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Finsler spaces with some special (α,β)-metric of Douglas type

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Abstract

The purpose of present paper is to considered a special (α,β)-metric under which various conditions reduces to a Douglas type Finsler space.

Keywords

Finsler space,(α,β)-metrics, Riemannian metric, One form differential,Randers metric,Douglas Space.

AMS Subject Classification

53C60,53B40.

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Contents

1. Introduction

In the year 1997, S. Bacso amd M. Matsomoto [4] introduced the notion of Douglas space as a generalization of Berwald space from the viewpoint of geodesic equations.It is remarkable that a Finsler space is a Douglas space or is of Douglas type, if and only if the Douglas tensor Vanishes identically.Further M. Matsumoto [11] has studied the conditions for some Finsler spaces with (α,β)-metric to be of Douglas type.

The theories of Finsler spaces with (α,β)-metric have contributed to the development of Finsler geometry [10] ,and Berwald spaces with (α,β)-metric have been studied by many authors [1, 7, 12] .Since Berwald space is also a kind of Douglas space, the impotant point of the paper [11] is to observe that,comparing with the condition of Berwald space, to what condition of Douglas spce relaxes.In continuous the present paper is to considered a special (α,β)-metric under which various conditions reduces to a Doglas type Finsler space.

2. Preliminaries

Let $\alpha(x,y)$ and $\beta(x,y)$ be a Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and a differentiable one - form $\beta=b_i(x)y^i$ in an n-dimensional differentiable manifold M^n. If a Finsler fundamental function in M^n is a function $L(\alpha,\beta)$ of α and β which is positively homogeneous of degree one, then the structure $F^n=(M^n,L(\alpha,\beta))$ is called a Finsler space with $(\alpha,\beta)-$ metric [10]. The space $R^n=(M^n),\alpha$ is called a Riemannian space associated with F^n [4]. In R^n, we have the christoffel symbols $\gamma^i_{jk}(x)$. and the covariant differentiation ∇ w.r.t $\gamma^i_{jk}(x)$ We shall use the symbols as follows:

$$r_{ij}=\frac{1}{2}(\nabla_j b_i+\nabla_i b_j), s_{ij}=\frac{1}{2}(\nabla_j b_i-\nabla_i b_j),$$
$$s^i_j=a^{ir}s_{rj}, s_j=b_r s^r_j.$$

It is to be noted that $s_{ij}=\frac{1}{2}(\partial_j b_i-\partial_i b_j)$. Throughout the paper the symbols ∂_j and $\dot{\partial}_j$ stand for $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ respectively. We are concerned with the Berwald connection $B\Gamma=(G^i_{jk},G^i_j)$ which is given by $2G^i(x,y)=g^{ij}(y^r\dot{\partial}_j\partial_r F)$, where $F=\frac{L^2}{2}, G^i_j=\dot{\partial}_j G^i$ and $G^i_{jk}=\dot{\partial}_k G^i_j$. The Finsler space F^n is said to be of Douglas type or called a Douglas space [4] if $D^{ij}=G^i(x,y)y^j-G^j(x,y)y^i$ are homogeneous polynomial in

y^i of degree three. It has been shown that F^n is of Douglas type iff Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes identically, where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the hv-curvature tensor of the Berwald connection $B\Gamma, G_{ij} = G^r_{ijr}$ and $G_{ijk} = \dot{\partial}_k G_{ij}$. [3] Now we consider the function $G^i(x,y)$ of F^n with (α,β) -metric. According to [8, 12] they are written in the form

$$2G^i = \gamma^i_{00} + 2B^i, B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L\alpha}s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*(\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i) \quad (2.1)$$

where we put,

$$E = \frac{\beta L_\beta}{L}C^*, C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha r^2 L_{\alpha\alpha})}, \quad (2.2)$$
$$b^i = a^{ij}b_j, r^2 = b^2\alpha^2 - \beta^2, b^2 = a^{ij}b_i b_j$$

and the subscript α and β in L denote the partial differentiation w.r.t α and β respectively.

Since $\gamma^i_{00} = \gamma^i_{jk}y_j y_k$ is homogeneous polynomial in (y^i) of degree two, we have [11].

Proposition 2.1. *A Finsler space F^n with (α,β) - metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are homogeneous polynomials in y^i of degree three.*
Equation (2.1) gives

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i) \quad (2.3)$$

Here we state the following lemma for the latter frequent use [6].

Lemma 2.2. *If $\alpha^2 \equiv 0(mod\beta)$, i.e. $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

Through out the paper, we shall say "homogeneous polynomial (s) in y^i of degree r" as $hp(r)$ for brevity. Thus γ^i_{00} are $hp(2)$ and if the space is of Douglas type then D^{ij} and B^{ij} are $hp(3)$. Also, we have assumed that $\alpha^2 \not\equiv 0(mod\beta)$, throughout the paper.

