



Uniform practical stability of perturbed impulsive differential system in terms of two measures

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Abstract

In the present paper, a perturbed impulsive differential system is investigated for uniform practical stability. For the same, sufficient conditions have been obtained in terms of two measures by using Lyapunov-like function. An example has also been given to illustrate the derived results.

Keywords

Perturbed Impulsive differential system, Practical Stability, Two Measures, Lyapunov function.

AMS Subject Classification

34D20, 34K27.

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Contents

1	Introduction	142
2	Preliminaries	143
3	Main Results	143
4	Example	144
5	Conclusion	145
	References	145

1. Introduction

The theory of impulsive differential equations has been emerged as an important field of investigation owing to its wide applications in the area of control theory, population dynamics etc., and a considerable success has been achieved in this area in the past [7, 11, 14, 17]. The dynamical systems which undergoes a sudden changes at specific moment of times are known as impulsive systems. There are certain situations in real world, in which continuous perturbations influence the impulsive differential system. These impulsive differential systems, which are abrupted by continuous perturbations, provide more generalized framework to explore the characteristics of a variety of problems pertaining to real world e.g. control models for vaccination in the contagious diseases, control models for economic problems etc. Many researchers have investigated perturbed impulsive differential

systems in the past and various results have been deduced regarding stability of these systems [1–3, 9, 13].

Stability is one of the major issue in the field of differential systems. In many real world applications, it is necessarily required that the state of a system may not be stable, yet, the system may fluctuate closely to the desired state and its outcome is reasonably good. For example, a projectile, which may fluctuate about an unstable trajectory, yet its final path, may be acceptable. Similarly, the effectiveness of vaccines in certain infectious diseases may not give complete results but it is practically acceptable. In these cases, it is appropriate to use the practical stability which stabilizes a system into certain subsets of a phase space. The practical stability has been investigated extensively in the past for different types of differential systems by many researchers [4, 16]. In order to combine various types of stability concepts and to have a common platform to explore the stability theory, the potential of stability concepts in terms of two measures has been successfully demonstrated [6, 8, 15, 18]. In terms of two measure, many results have been carried out in the past to explore the practical stability by various researchers[5, 10, 12, 19].

In the present paper, the main motive is to investigate the practical stability of a impulsive differential system which is perturbed, by using Lyapunov - like function in terms of two measures. The paper is arranged into three sections: In preliminaries, we have introduced some basic definitions and notations whereas in main results, we have mentioned some conditions to bring the uniform practical stability of perturbed

impulsive differential system. Finally, in the last section, an appropriate example has been given to illustrate our results.

2. Preliminaries

Let R^n denotes an n dimensional Euclidean space and Ω is a domain in R^n which contains the origin.

Consider an impulsive differential system :

$$\begin{cases} \dot{x} = f(t, x), & t \neq t_i, \quad i = 1, 2, 3, \dots \\ \Delta x = I_i(x), & t = t_i \\ x(t_0) = x_0 \end{cases} \quad (1)$$

where

- (1) $t_0 (= 0) < t_1 < t_2 < t_3 < \dots < t_i < t_{i+1} < \dots$ and $t_i \rightarrow +\infty$ as $i \rightarrow +\infty$
- (2) $f : R^+ \times \Omega \rightarrow R^n$ is a piecewise continuous function in time t . The function f is Lipschitz in the local neighborhood of x and has discontinuities of first kind at $t = t_i$.
- (3) $I_i : \Omega \rightarrow R^n$ such that $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$

If the system (1) is subjected to small perturbation, the impulsive differential system under perturbation effect becomes [4]:

$$\begin{cases} \dot{x} = f(t, x) + g(t, x), & t \neq t_i, \quad i = 1, 2, 3, \dots \\ \Delta x = I_i(x) + J_i(x), & t = t_i \\ x(t_0) = x_0 \end{cases} \quad (2)$$

where

- (4) $g : R^+ \times \Omega \rightarrow R^n$. Likewise function f , this function g is also Lipschitz in the local neighbourhood of x .
- (5) $J_i : \Omega \rightarrow R^n$ is a locally Lipschitz function.

