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Rainbow coloring in some corona product graphs

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Abstract

Let G be a non-trivial connected graph on which is defined a coloring $c : E(G) \to \{1,2,\cdots,k\}, k \in N$ of the edges of G, where adjacent edges may be colored the same. A path P in G is called a rainbow path if no two edges of P are colored the same. G is said to be rainbow-connected if for every two vertices u and v in it, there exists a rainbow $u-v$ path. The minimum k for which there exist such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper we determine $rc(G)$ for some corona product graphs.

Keywords

Diameter, Edge-coloring Rainbow path, rainbow connection number, Rainbow critical graph, corona product.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

All graphs in this paper are finite, non-trivial, simple, connected and undirected graphs. Coloring problems related to vertex coloring and edge coloring are one of the interesting problems in graph theory and many results exist in literature. A path in a graph is called a rainbow path if no two edges in it are colored the same. Let G be a graph on which is defined a coloring $c : E(G) \to \{1,2,\cdots,k\}, k \in N$of the edges of G, where adjacent edges may be colored the same. A path P in G is called a rainbow path if no two edges of P are colored the same. G is said to be rainbow connected if for every two vertices u and v in it, there exists a rainbow $u-v$ path. The minimum k for which there exist such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. Clearly, every rainbow connected graph is a connected graph, and conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color.

The concept of rainbow coloring was introduced by Chartrand et.al. in [1]. For graph products like direct and strong product graphs, Gologranc et.al. in [2] investigated the bounds with respect to rainbow coloring. For other results related to the bounds we refer [3],[4],[5],[6] and for the exact values of $rc(G)$ for various graphs, we refer [7], [8], [9], [10], [11] and [12].

2. Preliminaries

In this section, we recall some definitions which will be used throughout the paper.

Definition 2.1. *Let G and H be two graphs. The corona product of G and H, denoted by $G \circ H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, where $1 \leq i \leq |V(G)|$.*

The corona product graph $K_4 \circ P_3$ is shown in figure 1 below.

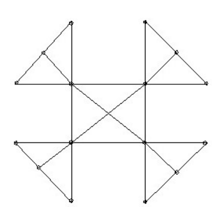

Figure 1. The graph $K_4 \circ P_3$

Definition 2.2. *A connected graph G which has a given property, but $G-e$ does not, for every edge of G is called a critical graph with respect to the given property.*

We say that a connected graph G is partially critical or $P-$ critical with respect to some edge E in G or a subset of edges in G.

3. Main Results

For the path graph P_n we know that $rc(P_n) = n-1$ and for the complete graph K_n we know that $rc(K_n) = 1$.

In our first result, we determine the rainbow connection number of the corona product $P_n \circ K_n$.

Theorem 3.1. *Let $G = P_n \circ K_n$. Then for $n \geq 2$, $rc(G) = src(G) = 2n-1$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and let the vertex set of the i copies of K_n namely $(K_n)_i$ be $V\{(K_n)_i\} = \{u_{ij} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$.

By definition of the corona product, each vertex of P_n is adjacent to every vertex of a copy of K_n, that is, for $1 \leq i \leq n$ the vertex $v_i \in V(P_n)$ is adjacent to the vertices of the set $\{u_{ij} : 1 \leq j \leq n\}$ in the i^{th} copy of K_n.

Let $E(P_n \circ K_n) = \{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(P_n) = \{e_i = (v_i, v_{i+1}); 1 \leq i \leq n-1\}$, E_2 be the edge set of $(K_n)_i$ for $1 \leq i \leq n$ and $E_3 = \{(e_k)_i = (v_i, u_{ij}); 1 \leq i \leq n, 1 \leq k \leq n$ and $0 \leq j \leq n-1\}$.

We assign a rainbow coloring to the edges of $P_n \circ K_n$ as follows:

For $1 \leq i \leq n$ assign the color i to the edges of $(K_n)_i$ and to the edges $(e_k)_i$ of $P_n \circ K_n$ and for $1 \leq j \leq n-1$, assign the color $j+n$ to the edges of (P_n) of $P_n \circ K_n$. From this assignment of colors, it is clear that

$rc(P_n \circ K_n) \leq 2n-1$ ———(i)

To prove $rc(P_n \circ K_n) \geq 2n-1$, we assume that $rc(P_n \circ K_n) = 2n-2$.

