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Monotone iterative technique for weakly coupled system of finite difference equations of time degenerate parabolic problems

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Abstract
The purpose of this paper is to develop monotone iterative technique for finite difference system, which correspond to a class of semilinear weakly coupled system of time degenerate parabolic initial boundary value problems and prove existence-comparison result.

Keywords
Monotone iterative technique, Finite difference equations,Dirichlet initial boundary value problem, Existence-comparison result.

AMS Subject Classification
26A33, 33E12 34A08, 35R11.

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Contents

1. Introduction

The monotone iterative technique is one of the well known method employed successfully in the study of existence comparison of solution of initial boundary value problem (IBVP) for nonlinear partial differential equations.In 1972,Sattinger [13] first developed monotone technique for nonlinear parabolic as well as elliptic boundary value problems. His work, later on refined and extended by many researchers and series of papers appeared in the literature. An excellent account of these results are given in the elegant books by Ladde, Lakshmikantham and Vatsala [7], Pao [10] and Leung [8].

In 1985, Pao [9] introduced monotone technique for finite difference equations of nonlinear parabolic and elliptic boundary value problems. A series of papers appeared in the literature for reaction diffusion problems under different conditions (see [3],[11], [12])and references therein. Recently Dhaigude et.al[4]developed monotone technique for discrete weakly coupled system of finite difference equations of parabolic type. In this paper, the backward approximation for the spatial derivative terms are used and the monotone technique is developed, using the notion of upper-lower solutions for weakly coupled system of finite difference equations which corresponds to weakly coupled system of semilinear time degenerate parabolic Dirichlet IBVP when the reaction functions $f^{(1)}_{i,n}(u_{i,n},v_{i,n})$ and $f^{(2)}_{i,n}(u_{i,n},v_{i,n})$ are assumed to be quasimonotone nonincreasing. Positivity lemma is the main ingredient used in the proof of these results.

We organize the paper as follows: In section 2, Dirichlet IBVP for finite difference system is formulated from the corresponding continuous semilinear problem under consideration.The notion of upper lower solution is introduced. Section 3 is devoted for the construction of monotone scheme for the discrete Dirichlet IBVP. Using upper and lower solutions as distinct initial iterations, two monotone convergent sequences are constructed, which converge monotonically from above and below to maximal and minimal solutions respectively. Existence-comparison result for the solution of discrete Dirichlet IBVP are proved in the last section.

2. Upper Lower Solutions

In this section,we obtain the discrete version of the Dirichlet Initial Boundary Value Problem (IBVP) for weakly coupled system of semilinear time degenerate parabolic equations.We consider the time degenerate Dirichlet IBVP for weakly coupled system of semilinear time degenerate parabolic equations.

$$\begin{aligned} d^{(1)}(x,t)u_t - D^{(1)}(x,t)\nabla^2 u = f^{(1)}(x,t,u,v);\\ d^{(2)}(x,t)v_t - D^{(2)}(x,t)\nabla^2 v = f^{(2)}(x,t,u,v); \end{aligned} \quad in\ D_T \quad (2.1)$$

boundary conditions

$$\begin{aligned} u(x,t) = h^{(1)}(x,t);\\ v(x,t) = h^{(2)}(x,t); \end{aligned} \quad on\ S_T \quad (2.2)$$

initial conditions

$$\begin{aligned} u(x,0) = \psi^{(1)}(x);\\ v(x,0) = \psi^{(2)}(x); \end{aligned} \quad in\ \Omega \quad (2.3)$$