In order to study the stability of these perturbed impulsive differential system, firstly we will discuss some of the definitions as given below:

Definition 2.1 [8]. Let the Lyapunov function $V : R^+ \times \Omega \rightarrow R^+$ belongs to the class v_0 such that:

- (a) V is continuous function on each of the sets $[t_{i-1}, t_i) \times \Omega$; $\forall x \in \Omega$ and there exists a limit $\lim_{(t,y) \rightarrow (t_i^-, x)} V(t, y) = V(t_i^-, x)$,
- (b) V is Lipschitz in the local neighbourhood of x , where $x \in \Omega$ and $V(t, 0) = 0$,
- (c) The right hand derivative of $V \in v_0$ is given by:

$$V'(t, x) = \lim_{s \rightarrow 0^+} \sup \frac{1}{s} \{V(t+s, x+sf(t, x)) - V(t, x)\}.$$

Definition 2.2 The set K and Γ be defined as the class of continuous functions:

$$K = \{\phi \in C(R^+, R^+) : \text{strictly increasing and } \phi(0) = 0\},$$

$$\Gamma = \{h \in C(R^+ \times R^n, R^+), \inf h(t, x) = 0\}.$$

Definition 2.3 We have considered the following definitions for our investigations from V. Lakshmikantham and X. Liu [8]. Let both h_0 and h belong to Γ , then

- (a) h_0 is finer than h , iff there is a constant $P > 0$ and a function $\varphi \in K : h(t, x) \leq \varphi(h_0(t, x))$ whenever $h_0(t, x) < P$.
- (b) h is positive definite, iff there is a $S > 0$ and a function $\beta \in K : \beta(h(t, x)) \leq V(t, x)$ whenever $h(t, x) < S$.
- (c) h_0 is decrescent, iff there is a $P > 0$ and a function $\alpha \in K : V(t, x) \leq \alpha(h_0(t, x))$ whenever $h_0(t, x) < P$.
- (d) $S(h, \rho) = \{(t, x) \in R^+ \times \Omega, h(t, x) < \rho\}$.

Definition 2.4 [19]. The equilibrium point of system (1) is

- (i) practically stable w.r.t. two measures - (h_0, h) if for given $0 < P < Q$, we have $h_0(t_0, x_0) < P$ implies $h(t, x) < Q, t \geq t_0$ for some $t_0 \in R^+$
- (ii) uniformly practically stable w.r.t. two measures - (h_0, h) if condition (i) holds for every $t_0 \in R^+$.

3. Main Results

To investigate the uniform practical stability of the system (2) in terms of two measures by Lyapunov method, we will consider the following assumptions:

Suppose the system (1) is uniformly practically stable with Lyapunov function $V \in v_0$ which satisfy the following conditions:

- (A1) Let $h_0, h \in \Gamma$ such that h_0 is finer than h for $\varphi \in K$ and $P > 0$.
- (A2) There will be a Lyapunov function V so that $\beta(h(t, x)) \leq V(t, x) \leq \alpha(h_0(t, x))$ for $\alpha, \beta \in K$ and $V'(t, x) \leq -p(h_0(t, x))$, for $(t, x) \in S(h, \rho), p \in K, t \neq t_i$
- (A3) $\|V(t, x) - V(t, y)\| \leq M\|x - y\|$ where $(t, x), (t, y) \in S(h, \rho)$ such that $M > 0$
- (A4) $V(t_i, x(t_i^-) + I_i(x(t_i^-))) \leq V(t_i^-, x(t_i^-))$ for $t = t_i$

Theorem 3.1 Let (A1 - A4) holds. In addition to these assumptions, let the below mentioned criteria also hold:

- (I) given $0 < P < Q$;
- (II) $\|g(t, x)\| \leq qp(h_0(t, x))$, here q is sufficiently small but positive constant;
- (III) $\|J_i(x)\| \leq \frac{d_i}{M} V(t_i^-, x(t_i^-))$ with $d_i \geq 0$, and $\sum d_i < \infty$;



(IV) $\varphi(P) < Q, N\alpha(P) < \beta(Q)$, with
 $N = \prod_{i \rightarrow \infty} (1 + d_i) < \infty$

Then, the perturbed system (2) is (h_0, h) - uniformly practically stable.