3. Special (α,β) metric

We shall apply the proposition (2.1) to the (α,β) - metric

$$L = \frac{b_1\alpha^3 + b_2\alpha^2\beta + b_3\alpha\beta^2 + b_4\beta^3}{a_1\alpha^2 + a_2\alpha\beta + a_3\beta^2}$$

Where a's and b's are constants. It is obvious that by homothetic change of α and β this kind of the metric may be classified as follows:
(I) If $a_1 \neq 0, a_2 = 0, a_3 = 0$, we have the Rander's metric $L = \alpha + \beta$ (for $b_3 = b_4 = 0$) the metric $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}(for\ b_4 = 0, b_3 \neq 0)$

$$L = c_1\alpha + c_2\beta + c_3\frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}(for\ b_4 \neq 0, b_3 \neq 0) \quad (3.1)$$

The metric (3.1) is approximate Matsumoto metric of second order.
(II) If $a_2 \neq 0, a_1 = 0, a_3 = 0$, we have the Rander's metric $L = \alpha + \beta$(for $b_1 = b_4 = 0$) the metric $L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}(for\ b_4 = 0, b_1 \neq 0)$

$$L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}(for\ b_4 \neq 0, b_1 \neq 0) \quad (3.2)$$

(III) If $a_3 \neq 0, a_1 = 0, a_2 = 0$, we have the Rander's metric $L = \alpha + \beta(for b_1 = b_2 = 0)$ the metric $L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ (for $b_1 = 0, b_2 \neq 0$)

$$L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\alpha^2}{\beta}(for b_1 \neq 0, b_2 \neq 0) \quad (3.3)$$

Theorem 3.1. *A Randers space is of Douglas type, if and only if $s_{ij} = 0$. Then $2G^i = \gamma^i_{00} + \frac{r_{00}y^i}{L}$.*
As the metric $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ we have [14].

Theorem 3.2. *A Finsler space F^n with $(\alpha,\beta), L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ for which $c_2 \neq 0, b^2 \neq c_1 and \alpha^2 \not\equiv 0(mod\beta)$, is a Douglas space if and only if there exist a scalar function h(x) such that*

$$\nabla_j b_i = h(x)[(c_1 + 2b^2)a_{ij} - 3b_i b_j]$$

holds. In particular if $h(x) = 0$, then F^n is a Berwald space.
As for metric $L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$, we have [14].

Theorem 3.3. *Let F^n be a Douglas space with metric $L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0(mod\beta)$, then there exist a scalar function u(x) and a tensor function $v_{ij}(x)$ such that $\nabla_j b_i (= r_{ij} + s_{ij})$*

where $r_{ij} = \frac{c_2}{2c_1}(b_i s_j + b_j s_i) - u a_{ij}$ and
$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) - \frac{4}{n-1}v_{ij}$
Now as for metric $L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha+\beta}$, we have [14].

Theorem 3.4. *Let F^n be a Douglas space with(α,β) metric $L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha+\beta}$ for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0(mod\beta)$, then $(\nabla_j b_i - \nabla_i b_j) = \frac{\mu}{k_0(n-1)}(b_i s_j - b_j s_i)$ where $\mu = 2nc_0(c_2 - c_1) - \frac{k_0}{b^2}, c_0 = c_1 - c_2 + c_3$ and $k_0 = (c_2 - c_1)(c_1 + 2c_0 b^2)$*
We shall discuss the condition for F^n with metrics $(3.1), (3.2)$ and (3.3) to be of Douglas type in the following articles.

4. Finsler space with the metric $L=c_1\alpha+c_2\beta+c_3\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$

For the metric $L=c_1\alpha+c_2\beta+c_3\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}(for\ \ b_4\neq 0, b_3\neq 0)$, we have

$$L_\alpha=\frac{c_1\alpha^3-c_3\alpha\beta^2-2\beta^3}{\alpha^3}, L_\beta=\frac{c_2\alpha^2+2c_3\alpha\beta+3\beta^2}{\alpha^2},$$
$$L_{\alpha\alpha}=\frac{2\alpha\beta^2c_3+6\beta^3}{\alpha^4}$$

Therefore the value of C^* given in (2.2) becomes

$$C^*=\frac{\alpha\beta}{2}\left[\frac{r_{00}(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)-2s_0(c_2\alpha^2+2\alpha\beta c_3+3\beta^2)\alpha^2}{c_1\alpha^3\beta^2-c_3(3\alpha\beta^4-2b^2\beta^2\alpha^3)-8\beta^5+6b^2\alpha^2\beta^3}\right]$$