Then, for a proper rainbow coloring, $2n-2$ colors must be assigned to the edges of $(P_n \circ K_n)$. Since $P_n \circ K_n$ has n copies of K_n, we assign n colors to the n copies of K_n and assign the remaining $n-2$ colors to $n-1$ edges of P_n. An easy check shows that at least two of the edges of P_n are colored with the same colors.

This implies that at least one path in $P_n \circ K_n$ is not rainbow connected, which is a contradiction.

Thus $rc(P_n \circ K_n) \geq 2n-1$ ———(ii)

From (i) and (ii), it follows that

$rc(P_n \circ K_n) = 2n-1$

(An illustration for the assignment of rainbow colors in $P_4 \circ K_4$ is provided in figure 2.)

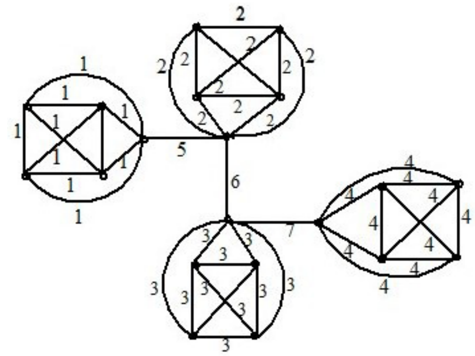

Figure 2. Rainbow coloring in the graph $P_4 \circ K_4$

Further, for any distinct pair of vertices u and v a rainbow $u-v$ geodesic requires the same number of colors.

Hence $src(G) = 2n-1$.

Hence the proof. □

Remark: Deletion of any edge from E_1 disconnects the graph G. Hence G is not $P-$critical with respect to E_1 . For the edge sets E_2 and E_3 we have following corollary.

3.0.1 corollary

Let $G = (P_n \circ K_n)$. Then for $n \geq 2$, G is not $p-$critical with respect to E_2 and E_3.

Proof. Let $e = (x, y)$ be any edge in E_2. If we follow a coloring as in theorem 2.1, it is clear that the edges in E_2 can be colored by one color. Now the deletion of the edge e from E_2 will give $d(x, y) = n-2$. Let P be the path from x to y in the set E_2. Then, since two edges in path P have same color, a $x-y$ rainbow path in E_2 of G is not possible. This holds for every edge e in E_2. Hence, to obtain a rainbow path, one more color is required other than $2n-1$ colors already assigned in G. This holds for every $e \in E_2$ of G. Therefore, $rs(E_2) - e = 2n-1+1 = 2n$.

This shows that each $(K_n)_i$ of G is $p-$critical with respect to E_2.

A similar proof follows for the edges in E_3. □

For the corona product of a path and cycle graph, we have the following result.

Theorem 3.2. *Let $G = P_n \circ C_n$. Then for $n \geq 3$, $rc(G) = src(G) = 2n-1$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and let the vertex set of the i copies of C_n namely $(C_n)_i$ be $V\{(C_n)_i\} = \{u_{ij} : 1 \leq i \leq n, 0 \leq j \leq n-1\}$.

By definition of the corona product, each vertex of P_n is adjacent to every vertex of copy of C_n, that is for $1 \leq i \leq n$ the vertex $v_i \in V(P_n)$ is adjacent to the vertices of the set $\{u_{ij} : 0 \leq j \leq n-1\}$ in the i^{th} copy of C_n.

Let $E(P_n \circ C_n) = \{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(P_n) = \{e_i = (v_i, v_{i+1}); 1 \leq i \leq n-1\}$, $E_2 = E(C_n)_i = \{e_{ij} = (u_{ij}, u_{ij+1}); 1 \leq$

$i \leq n, 0 \leq j \leq n-1\}$ where computation for index j is under modulo n and $E_3 = \{(e_k)_i = (v_i, u_{ij}); 1 \leq i \leq n, 1 \leq k \leq n$ and $0 \leq j \leq n-1\}$.

We assign $2n-1$ colors to the edges of $P_n \circ C_n$ as follows.

We have the following cases.

Case 1: $n = 3$.

In this case, for $1 \leq i \leq 3$ we assign the color i to the edges of $(C_3)_i$ and $(e_k)_i$ and for $1 \leq i \leq 2$ we assign the color $i+3$ to the edges e_i. Then, clearly $r_c(P_3 \circ C_3) = 5$.

Case 2: $n \geq 4$

For $1 \leq i, k \leq n$ we assign the color i to the edges $(e_k)_i$ and for $1 \leq i \leq n-1$ we assign the color $n+i$ to the edges e_i. Further,

Subcase 1: n is even:

In this case, adjacent edges of $(C_n)_i$ for each i are colored recursively with the colors $\{1, 2, 3, \cdots, \frac{n}{2}\}$, and, after $\{\frac{n}{2}\}$ the same order is followed until the last edge.