Proof: Let the Lyapunov function V for the system (1) fulfills all the assumptions (A1 - A4). Here we will utilize the same Lyapunov function V to study the uniform practical stability of the perturbed system (2). Then the derivative of Lyapunov function V w.r.t. perturbed system (2) is given by:

$$\begin{aligned} V'(t, x) &= \lim_{s \rightarrow 0^+} \sup \frac{1}{s} \left[V(t+s, x+sf(t, x)+sg(t, x)) \right. \\ &\quad \left. - V(t, x) \right] \\ &\leq M \|g(t, x)\| + V'(t, x) \\ &\leq M \|g(t, x)\| - p(h_0(t, x)) \\ &\leq Mqp(h_0(t, x)) - p(h_0(t, x)) \\ &\leq M \frac{l}{M} p(h_0(t, x)) - p(h_0(t, x)) \\ &\quad \text{where } q = \frac{l}{M} \text{ for } l \in (0, 1) \\ &= (l-1)p(h_0(t, x)) \\ &< 0 \end{aligned}$$

For the perturbed system (2) at $t = t_i$, we obtain

$$\begin{aligned} V(t_i, x(t_i)) &= V(t_i, x(t_i^-) + I_i(x(t_i^-)) + J_i(x(t_i^-))) \\ &\leq M \|J_i(t_i^-)\| + V(t_i^-, x(t_i^-)) \\ &= (1 + d_i)V(t_i^-, x(t_i^-)) \end{aligned}$$

Let $h_0(t_0, x_0) < P$.

Therefore, as per definition 3(a), we have

$$h(t_0, x_0) \leq \varphi(h_0(t_0, x_0)) < \varphi(P) < Q$$

Now, we will verify that

$$V(t, x) \leq N\alpha(P); \quad \forall t \geq t_0 \quad (3)$$

In order to get (3), firstly we have to verify that

$$V(t, x(t)) \leq \alpha(P); \quad t_0 \leq t < t_1 \quad (4)$$

Clearly $V(t_0, x(t_0)) \leq \alpha(h_0(t_0, x_0)) < \alpha(P)$

Hence the result holds for $t = t_0$.

Let if possible, condition (4) doesn't hold true for $t > t_0$, then there exists a $\hat{t} \in (t_0, t_1)$ such that $V(\hat{t}, x(\hat{t})) > \alpha(P)$

Let $\bar{t} = \inf \{t : V(t, x(t)) > \alpha(P), t \in (t_0, t_1)\}$

As, $V(t, x)$ is a continuous function, we have

$V(\bar{t}, x(\bar{t})) = \alpha(P)$ which implies $V'(\bar{t}, x) \geq 0$

Hence, a contradiction arises, as it is already proved that $V'(t, x) < 0$, so (4) holds.

Consider,

$$\begin{aligned} V(t_1, x(t_1)) &= V(t_1, x(t_1^-) + I_i(x(t_1^-)) + J_i(x(t_1^-))) \\ &\leq (1 + d_1)V(t_1^-, x(t_1^-)) \\ &\leq (1 + d_1)\alpha(P) \end{aligned}$$

Now, in the next part, we have to verify that

$$V(t, x(t)) \leq (1 + d_1)\alpha(P); \quad t_1 < t < t_2 \quad (5)$$

If condition (5) doesn't hold true, then there exist a $\hat{u} \in (t_1, t_2)$, such that

$$V(\hat{u}, x(\hat{u})) > (1 + d_1)\alpha(P)$$

Let $\bar{u} = \inf \{t : V(t, x(t)) > (1 + d_1)\alpha(P), t \in (t_1, t_2)\}$

Again, from the continuity of $V(t, x)$, we have

$$V(\bar{u}, x(\bar{u})) = (1 + d_1)\alpha(P) \text{ which implies } V'(\bar{u}, x(\bar{u})) \geq 0.$$

Thus, we get a contradiction as $V'(t, x) < 0$, so (5) holds.