From (2.3), we have

$$\begin{cases} B^{ij}=\alpha^2\frac{(c_2\alpha^2+2c_3\alpha\beta+3\beta^2)}{(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)}(s_0^iy^j-s_0^jy^i)\\ +\alpha^2\frac{c_3\alpha\beta^2+3\beta^3}{(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)}\\ \times\left[\frac{r_{00}(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)-2s_0(c_2\alpha^2+2\alpha\beta c_3+3\beta^2)\alpha^2}{c_1\alpha^3\beta^2-c_3(3\alpha\beta^4-2b^2\beta^2\alpha^3)-8\beta^5+6b^2\alpha^2\beta^3}\right]\\ (b^iy^j-b^jy^i)\end{cases}\quad(4.1)$$

Which may be written as

$$\begin{cases}(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)(c_1\alpha^3\beta^2-3c_3\alpha\beta^4+\\ 2c_3b^2\beta^2\alpha^3-8\beta^5+6b^2\alpha^2\beta^3)B^{ij}-\alpha^2(c_2\alpha^2\\ +2\alpha\beta c_3+3\beta^2)(c_1\alpha^3\beta^2-3c_3\alpha\beta^4\\ +2c_3b^2\beta^2\alpha^3-8\beta^5+6b^2\alpha^2\beta^3)(s_0^iy^j-s_0^jy^i)\\ -\alpha^2(c_3\alpha\beta^2+3\beta^3)[r_{00}(c_1\alpha^3-c_3\alpha\beta^2-2\beta^3)\\ -2s_0(c_2\alpha^2+2\alpha\beta c_3+3\beta^2)\alpha^2]\cdot(b^iy^j-b^jy^i)=0\end{cases}\quad(4.2)$$

Since α is irrational in (y^i), the equation (4.2) are divided into two equations as follows:

$$\begin{cases}[c_1^2\alpha^6-4c_1c_3\alpha^4\beta^2+2c_1c_3b^2\alpha^4+3c_3^2\alpha^2\beta^4\\ -2c_3^2b^2\beta^2\alpha^4+16\beta^6-12b^2\alpha^2\beta^4]B^{ij}\\ -\alpha^2[6c_2b^2\alpha^4\beta+2c_1c_3\alpha^4\beta-8c_2\alpha^2\beta^3\\ -6c_3^2\alpha^2\beta^3-24\beta^5+18b^2\alpha^2\beta^3+\\ 4c_3^2b^2\beta\alpha^4](s_0^iy^j-s_0^jy^i)-\alpha^2[c_1c_3\alpha^4r_{00}-c_3^2\beta^2\\ -2s_0(2c_3^2\alpha^2\beta+3c_2\alpha^2\beta+9\beta^3)\alpha^2-\\ 6\beta^4r_{00}](b^iy^j-b^jy^i)=0\end{cases}\quad(4.3)$$

And

$$\begin{cases}[-10c_1\beta^3\alpha^2+6c_1b^2\beta\alpha^4+14c_3\beta^5\\ -10c_3b^2\alpha^2\beta^3]B^{ij}-\alpha^2[c_1c_2\alpha^4-3c_2c_3\alpha^2\beta^2\\ +2c_2c_3b^2\alpha^4-25c_3\beta^4+18c_3b^2\alpha^2\beta^2+3c_1\alpha^2\beta^2]\\ (s_0^iy^j-s_0^jy^i)-\alpha^2[-5c_3\beta^3r_{00}-2s_0(c_2c_3\alpha^2\\ +9c_3\beta^2)\alpha^2+3c_1\alpha^2\beta r_{00}](b^iy^j-b^jy^i)\end{cases}\quad(4.4)$$

Only term $14c_3\beta^5B^{ij}$ of (4.4) seemingly does not contain α^2. Therefore there exists a $hp(6)K^{ij}_{(6)}$ such that it is equal to $\alpha^2K^{ij}_{(6)}$.
Hence we have $B^{ij}=\alpha^2K^{ij}$ where we put $K^{ij}_{(6)}=14c_3\beta^5K^{ij}$ with $hp(1)K^{ij}$.
Therefore (4.4) reduces to

$$\begin{cases}[6c_1b^2\beta\alpha^4+14c_3\beta^5-10c_1\beta^3\alpha^2-10c_3b^2\alpha^2\beta^3]k^{ij}\\ -[c_1c_2\alpha^4-3c_2c_3\alpha^2\beta^2+2c_2c_3b^2\alpha^4-25c_3\beta^4\\ +18c_3b^2\alpha^2\beta^2+3c_1\alpha^2\beta^2](s_0^iy^j+s_0^jy^i)\\ -[-5c_3\beta^3r_{00}-2s_0(c_2c_3\alpha^2+9c_3\beta^2)\alpha^2\\ +3c_1\alpha^2\beta r_{00}](b^iy^j-b^jy^i)=0\end{cases}\quad(4.5)$$