where Ω is a bounded domain in $\mathbb{R}^n, (n = 1,2,\dots)$ with boundary $\partial\Omega$. Suppose that the functions $d^{(1)}(x,t), d^{(2)}(x,t)$ are nonnegative in D_T. However we will not assume that $d^{(1)}(x,t)$ and $d^{(2)}(x,t)$ are bounded away from zero. Since the coefficients $d^{(1)}(x,t) = 0, d^{(2)}(x,t) = 0$ for some $(x,t) \in D_T$ and hence the system is time degenerate.The coefficients $D^{(1)}(x,t), D^{(2)}(x,t)$ are positive in D_T. The functions $f^{(1)}(x,t,u,v)$, $f^{(2)}(x,t,u,v)$ are in general nonlinear in u,v and depend explicitly on (x,t).They are quasimonotone nonincreasing. Now, we write the discrete version of the above continuous IBVP (2.1)- (2.3) by converting it into finite difference equations as in [[1],[6]]. Let i=(i_1,i_2,...,i_p) be a multiple index with i_ν= 0,1,2,...,M_ν +1 and let x_i = (x_{i_1},x_{i_2},...,x_{i_p}) be an arbitrary mesh point in Ω_p where M_ν is the total number of interior mesh points in the x_{i_ν} co-ordinate direction.Denoted by Ω_p, $\bar{\Omega}_p$, $\partial\Omega_p$, Λ_p and S_p the sets of mesh points in Ω, $\bar{\Omega}$, $\partial\Omega$, $\Omega\times$(0,T) and $\partial\Omega\times$(0,T)respectively and $\bar{\Lambda}_p$ denote the set of all mesh points in $\bar{\Omega}\times$ [0,T] where $\bar{\Omega}$ is the closure of Ω.Let (i,n) be used to represent the mesh point (x_i,t_n). Set

$$\begin{aligned}
&u_{i,n} \equiv u(x_i,t_n), v_{i,n} \equiv v(x_i,t_n), d^{(1)}_{i,n} \equiv d^{(1)}(x_i,t_n),\\
&d^{(2)}_{i,n} \equiv d^{(2)}(x_i,t_n); D^{(1)}_{i,n} \equiv D^{(1)}(x_i,t_n), D^{(2)}_{i,n} \equiv D^{(2)}(x_i,t_n),\\
&h^{(1)}_{i,n} \equiv h^{(1)}(x_i,t_n), h^{(2)}_{i,n} \equiv h^{(2)}(x_i,t_n); u^{(1)}_{i,0} \equiv u^{(1)}(x_i,0),\\
&u^{(2)}_{i,0} \equiv u^{(2)}(x_i,0), v^{(1)}_{i,0} \equiv v^{(1)}(x_i,0), v^{(2)}_{i,0} \equiv v^{(2)}(x_i,0);\\
&\psi^{(1)}_i \equiv \psi^{(1)}(x_i), \psi^{(2)}_i \equiv \psi^{(2)}(x_i);\\
&f^{(1)}_{i,n}(u_{i,n},v_{i,n}) \equiv f^{(1)}(x_i,t_n,u_{i,n},v_{i,n}),\\
&f^{(2)}_{i,n}(u_{i,n},v_{i,n}) \equiv f^{(2)}(x_i,t_n,u_{i,n},v_{i,n})
\end{aligned}$$

Let $k_n = t_n - t_{n-1}$ be the n^{th} time increment for $n = 1,2,...N$ and h_ν be the spatial increment in the x_{i_ν} co-ordinate direction. Let e_ν= (0,...,1...,0) be the unit vector in $\mathbb{R}^n$ where the constant 1 appears in the ν^{th} component and zero elsewhere.The standard second order difference approximation is (see [1],[6])

$$\begin{aligned} \triangle^{(\nu)}u(x_i,t_n) = h_\nu^{(-2)}[u(x_i + h_\nu e_\nu, t_n) - 2u(x_i,t_n)\\ + u(x_i - h_\nu e_\nu, t_n)] \end{aligned}$$

and usual backward difference approximation for u_t is $k_n^{-1}(u_{i,n} - u_{i,n-1})$.

Then the continuous Dirichlet IBVP (2.1)- (2.3) for time degenerate parabolic partial differential equations becomes

$$\begin{aligned}
&d^{(1)}_{i,n}k_n^{-1}(u_{i,n} - u_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}u_{i,n} = f^{(1)}_{i,n}(u_{i,n},v_{i,n});\\
&\qquad\qquad (i,n) \in \Lambda_p\\
&d^{(2)}_{i,n}k_n^{-1}(v_{i,n} - v_{i,n-1}) - \sum_{\nu=1}^{p} D^{(2)}_{i,n}\triangle^{(\nu)}v_{i,n} = f^{(2)}_{i,n}(u_{i,n},v_{i,n});
\end{aligned} \quad (2.4)$$

boundary conditions

$$\begin{aligned} u_{i,n} = h^{(1)}_{i,n};\\ v_{i,n} = h^{(2)}_{i,n}; \end{aligned} \quad (i,n) \in S_p \quad (2.5)$$

initial conditions

$$\begin{aligned} u_{i,0} = \Psi^{(1)}_i;\\ v_{i,0} = \Psi^{(2)}_i; \end{aligned} \quad i \in \Omega_p \quad (2.6)$$