Subcase 2: n is odd:

In this case, adacent edges of $(C_n)_i$ for each i are colored recursively with the colors $\{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$, and, after $\lceil \frac{n}{2} \rceil$ the same order is followed until the last edge.

Hence $rc(P_n \circ C_n) \leq 2n-1$ ———(i)

To prove $rc(P_n \circ C_n) \geq 2n-1$, we assume that $rc(P_n \circ C_n) = 2n-2$. Then, for a proper rainbow coloring, $2n-2$ colors must be assigned to the edges of $(P_n \circ C_n)$. Since $P_n \circ C_n$ has n copies of C_n, we assign n colors to the n copies of C_n and assign the remaining $n-2$ colors to $n-1$ edges of P_n. An easy check shows that at least two of the edges of P_n are colored with the same colors.

This implies that at least one path in $P_n \circ C_n$ is not rainbow connected, which is a contradiction.

Thus $r_c(P_n \circ C_n) \geq 2n-1$ ———(ii)

From (i) and (ii), it follows that

$rc(P_n \circ C_n) = 2n-1$.

(An illustration for the assignment of rainbow colors in $P_5 \circ C_5$ is provided in figure 3)

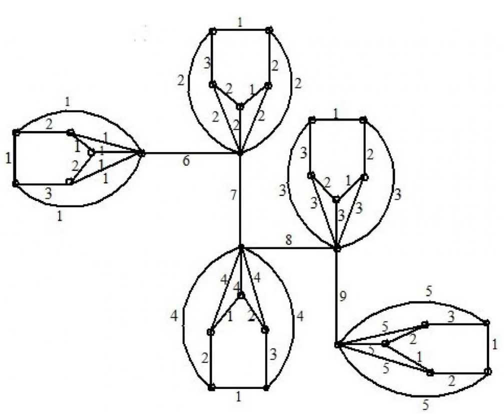

Figure 3. Rainbow coloring in the graph $P_5 \circ C_5$

Further, for any distinct pair of vertices u and v a rainbow $u-v$ geodesic requires the same number of colors.

Hence $src(G) = 2n-1$

Hence the proof.

□

For the corona product of path and star graph, we have the following result.

Theorem 3.3. *Let $G = P_n \circ K_{1,n}$. Then for $n \geq 3$, $rc(G) = src(G) = 2n-1$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and let the vertex set of the i copies of $K_{1,n}$ namely $(K_{1,n})_i$ be $V\{(K_{1,n})_i\} = \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n+1\}$.

By definition of the corona product, each vertex of P_n is adjacent to every vertex of copy of $K_{1,n}$, that is for $1 \leq i \leq n$ the vertex $v_i \in V(P_n)$ is adjacent to the vertices of the set $\{u_{ij} : 1 \leq j \leq n+1\}$ in the i^th copy of $K_{1,n}$.

Let $E(P_n \circ K_{1,n}) = \{E_1 \cup E_2 \cup E_3\}$; where $E_1 = E(P_n) = \{e_i : e_i = (v_i, v_{i+1}); 1 \leq i \leq n-1\}$. $E_2 = E(K_{1,n})_i = \{e_{ij} = (u_{i1}, u_{ij+1}); 1 \leq i \leq n, 1 \leq j \leq n\}$ and $E_3 = \{(e_k)_i = (v_i, u_{ik}); 1 \leq i \leq n, 1 \leq k \leq n+1\}$.

We assign a rainbow coloring to the edges of $E(P_n \circ K_{1,n})$ as follows;

For $1 \leq j \leq n-1$, assign the color $j+n$ to the edges of (P_n).

For $1 \leq i \leq n$, assign the colors $\{1, 2, \cdots, n\}$ to the edges of $(K_{1,n})_i$ and for $1 \leq i \leq n$, assign the color i to the edges $(e_k)_i$. From this assignment of colors, it is clear that

$rc(P_n \circ K_{1,n}) \leq 2n-1$ ....(i)

To prove $rc(P_n \circ K_{1,n}) \geq 2n-1$, we assume that $rc(P_n \circ K_{1,n}) = 2n-2$. Then,for proper rainbow coloring, $2n-2$ colors must be assigned to the edges of $(P_n \circ K_{1,n})$.Since $(P_n \circ K_{1,n})$ has n copies of $K_{1,n}$, we assign n colors to the n copies of $K_{1,n}$ and assign the remaining $n-2$ colors to $n-1$ edges of P_n.An easy check shows that at least two of the edges of P_n are colored with same colors.