The term in (4.5) which seemingly does not contain β are

$$-(c_1c_2\alpha^4+2c_2c_3b^2\alpha^4)(s_0^iy^j+s_0^jy^i)+2c_2c_3s_0\alpha^4(b^iy^j-b^jy^i)$$

Hence we must have $hp(1)m^{ij}$ Such that above is equal to $c_2\alpha^4\beta m^{ij}$.
Therefore we have

$$\begin{cases}-(c_1+2c_3b^2)(s_0^iy^j+s_0^jy^i)+2c_3s_0(b^iy^j\\ -b^jy^i)=\beta m^{ij}\end{cases}\quad(4.6)$$

By putting $m^{ij}=m^{ij}_k(x)y^k$ then equation (4.6) may be written as

$$\begin{cases}-(c_1+2c_3b^2)[s_h^i\delta_k^j+s_k^i\delta_h^j-s_h^j\delta_k^i-s_k^j\delta_h^i]\\ +2c_3[(s_h\delta_k^j+s_k\delta_h^j)b^i-(s_h\delta_k^i+s_k\delta_h^i)b^j]\\ =b_hm_k^{ij}+b_km_h^{ij}\end{cases}\quad(4.7)$$

Contracting (4.7) by $j=k$, we get

$$\begin{cases}n[-(c_1+2c_3b^2)s_h^i+2c_3s_hb^i]\\ =b_hm_r^{ir}+b_rm_h^{ir}\end{cases}\quad(4.8)$$

Transvecting (4.7) by b_jb^h, we obtain

$$\begin{cases}-(c_1+2c_3b^2)[b^2s_k^i-s^ib_k-b^is_k]\\ =b^2b_rm_k^{ir}+b_kb_rm_s^{ir}b^s\end{cases}\quad(4.9)$$

Further transvecting (4.9) with b^k, we get

$$b_rm_s^{ir}b^s=(c_1+2c_3b^2)s^i,\text{ provided } b^2\neq 0$$

Thus (4.9) gives

$$b^2b_rm_k^{ir}=(c_1+2c_3b^2)(b^is_k-b^2s_k^i)\quad(4.10)$$

From (4.8) we have

$$b_hm_r^{ir}=n[-(c_1+2c_3b^2)s_h^i+2c_3s_hb^i]-b_rm_h^{ir}\quad(4.11)$$

Using (4.10) in (4.11), we have

$$b_hm_r^{ir}=-(n-b^2)k_0s_h^i+\mu b^is_h\quad(4.12)$$

Where $k_0=c_1+2c_3b^2$ and $\mu=2c_3(1-b^2)-c_1$
If we put $m_r^{ir}=\mu m^i$ then (4.12) gives

$$(n-b^2)k_0s_h^i=\mu[b^is_h-b_hm^i]$$

Or equivalently

$$s_{ij}=\frac{\mu}{(n-b^2)k_0}(b_is_j-b_jm_i)$$

Since s_{ij} is skew-symmetric, we have $m_i = s_i$ i.e. m_i and s_i have the same direction, therefore we have

$$s_{ij} = \frac{\lambda}{(n-b^2)k_0}(b_i s_j - b_j s_i) \tag{4.13}$$

Theorem 4.1. *Let F^n be a Douglas Space with (α,β) metric $L = c_1\alpha + c_2\beta + c_3\frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0(mod\beta)$ then*

$$(\nabla_j b_i - \nabla_i b_j) = \tfrac{\mu}{(n-b^2)k_0}(b_i s_j - b_j s_i),$$

Where $\mu = 2c_3(1-b^2) - c_1$ and $k_0 = c_1 + 2c_3 b^2$.