In this way we have obtained the discrete Dirichlet IBVP (2.4)-(2.6) for time degenerate parabolic partial differential equations.The functions $f^{(1)}_{i,n}$and$f^{(2)}_{i,n}$ are quasimonotone nonincreasing.We define them as follows:

Definition 2.1. *The C^1-functions $f^{(1)}_{i,n}$andf$^{(2)}_{i,n}$ are said to be quasimonotone nonincreasing if*

$$f^{(1)}_{i,n}(u_{i,n},v_{i,n}) \geq f^{(1)}_{i,n}(u_{i,n},v'_{i,n})\,for\ v'_{i,n} \geq v_{i,n}$$

and

$$f^{(2)}_{i,n}(u_{i,n},v_{i,n}) \geq f^{(2)}_{i,n}(u'_{i,n},v_{i,n})\,for\ u'_{i,n} \geq u_{i,n}$$

respectively

Now we define upper and lower solutions of the discrete time degenerate Dirichlet IBVP (2.4)-(2.6).

Definition 2.2. *The functions $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ with $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \geq (\hat{u}_{i,n}, \hat{v}_{i,n})$ are called ordered upper and lower solutions of discrete time degenerate Dirichlet IBVP* (2.4)-

(2.6) *if they satisfy differential inequalities*

$$d_{i,n}^{(1)}k_n^{-1}(\tilde{u}_{i,n}-\tilde{u}_{i,n-1})-\sum_{\upsilon=1}^{p}D_{i,n}^{(1)}\triangle^{(\upsilon)}\tilde{u}_{i,n} \geq f_{i,n}^{(1)}(\tilde{u}_{i,n},\tilde{v}_{i,n});$$

$$d_{i,n}^{(1)}k_n^{-1}(\hat{u}_{i,n}-\hat{u}_{i,n-1})-\sum_{\upsilon=1}^{p}D_{i,n}^{(1)}\triangle^{(\upsilon)}\hat{u}_{i,n} \leq f_{i,n}^{(1)}(\hat{u}_{i,n},\hat{v}_{i,n});$$

$$(i,n)\in\Lambda_p$$

$$d_{i,n}^{(2)}k_n^{-1}(\tilde{v}_{i,n}-\tilde{v}_{i,n-1})-\sum_{\upsilon=1}^{p}D_{i,n}^{(2)}\triangle^{(\upsilon)}\tilde{v}_{i,n} \geq f_{i,n}^{(2)}(\tilde{u}_{i,n},\tilde{v}_{i,n});$$

$$d_{i,n}^{(2)}k_n^{-1}(\hat{v}_{i,n}-\hat{v}_{i,n-1})-\sum_{\upsilon=1}^{p}D_{i,n}^{(2)}\triangle^{(\upsilon)}\hat{v}_{i,n} \leq f_{i,n}^{(2)}(\hat{u}_{i,n},\hat{v}_{i,n});$$

boundary conditions

$$\left.\begin{aligned}\tilde{u}_{i,n}\geq h_{i,n}^{(1)}\geq\hat{u}_{i,n}\\ \tilde{v}_{i,n}\geq h_{i,n}^{(2)}\geq\hat{v}_{i,n}\end{aligned}\right\}\quad (i,n)\in S_p$$

initial conditions

$$\left.\begin{aligned}\tilde{u}_{i,0}\geq \Psi_i^{(1)}\geq\hat{u}_{i,0}\\ \tilde{v}_{i,0}\geq \Psi_i^{(2)}\geq\hat{v}_{i,0}\end{aligned}\right\}\quad i\in\Omega_p$$

Definition 2.3. *For any ordered upper and lower solutions $(\tilde{u}_{i,n},\tilde{u}_{i,n})$ and $(\hat{v}_{i,n}\ ,\hat{v}_{i,n})$ the sector is denoted by $S_{i,n}$ and is defined as*

$$S_{i,n}=\{(u_{i,n},v_{i,n})\in\bar{\Lambda}_p:(\hat{u}_{i,n},\hat{v}_{i,n})\leq(u_{i,n},v_{i,n})\leq(\tilde{u}_{i,n},\tilde{v}_{i,n})\} \tag{2.7}$$