This implies that at least one path in $(P_n \circ K_{1,n})$ is not rainbow connected,which is a contradiction.

Thus $rc(P_n \circ K_{1,n}) \geq 2n-1$...............(ii)

From (i) and (ii) it follows that

$rc(P_n \circ K_{1,n}) = 2n-1$.

(An illustration for the assignment of rainbow colors in $P_5 \circ K_{1,5}$ is provided in figure 4.)

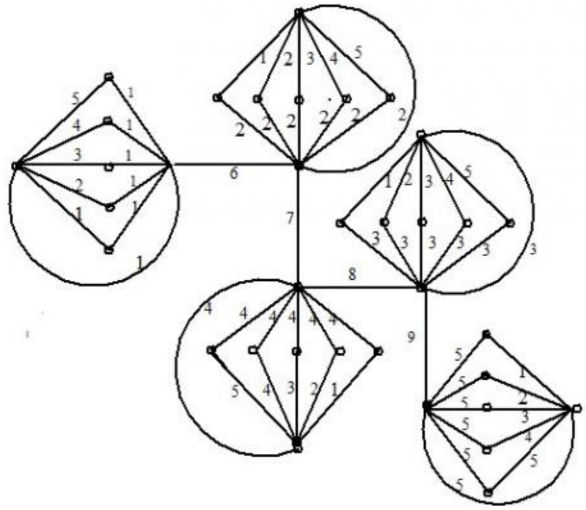

Figure 4. Rainbow coloring in the graph $P_5 \circ K_{1,5}$

Further, for any distinct pair of vertices u and v a rainbow $u-v$ geodesic requires the same number of colors.

Hence $src(G) = 2n-1$.

Hence the proof.

□

For the corona product of star and complete graph, we have the following result.

Theorem 3.4. *Let $G = K_{1,n} \circ K_n$. Then for $n \geq 2$, $rc(G) = src(G) = 2n+1$.*

Proof. Let the vertex set $V(k_{1,n}) = \{v_i : 1 \leq i \leq n+1\}$. Let vertex set of the i copies of K_n namely $(K_n)_i$ be $V\{(K_n)_i\} = \{u_{ij} : 1 \leq i \leq n+1, 1 \leq j \leq n\}$.

By definition of corona graph, each vertex of $K_{1,n}$ is adjacent to every vertex of copy of K_n, that is for $1 \leq i \leq n+1$ the vertex $v_i \in V(K_{1,n})$ is adjacent to the vertices of the set $\{u_{ij} : 1 \leq j \leq n\}$ in the i^{th} copy of K_n.

Let $E(K_{1,n} \circ K_n) = \{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(K_{1,n}) = \{e_i = (v_1, v_{i+1}); 1 \leq i \leq n-1\}$, E_2 be the edge set of $(K_n)_i$ for $1 \leq i \leq n$ and $E_3 = \{(e_k)_i = (v_i, u_{ij}); 1 \leq i \leq n, 1 \leq j,k \leq n\}$.

We assign $2n-1$ colors to the edges of $K_{1,n} \circ K_n$ as follows:

For $1 \leq i \leq n+1$, assign the color i to the edges of $(K_n)_i$ and to the edges $(e_k)_i$ and for $1 \leq j \leq n$, assign the color $j+n+1$ to the edges of $(K_{1,n})$ of $K_{1,n} \circ K_n$. From this assignment of colors, it is clear that

$rc(K_{1,n} \circ K_n) \leq 2n+1$ ———(i)

To prove $rc(K_{1,n} \circ K_n) \geq 2n+1$, we assume that $rc(K_{1,n} \circ K_n) = 2n$. Then, for a proper rainbow coloring, $2n$ colors must be assigned to the edges of $(K_{1,n} \circ K_n)$. Since $K_{1,n} \circ K_n$ has $n+1$ copies of K_n, we assign $n+1$ colors to the $n+1$ copies of K_n and assign the remaining $n-1$ colors to n edges of $K_{1,n}$. An easy check shows that at least two of the edges of K_1, n are colored with the same colors.

This implies that at least one path in $K_{1,n} \circ K_n$ is not rainbow connected, which is a contradiction.