5. Finsler space with metric $L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

For the metric $L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}(for\ \ b_4 \neq 0, b_1 \neq 0)$, we have

$$L_\alpha = \frac{c_1\beta\alpha^2 + 2c_3\alpha^{3^{\text{ffl}}} - \beta^3}{\alpha^2\beta}, L_\beta = \frac{c_2\alpha\beta^2 - c_3\alpha^3 + 2\beta^3}{\alpha\beta^2},$$

$$L_{\alpha\alpha} = \frac{2c_3\alpha^3 + 2\beta^3}{\beta\alpha^3}$$

Then the value of C^* given in (2.2) becomes

$$C^* = \tfrac{\alpha}{2}\left[\tfrac{\beta r_{00}(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3) - 2s_0(c_2\alpha\beta^2 - c_3\alpha^2 + 2\beta^3)\alpha^2}{c_1\beta^3\alpha^2 - 3\beta^5 + 2c_3b^2\alpha^5 + 2b^2\alpha^2\beta^3}\right]$$

Therefore from (2.3), we have

$$\begin{cases} B^{ij} = \frac{(c_2\alpha^3\beta^2 - c_3\alpha^5 + 2\beta^3\alpha^2)}{\beta(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3)}(s^i_0 y^j - s^j_0 y^j) \\ + \frac{\alpha^2(c_3\alpha^3 + \beta^3)}{\beta(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3)} \\ \times\left[\frac{\beta r_{00}(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3) - 2s_0(c_2\alpha\beta^2 - c_3\alpha^2 + 2\beta^3)\alpha^2}{c_1\beta^3\alpha^2 - 3\beta^5 + 2c_3b^2\alpha^5 + 2b^2\alpha^2\beta^3}\right] \\ (b^i y^j - b^j y^i) \end{cases} \tag{5.1}$$

This equation may be written as

$$\begin{cases} \beta(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3)(c_1\beta^3\alpha^2 - 3\beta^5 \\ +2c_3b^2\alpha^5 + 2b^2\alpha^2\beta^3)B^{ij} - (c_2\alpha^3\beta^2 - c_3\alpha^5 \\ +2\beta^3\alpha^2)(c_1\beta^3\alpha^2 - 3\beta^5 + 2c_3b^2\alpha^5 \\ +2b^2\alpha^2\beta^3)(s^i_0 y^j - s^j_0 y^i) - \alpha^2(c_3\alpha^3 \\ +\beta^3)[\beta r_{00}(c_1\beta\alpha^2 + 2c_3\alpha^3 - \beta^3) - \\ 2s_0(c_2\alpha\beta^2 - c_3\alpha^2 + 2\beta^3)\alpha^2](b^i y^j - b^j y^i) = 0 \end{cases} \tag{5.2}$$

Since α is irrational in y^i, the equation (5.2) are divide into two equation as follows:

$$\begin{cases} \beta[c_1^2\alpha^4\beta^4 - 4c_1\beta^6\alpha^2 + 2c_1b^2\alpha^4\beta^4 + 4c_3^2b^2\alpha^8 \\ -2b^2\alpha^2\beta^6 + 3\beta^8]B^{ij} - [2c_2c_3b^2\alpha^8\beta^2 - 2c_3^2b^2\alpha^{10} \\ +2c_1\alpha^4\beta^6 - 6\beta^8\alpha^2 + 4b^2\beta^6\alpha^4](s^i_0 y^j \\ -s^j_0 y^i) - \alpha^2[2c_3^2\beta r_{00}\alpha^6 + c_1\beta^5 r_{00}\alpha^2 - \beta^7 r_{00} \\ -2s_0(c_2c_3\beta^2\alpha^4 - c_3^2\alpha^6 + 2\beta^6)\alpha^2](b^i y^j - b^j y^i) = 0 \end{cases} \tag{5.3}$$

And

$$\begin{cases} \beta[2c_1c_3b^2\beta\alpha^7 + 2c_1c_3\beta^3\alpha^5 - 6c_3\alpha^3\beta^5 \\ +2c_3b^2\alpha^5\beta^3]B^{ij} - [c_1c_2\alpha^5\beta^5 - 3c_2\alpha^3\beta^7 \\ +2c_2b^2\beta^5\alpha^5 - c_1c_2\beta^3\alpha^7 + 3c_3\alpha^5\beta^5 + 2c_3b^2\alpha^7\beta^3] \\ (s^i_0 y^j - s^j_0 y^i) - \alpha^2[c_1c_3\beta^2 r_{00}\alpha^5 + c_3 r_{00}\alpha^3\beta^4 \\ -2s_0(c_3\alpha^3\beta^3 + c_2\alpha\beta^5)\alpha^2](b^i y^j - b^j y^i) = 0 \end{cases} \tag{5.4}$$