Suppose there exist nonnegative constants $c_{i,n}^{(1)}$ and $c_{i,n}^{(2)}$ such that the function ($f_{i,n}^{(1)}$ and $f_{i,n}^{(2)}$) satisfies the following one sided Lipschitz condition for $u_{i,n}\geq u'_{i,n}$ and $v_{i,n}\geq v'_{i,n}$

$$\begin{aligned}f_{i,n}^{(1)}(u_{i,n},v_{i,n})-f_{i,n}^{(1)}(u'_{i,n},v_{i,n})\geq -c_{i,n}^{(1)}(u_{i,n}-u'_{i,n})\\ f_{i,n}^{(2)}(u_{i,n},v_{i,n})-f_{i,n}^{(2)}(u_{i,n},v'_{i,n})\geq -c_{i,n}^{(2)}(v_{i,n}-v'_{i,n})\end{aligned} \tag{2.8}$$

Adding $c_{i,n}^{(1)}u_{i,n}$ and $c_{i,n}^{(2)}v_{i,n}$ on both sides of equation in (2.4) respectively, we get

$$d_{i,n}^{(1)}k_n^{-1}(u_{i,n}-u_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(1)}\triangle^{(\nu)}u_{i,n}+c_{i,n}^{(1)}u_{i,n} = c_{i,n}^{(1)}u_{i,n}+f_{i,n}^{(1)}(u_{i,n},v_{i,n})$$

$$d_{i,n}^{(2)}k_n^{-1}(v_{i,n}-v_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}v_{i,n}+c_{i,n}^{(2)}v_{i,n} = c_{i,n}^{(2)}v_{i,n}+f_{i,n}^{(2)}(u_{i,n},v_{i,n})$$

Suppose

$$L[u_{i,n}]\equiv d_{i,n}^{(1)}k_n^{-1}(u_{i,n}-u_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(1)}\triangle^{(\nu)}u_{i,n} + c_{i,n}^{(1)}u_{i,n}$$

$$L[v_{i,n}]\equiv d_{i,n}^{(2)}k_n^{-1}(v_{i,n}-v_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}v_{i,n} + c_{i,n}^{(2)}v_{i,n}$$

Now we develop monotone scheme for the discrete time degenerate Dirichlet IBVP (2.4)-(2.6). The discrete version of the positivity lemma in [2] for the continuous problem play an important role in the construction of monotone and convergent sequences.We state it as follows:

Lemma 2.4. ***(Positivity Lemma)*** *Suppose $w_{i,n}$ satisfies the following inequalities*

$$d_{i,n}k_n^{-1}(w_{i,n}-w_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}\Delta^{(\nu)}w_{i,n}+c_{i,n}w_{i,n} \geq 0;(i,n)\in\Lambda_p$$

$$Bw_{i,n}\geq 0;(i,n)\in S_p$$

$$w_{i,0}\geq 0; i\in\Omega_p$$

where $Bw_{i,n}=\alpha(x_i,t_n)|x_i-\hat{x}_i|^{-1}[w(x_i,t_n)-w(\hat{x}_i,t_n)] +\beta(x_i,t_n)w(x_i,t_n)$, $c_{i,n}\geq 0, d_{i,n}\geq 0, \hat{x}_i$ is a suitable point in Ω_p and $|x_i-\hat{x}_i|$ is the distance between x_i and $\hat{x}_i$ then

$$w_{i,n}\geq 0 \quad in \quad \bar{\Lambda}_p.$$

3. Monotone Iterative Technique

We choose suitable initial iterations $(u_{i,n}^{(0)},v_{i,n}^{(0)})$ as either $(\tilde{u}_{i,n},\tilde{v}_{i,n})$ or $(\hat{u}_{i,n},\hat{v}_{i,n})$ and construct a sequence of iterations $\{u_{i,n}^{(m)},v_{i,n}^{(m)}\}$ from the following iterative scheme.