This implies that at least one path in $K_{1,n} \circ K_n$ is not rainbow connected, which is a contradiction.

Thus $rc(K_{1,n} \circ K_n) \geq 2n+1$ ———(ii)

From (i) and (ii), it follows that

$rc(K_{1,n} \circ K_n) = 2n+1$.

(An illustration for the assignment of rainbow colors in $K_{1,3} \circ K_3$ is provided in figure 5.)

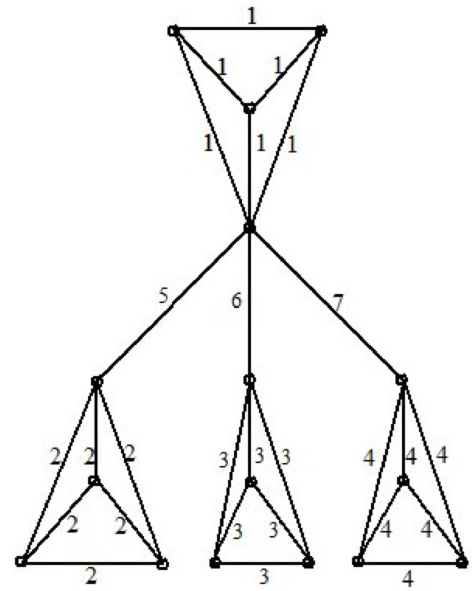

Figure 5. Rainbow coloring in the graph $K_{1,3} \circ K_3$

Further, for any distinct pair of vertices u and v a rainbow $u-v$ geodesic requires the same number of colors.

Hence $src(G) = 2n+1$.

Hence the proof.

□

3.1 p-critical corona product graphs

In this section, we examine the p-criticalness property of the corona product graphs discussed in the previous section. We begin with the graph G described in Theorem 2.1. Deletion of any edge from E_1 disconnects G. Hence G is not $p-critical$ with respect to E_1. For the edge sets E_2 and E_3 we have the following result.

Lemma 3.5. *Let $G = P_n \circ K_n$. Then for $n \geq 2$, G is rainbow $p-critical$ with respect to E_2 and E_3.*

Proof: Let $e = (x,y)$ be any edge in E_2. If we follow a coloring as in theorem 2.1, it is clear that the edges in E_2 can be colored by one color. Now deletion of the edge e from E_2 will give $d(x,y) = n-2$. Let P be the path from x to y in the set E_2. Then, since two edges in path P have same color, a $x-y$ rainbow path in E_2 of G is not possible. This holds for every edge e in E_2. Hence, to obtain a rainbow path , one more color is required other than the $2n-1$ colors already *assigned in G. This holds for every $e \in E_2$ of G. Therefore, $rc\{E_2 - e\} = 2n - 1 + 1 = 2n$.*

This shows that each G is p-critical with respect to E_2.

A similar proof follows for the edges in E_3.

The graphs in theorems 2.2 and 2.3 are critical with respect to the edge sets E_2 and E_3. We state these properties in lemmas 2.2 and 2.3 whose proofs are similar to the proof given in lemma 2.1.

Lemma 3.6. *Let $G = P_n \circ C_n$. Then for $n \geq 3$, G is rainbow $p-$critical with respect to E_2 and E_3.*

Lemma 3.7. *Let $G = P_n \circ K_{1,n}$. Then for $n \geq 3$, G is rainbow $p-$critical with respect to E_2 nd E_3.*

For the graph G in theorem 2.4, we have the following result.

Lemma 3.8. *Let $G = K_{1,n} \circ K_n$. Then for $n \geq 2$, G is rainbow $p-$critical with respect to E_2 nd E_3.*

Proof: Let $e = (x, y)$ be any edge in E_2. If we follow a coloring as in theorem 2.4, it is clear that the edges in E_2 can be colored by one color. Now let us delete the edge e from E_2. Let P be the path from x to y in the set E_2. Then, since two edges in path P have same color, a $x - y$ rainbow path in E_2 of G is not possible. This holds for every edge e in E_2. Hence, to obtain a rainbow path , one more color is required other than the $2n + 1$ colors already assigned in G. Therefore, $rc\{E_2 - e\} = 2n + 1 + 1 = 2n + 2$.

This shows that each G is p-critical with respect to E_2.

A similar proof follows for the edges in E_3.

Conclusion

In this paper, We obtain the rainbow connection number, strong rainbow connection number and p-criticalness property of some corona product graphs involving the path and complete graph, path and cycle graph, Path and star graph and star and complete graph.

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