Only the term $3\beta^9 B^{ij}$ of (5.3) seemingly does not contain α^2. Therefore there exists a $hp(10)K^{ij}_{(10)}$ such that it is equal to $\alpha^2 K^{ij}_{(10)}$.
Hence we have $B^{ij} = \alpha^2 K^{ij}$, where we put $K^{ij}_{(10)} = 3\beta^9 K^{ij}$ with $hp(1)K^{ij}$.
Therefore (5.3) reduces to

$$\begin{cases} \beta[c_1^2\beta^4\alpha^4 - 4c_1\beta^6\alpha^2 + 2c_1b^2\alpha^4\beta^4 + 4c_3^2b^2\alpha^8 \\ -2b^2\alpha^2\beta^6 + 3\beta^8]k^{ij} - [2c_2c_3b^2\alpha^6\beta^2 \\ -2c_3^2b^2\alpha^8 + 2c_1\alpha^2\beta^6 - 6\beta^8 + 4b^2\beta^6\alpha^2] \\ (s^i_0 y^j - s^j_0 y^i) - [2c_3^2\beta r_{00}\alpha^6 + c_1\beta^5 r_{00}\alpha^2 - \beta^7 r_{00} \\ -2s_0(c_2c_3\alpha^4\beta^2 - c_3^2\beta^6 + 2\beta^6)\alpha^2](b^i y^j - b^j y^i) = 0 \end{cases} \tag{5.5}$$

The term in (5.5) which seemingly does not contain β is

$$2c_3^2b^2\alpha^8(s^i_0 y^j - s^j_0 y^i) - 2s_0c_3^2\alpha^8(b^i y^j - b^j y^i)$$

Hence we must have $hp(1)m^{ij}$ such that above is equal to $-2c_3^2\alpha^8\beta m^{ij}$.
Therefore we have

$$\left\{ -b^2(s^i_0 y^j - s^j_0 y^i) + s_0(b^i y^j - b^j y^i) = \beta m^{ij} \right. \tag{5.6}$$

By putting $m^{ij} = m^{ij}_k(x)y^k$, then equation (5.6) may be written as

$$\begin{cases} -b^2[s^i_h\delta^j_k + s^i_k\delta^j_h - s^j_h\delta^i_k - s^j_k\delta^i_h] \\ +[(s_h\delta^j_k + s_k\delta^j_h)b^i - (s_h\delta^i_k + s_k\delta^i_h)b^j] \\ = b_h m^{ij}_k + b_k m^{ij}_h \end{cases} \tag{5.7}$$

Contracting (5.7) by $j = k$, we get

$$n[-b^2 s^i_h + s_h b^i] \tag{5.8}$$
$$= b_h m^{ir}_r + b_r m^{ir}_h \tag{5.9}$$

Transvecting (5.7) by $b_j b^h$, we obtain

$$\begin{cases} -b^2(b^2 s^i_k - s^i_k b_k - b^i s_k) \\ = b^2 b_r m^{ir}_k + b_k b_r m^{ir}_s b^s \end{cases} \tag{5.10}$$

Further contracting (5.9) by b^k, we get $b_r m^{ir}_s b^s = b^2 s^i$, provided $b^2 \neq 0$.
Thus (5.9) gives

$$b^2 b_r m^{ir}_k = b^2(b^i s_k - b^2 s^i_k) \tag{5.11}$$

Then (5.8) is written as

$$b_h m^{ir}_r = n[-b^2 s^i_h + s_h b^i] - b_r m^{ir}_h \tag{5.12}$$

Using (5.10) in (5.11), we get

$$b_h m^{ir}_r = -b^2(n-1)s^i_h + (n-1)s_h b^i \tag{5.13}$$

If we put $m^{ir}_r = (n-1)m^i$, then from (5.12) we have

$$b^2(n-1)s^i_h=(n-1)[b^i s_h-m^i b_h]$$
$$\Rightarrow b^2 s^i_h=[b^i s_h-m^i b_h]$$
or equivalently
$$s_{ij}=\tfrac{1}{b^2}[b_i s_j-b_j m_i]$$

Since s_{ij} is skew-symmetric, then we have $m_i=s_i$, therefore

$$s_{ij}=\tfrac{1}{b^2}[b_i s_j-b_j m_i]$$

Theorem 5.1. *Let F^n be a Douglas space with $(\alpha,\beta)-$ metric $L=c_1\alpha+c_2\beta+c_3\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ for which $b^2\neq 0$ and $\alpha^2\not\equiv 0(mod\beta)$, then*

$$(\nabla_j b_i-\nabla_i b_j)=\tfrac{1}{b^2}[b_i s_j-b_j s_i]$$

6. Finsler space with metric $L=c_1\alpha+c_2\beta+c_3\frac{\alpha^2}{\beta}+\frac{\alpha^2}{\beta}$

For the metric $L=c_1\alpha+c_2\beta+c_3\frac{\alpha^2}{\beta}+\frac{\alpha^2}{\beta}(for b_1\neq 0, b_2\neq 0)$, we have

$$L_\alpha=\tfrac{c_1\beta+2c_3\alpha+2\alpha}{\beta},\ L_\beta=\tfrac{c_3\beta^2-c_3\alpha^2-\alpha^2}{\beta^2},\ L_{\alpha\alpha}=\tfrac{2c_3+2}{\beta}$$

