+1} \setminus I_i$. Now consider $J_{i+1} = l(I_{i+1}) \cap \langle x_{i+1} \rangle \neq 0$, where $\langle x_{i+1} \rangle$ is the ideal generated by x_{i+1}. Here $J_i J_j = 0$ for $i < j$, a contradiction. □

Theorem 3.11. *:[6] Let N be a near-ring with unity and $l(I_1), l(I_2), ..., l(I_n)$ be the only essential N-subsets of N such that each I_i is an ideal. Then $\chi(\Gamma_s(N)) \le n+1$.*

Example 3.12. *Consider the set $Z_{(p^\infty)}$ of all rational numbers of the form $\frac{m}{p^k}$ such that $0 \le \frac{m}{p^k} < 1$, where p is a fixed prime number, n runs through all non negative integers. Then $Z(p^\infty)$ is a ring with respect to addition modulo 1 and multiplication defined as $ab = 0$ for all $a, b \in Z(p^\infty)$. It is to be noted that each subgroup of $Z(p^\infty)$ is an ideal of it and the only proper ideals of $Z(p^\infty)$ are of the form $I_{k-1} = \{0, \frac{1}{p^{k-1}}, \frac{2}{p^{k-2}}, ..., \frac{p^{k-1}-1}{p^{k-1}}\}$ for each positive integer k. Thus the ideals are in a chain $0 < I_1 < I_2 < .....$ and each I_i's are essential Z_{p^∞} is a reduced ring without unity. But here $l(I_{k-1}) = 0$ for all k which are not essential. Here $I_i I_j \neq 0$ for any i, j and $\chi(\Gamma_s(Z_{p^\infty})) = 2$.*

Example 3.13. *: Consider the set $M(N) = \begin{pmatrix} Z_2 & N \\ 0 & Z_2 \end{pmatrix}$ which is the set of elements of the form $\{\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\}$, where $n \in N$. Here $M(N)$ is a near-ring with respect to ordinary addition and multiplication($\bar{x}n = xn, \bar{x} \in Z_2$) with unity which is not strongly semi-prime as*

$$\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If N is not finite, then $M(N)$ has infinite invariant sets $I_i(i = 1,2,3,...)$ such *that* $l(I_i) = \{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mid n \in N\}$ *is essential and* $\chi(\Gamma_s(M(N))) = \infty$.

Theorem 3.14. *Let N be a near-ring such that $\Gamma_S(N)$ has no infinite clique, then N satisfies the a.c.c for annihilators of left N-subsets of N .*

Proof. Let $Ann(I_1) \subset Ann(I_2) \subset Ann(I_3) \subset \ldots\ldots$ be an ascending chain where I_i are left $N - subsets$ such that $Ann(I_i) \neq 0$. Suppose $Ann(I_{n-1}) \subset Ann(I_n)$. Let $x_n \in Ann(I_n)$ but $x_n \notin Ann(I_{n-1})$. Consider $J_n = I_{n-1}x_n$. Clearly J_n is also a left N-subset. Also $l(I_{n-1}) \subseteq l(I_{n-1x_n})$ which gives that $l(I_{n-1x_n}) \neq 0$. Thus $l(J_n) \neq 0$. For each n, $J_nJ_{n-1} = (I_{n-1}x_n)(I_nx_{n+1}) = I_{n-1}(x_nI_nx_{n+1}) = 0$. In fact $J_nJ_k = (I_{n-1}x_n)(I_{k-1}x_k) = I_{n-1}(x_nI_{k-1})x_k = 0$ for $n > k$ since $x_n \in Ann(I_n) \subset Ann(I_k)$. Thus $\{J_i | i > 1\}$ forms an infinite clique, a contradiction. □

The subset $C(N) = \{x_i \in N | x_in = nx_i, n \in N\}$ of a near-ring N is called the multiplicative centre of N

Theorem 3.15. *Let $\{x_i\} \subset C(N)$ and let $I =< \{x_1, x_2, ..., x_n >$. If there are only finite Ix_i, then $\Gamma_s(N)$ has an infinite clique.*

Proof. Let $Ix_i = Ix_j$ for some $x_i, x_j \in C(N)$. We can pick $y_i \in \{x_1, x_2, ...\}$ such that $y_iy_j = y_iy_k$ for $i < j < k$. Now $Iy_i's$ are left N-subsets of N. Also $l(I) \subseteq l(Iy_i)$ gives that $l(Iy_i) \neq 0$ as $l(I) \neq 0$. Consider $Z_{ij} = Iy_i - Iy_j$. Clearly Z_{ij} is a left N-subset of N. Also $l(Z_{ij}) \neq 0$, for any $x \in l(I)$, $xI = 0$ implies $xIy_i = 0$ and $xIy_j = 0$ giving thereby $x(Iy_i - Iy_j) = 0$. Thus $l(I) \subseteq l(Z_{ij})$ implies that $l(Z_{ij}) \neq 0$. Further $Z_{ij}Z_{kr} = (Iy_i - Iy_j)(Iy_k - Iy_r) \subseteq ((Iy_iy_k - Iy_jy_k) - (Iy_iy_r - Iy_jy_r) = 0$ for all $i < j < k < r$. Thus $\{Z_{12}, Z_{34}, ...\}$ is an infinite clique of $\Gamma_s(N)$. □

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