Now Consider

$$\begin{aligned} V(t_2, x(t_2)) &= V(t_2, x(t_2^-) + I_k(x(t_2^-)) + J_k(x(t_2^-))) \\ &\leq (1 + d_2)V(t_2^-, x(t_2^-)) \\ &\leq (1 + d_2)(1 + d_1)\alpha(P) \end{aligned}$$

By continuing the above procedure, we have

$$V(t, x(t)) \leq (1 + d_1)(1 + d_2)\dots(1 + d_i)\alpha(P); \quad t_i \leq t \leq t_{i+1}.$$

Hence, $V(t, x(t)) \leq N\alpha(P)$ where $N = \prod_{k \rightarrow \infty} (1 + d_k) < \infty$.

Now, from condition (IV) and equation (3), we have

$$V(t, x(t)) \leq N\alpha(P) < \beta(Q); \quad \forall t \geq t_0$$

As h is positive-definite by assumption (A2), we have

$$H(t, x(t)) \leq \beta^{-1}(V(t, x(t))) < \beta^{-1}(\beta(Q)) = Q; \quad t \geq t_0$$

Thus, the perturbed impulsive differential system (2) is uniformly practically stable as per the conditions mentioned in Theorem 1 of reference [19].

4. Example

Here we consider a simple example to verify our results for uniform practical stability.

Consider the Ricatti scalar equation under the effect of impulse:

$$\begin{cases} \dot{x} = -f(t)x + r(t)x^2, & t \neq t_i, \quad i = 1, 2, 3, \dots \\ \Delta x = \lambda_i x, & t = t_i \end{cases} \quad (6)$$

where $f: R^+ \rightarrow R^+, r: R^+ \rightarrow R, \lambda_i \geq 0$; f and h are continuous functions on $(t_{i-1}, t_i]$ with points of discontinuity of first kind at $t = t_i, i = 1, 2, 3, \dots$ and $|r(t)| < qf(t)$, where q is a small but



positive constant.

The following system is the nominal system of the above perturbed system:

$$\begin{cases} \dot{x} = -f(t)x, & t \neq t_i, \quad i = 1, 2, 3, \dots \\ \Delta x = \lambda_i x, & t = t_i \\ x(t_0^+) = x_0 \end{cases} \quad (7)$$

Proof: Let us define the functions as follows:

$$\begin{aligned} h_0(t, x) &= h(t, x) = x, \\ \text{The Lyapunov function } V(t, x) &= \frac{1}{2}x^2, \\ \text{The functions } \varphi(x) = x, \alpha(x) &= nx^2 \text{ and } \beta(x) = \frac{1}{n}x^2 \\ \text{where } n \in N \text{ and } n > 1. \end{aligned}$$

Let $0 < P < \frac{1}{n\sqrt{N}}Q$ where $N = \prod(1 + d_i)$ and $d_i \geq 0$ such that i is the total number of impulsive moments over the range of existence of solution.

For $h_0(t, x) < P$, clearly h_0 is finer than h .

Also, $\beta(h(t, x)) \leq V(t, x) \leq \alpha(h_0(t, x))$ is satisfied.

Let $x, y \in R^+$

Now, $|V(t, x) - V(t, y)| < M|x - y|$ where $M = |x + y| > 0$.

Let $q = \frac{l}{M}$ where $l \in (0, 1)$. Then clearly $q > 0$.

Let us define $p \in K$ such that $p(x) = f(t)x^2 \forall t$.

Clearly $V'(t, x) \leq -p(h_0(t, x))$ is satisfied.

$$\begin{aligned} \text{Consider } |g(t, x)| &= |r(t)x^2| \\ &\leq qf(t)x^2 \\ &= qp(h_0(t, x)) \end{aligned}$$

For $0 < P < \frac{1}{n\sqrt{N}}Q$, $\varphi(P) < Q$ and $N\alpha(P) < \beta(Q)$ are satisfied.

Hence, all the conditions of Theorem 1 are satisfied.

Therefore, the perturbed system (6) under consideration is uniformly practically stable in terms of two measures.

5. Conclusion

In this article, by using Lyapunov-like function, we have deduced some criteria to investigate the uniform practical stability in terms of two measures for the impulsive differential system which is perturbed. In the derived result, we obtained some conditions for perturbing terms which guarantee the uniform practical stability of the perturbed impulsive differential system. We have also verified our results with the help of an example.

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