Therefore the value of C^* given in (2.2) becomes

$$C^*=\tfrac{\alpha}{2}\left[\tfrac{\beta r_{00}(c_1\beta+2c_3\alpha+2\alpha)-2\alpha s_0(c_2\beta^2-c_3\alpha^2-\alpha^2)}{c_1\beta^3+2c_3\alpha^3b^2+2b^2\alpha^3}\right]$$

Then from (2.3), we have

$$\left\{\begin{array}{l} B^{ij}=\frac{\alpha(c_2\beta^2-c_3\alpha^2-\alpha^2)}{\beta(c_1\beta+2c_3\alpha+2\alpha)}(s^i_0y^j-s^j_0y^i)\\ +\frac{\alpha^3(c_3+1)}{\beta(c_1\beta+2c_3\alpha+2\alpha)}\\ \times\left[\frac{\beta r_{00}(c_1\beta+2c_3\alpha+2\alpha)-2\alpha s_0(c_2\beta^2-c_3\alpha^2-\alpha^2)}{c_1\beta^3+2c_3\alpha^3b^2+2b^2\alpha^3}\right]\\ \times(b^iy^j-b^jy^i)\end{array}\right. \quad (6.1)$$

Which may be written as

$$\left\{\begin{array}{l} \beta[c_1^2\beta^4+2c_1c_3b^2\alpha^3\beta+2c_1\beta b^2\alpha^3+2c_1c_3\alpha\beta^3\\ +4c_3^2b^2\alpha^4+8c_3b^2\alpha^4+2c_1\alpha\beta^3+4b^2\alpha^4]B^{ij}\\ -\alpha[c_1c_2\alpha^5+2c_2c_3b^2\beta^2\alpha^3+2c_2b^2\alpha^3\beta^2\\ -c_1c_3\alpha^2\beta^3-2c_3^2\alpha^5b^2-4c_3b^2\alpha^5-c_1\alpha^2\beta^3\\ -2b^2\alpha^5](s^i_0y^j-s^j_0y^i)-\alpha^3[c_1c_3\beta^2r_{00}\\ +2c_3^2\alpha\beta r_{00}+2c_3\alpha\beta r_{00}+c_1\beta^2r_{00}\\ +2c_3\alpha\beta r_{00}+2\alpha\beta r_{00}-2\alpha s_0(c_2c_3\beta^2-c_3^2\alpha^2\\ -2c_3\alpha^2+c_2\beta^2-\alpha^2)](b^iy^j-b^jy^i)=0\end{array}\right. \quad (6.2)$$

Since α is irrational in y^i, then the equation (6.2) are divided into two equations as follows:

$$\left\{\begin{array}{l} \beta[c_1^2\beta^4+4c_3^2\alpha^4b^2+8c_3b^2\alpha^4+4b^2\alpha^4]B^{ij}\\ -[2c_2c_3b^2\beta^2\alpha^4+2c_2b^2\alpha^4\beta^2-2c_3^2b^2\alpha^6\\ -4c_3b^2\alpha^6-2b^2\alpha^6](s^i_0y^j-s^j_0y^i)-[2c_3^2\alpha^4\beta r_{00}\\ +4c_3\alpha^4\beta r_{00}+2\alpha^4\beta r_{00}-2\alpha^4s_0(c_2c_3\beta^2\\ -c_3^2\alpha^2-2c_3\alpha^2-c_2\beta^2-\alpha^2)](b^iy^j-b^jy^i)=0\end{array}\right. \quad (6.3)$$

And

$$\left\{\begin{array}{l} \beta[2c_1c_3b^2\alpha^3\beta+2c_1b^2\alpha^3\beta+2c_1c_3\alpha\beta^3+2c_1\alpha\beta^3]\\ B^{ij}-[c_1c_2\alpha\beta^5-c_1c_3\alpha^3\beta^3-c_1\alpha^3\beta^3](s^i_0y^j-s^j_0y^i)\\ -[c_1c_2\alpha^3\beta^2r_{00}+c_1\alpha^3\beta^2r_{00}](b^iy^j-b^jy^i)=0\end{array}\right. \quad (6.4)$$

Only term $c_1^2\beta^5B^{ij}$ of (6.3) seemingly does not contain α^2. Therefore there exists a hp(6) $K^{ij}_{(6)}$ such that it is equal to $\alpha^2K^{ij}_{(6)}$.
Hence we have $B^{ij}=\alpha^2K^{ij}$, where we put $K^{ij}_{(6)}=c_1^2\beta^5K^{ij}$ with hp(1) K^{ij}.
Therefore (6.3) reduces to

$$\left\{\begin{array}{l} \beta[c_1^2\beta^4+4c_3^2\alpha^4b^2+8c_3b^2\alpha^4+4b^2\alpha^4]K^{ij}\\ -[2c_2c_3b^2\beta^2\alpha^2+2c_2b^2\alpha^2\beta^2-2c_3^2\alpha^4b^2\\ -4c_3b^2\alpha^4-2b^2\alpha^4](s^i_0y^j-s^j_0y^i)-[2c_3^2\alpha^2\beta r_{00}\\ +4c_3\alpha^2\beta r_{00}+2\alpha^2\beta r_{00}-2\alpha^2s_0(c_2c_3\beta^2\\ -c_3^2\alpha^2-2c_3\alpha^2-c_2\beta^2-\alpha^2)](b^iy^j-b^jy^i)=0\end{array}\right. \quad (6.5)$$

The terms in (6.5) which seemingly does not contain β are

$$-2\alpha^4[-c_3^2b^2-2c_3b^2-b^2](s^i_0y^j-s^j_0y^i)+2\alpha^4s_0[-c_3^2-2c_3-1](b^iy^j-b^jy^i)$$

We can write this in the form

$$-2\alpha^4\{-(c_3+1)^2\}b^2(s^i_0y^j-s^j_0y^i)+2\alpha^4s_0\{-(c_3+1)^2\}(b^iy^j-b^jy^i)$$

Hence we must have hp(1) m^{ij} such that above is equal to $2\alpha^4\{-(c_3+1)^2\}\beta m^{ij}$.
Then we have

$$\left\{\begin{array}{l} -b^2(s^i_0y^j-s^j_0y^i)+s_0(b^iy^j-b^jy^i)\\ =\beta m^{ij}\end{array}\right. \quad (6.6)$$

By putting $m^{ij}=m^{ij}_k(x)y^k$, equation (6.6) may be written as

$$\left\{\begin{array}{l} -b^2[s^i_h\delta^j_k+s^i_k\delta^j_h-s^j_h\delta^i_k-s^j_k\delta^i_h]\\ +[(s_h\delta^j_k+s_k\delta^j_h)b^i-(s_h\delta^i_k+s_k\delta^i_h)b^j]\\ =b_hm^{ij}_k+b_km^{ij}_h\end{array}\right. \quad (6.7)$$

Contracting (6.7) by $j=k$ we get

$$n[-b^2s^i_h+s_hb^i]=b_hm^{ir}_r+b_rm^{ir}_h \quad (6.8)$$

Transvecting (6.8) by b_jb^h we get

$$-b^2(b^2s^i_k-s^ib_k-b^is_k)=b^2b_rm^{ir}_k+b_kb_rm^{ir}_sb^s \quad (6.9)$$

Further transvecting (6.9) by b^k, we get

$$b_rm^{ir}_sb^s=b^2s^i \qquad \text{provided } b^2\neq 0.$$

Thus (6.9) gives

$$b^2b_rm^{ir}_k=b^2(b^is_k-b^2s^i_k).$$

Therefore we get

$$b_rm^{ir}_k=(b^is_k-b^2s^i_k) \quad (6.10)$$

Equation (6.8) may be written as

$$b_hm^{ir}_r=n[-b^2s^i_h+s_hb^i]-b_rm^{ir}_h \quad (6.11)$$

Using (6.10) in (6.11) we have

$$b_hm^{ir}_r=-b^2(n-1)s^i_h+\mu b^is_h \quad (6.12)$$

Where $mu=(n+1)$
If we put $m^{ir}_r=\mu m^i$ then equation (6.12) gives

$$b^2(n-1)s^i_h = \mu[b^i s_h - m^i b_h]$$

Or equivalently

$$s_{ij} = \frac{\mu}{b^2(n-1)}(b_i s_j - b_j m_i)$$

Since s_{ij} is skew symmetric, we have $m_i = s_i$ i.e. m_i and s_i have same direction, therefore

$$s_{ij} = \frac{\lambda}{b^2(n-1)}(b_i s_j - b_j s_i) \tag{6.13}$$

Theorem 6.1. *Let F^n be a Douglas Space with (α,β)-metric $L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\alpha^2}{\beta}$ for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0(mod\beta)$, then*

$$(\nabla_j b_i\text{-}\nabla_i b_j) = \frac{\mu}{b^2(n-1)}(b_i s_j - b_j s_i),$$

Where $\mu = n+1$.

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