$$\left.\begin{aligned}L[u_{i,n}^{(m)}]=&c_{i,n}^{(1)}u_{i,n}^{(m-1)}+f_{i,n}^{(1)}(u_{i,n}^{(m-1)},v_{i,n}^{(m-1)}),\\ L[v_{i,n}^{(m)}]=&c_{i,n}^{(2)}v_{i,n}^{(m-1)}+f_{i,n}^{(2)}(u_{i,n}^{(m)},v_{i,n}^{(m-1)}),\end{aligned}\right.\quad (i,n)\in\Lambda_p \tag{3.1}$$

$$\left.\begin{aligned}\bar{u}_{i,n}^{(m)}=&h_{i,n}^{(1)}\\ \underline{v}_{i,n}^{(m)}=&h_{i,n}^{(2)},\end{aligned}\right.\quad (i,n)\in S_p \tag{3.2}$$

$$\begin{aligned}\bar{u}_{i,0}^{(m)} &= \psi_i^{(1)} \\ \underline{v}_{i,0}^{(m)} &= \psi_i^{(2)},\end{aligned} \quad i \in \Omega_p \tag{3.3}$$

It is a system of linear algebraic equations. Here m =1 and with suitable choice of initial iterations $(\bar{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \hat{v}_{i,n})$. we get $(\bar{u}_{i,n}^{(1)})$ from first equation.Put this value in second equation,we get first iteration $(\bar{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)})$. Repeat the process for $m = 2,3,....$ .We construct a sequence $\{\bar{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$. Further,suppose m =1 and with suitable choice of initial iterations $(\underline{u}_{i,n}^{(0)}, \bar{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \tilde{v}_{i,n})$. we get $(\underline{u}_{i,n}^{(1)})$ from first equation.Put this value in second equation,we get first iteration $(\underline{u}_{i,n}^{(1)}, \bar{v}_{i,n}^{(1)})$. Repeat the process for m=2,3,.... .We construct a sequence $\{\underline{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m)}\}$. An interesting point about the above iterative scheme is that the component $(\bar{u}_{i,n}^{(1)})$ from first equation is used immediately in second equation,to obtain $(\bar{v}_{i,n}^{(1)})$. We note the above kind of iteration is similar to the Gauss-Seidal iterative method for algebraic systems.It has the advantage of obtaining faster convergent sequences.In the following,we obtain the monotone property of the sequences when initial iteration is either an upper solution or a lower solution.

Lemma 3.1 (Monotone Property)**.** *Suppose that*

(i) $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ are ordered upper and lower solutions of discrete time degenerate Dirichlet IBVP (2.4)-(2.6)

(ii)the function $(f_{i,n}^{(1)}, f_{i,n}^{(2)})$ is quasimonotone nonincreasing in $S_{i,n}$,

(iii)the function $(f_{i,n}^{(1)}, f_{i,n}^{(2)})$ satisfies the one sided Lipschitz condition (2.8)

Then the sequences $\{\bar{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$ and $\{\underline{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m)}\}$ obtained from iteration process (3.1)- (3.3) *with $(\bar{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \hat{v}_{i,n})$ and $(\underline{u}_{i,n}^{(0)}, \bar{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \tilde{v}_{i,n})$ possess the mixed monotone property in the sense that their components $\bar{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m)}$ and $\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}$, satisfy the following relation in $\bar{\Lambda}_p$*

$$\hat{u}_{i,n} \le \underline{u}_{i,n}^{(1)} \le \cdots \le \underline{u}_{i,n}^{(m+1)} \le \bar{u}_{i,n}^{(m+1)} \le \cdots \le \bar{u}_{i,n}^{(1)} \le \tilde{u}_{i,n} \tag{3.4}$$

$$\hat{v}_{i,n} \le \underline{v}_{i,n}^{(1)} \le \cdots \le \underline{v}_{i,n}^{(m+1)} \le \bar{v}_{i,n}^{(m+1)} \le \cdots \le \bar{v}_{i,n}^{(1)} \le \tilde{v}_{i,n} \tag{3.5}$$

for all m

Proof. Define $w_{i,n} = \bar{u}_{i,n}^{(0)} - \bar{u}_{i,n}^{(1)} = \tilde{u}_{i,n} - \bar{u}_{i,n}^{(1)}$ where $\bar{u}_{i,n}^{(0)} = \tilde{u}_{i,n}$.

Consider $L[w_{i,n}] = L[\tilde{u}_{i,n}] - L[\bar{u}_{i,n}^{(1)}]$

$$\begin{aligned}L[w_{i,n}] =& d_{i,n}^{(1)} k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} \tilde{u}_{i,n} \\ &+ c_{i,n}^{(1)} \tilde{u}_{i,n} - [d_{i,n}^{(1)} k_n^{-1}(\bar{u}_{i,n}^{(1)} - \bar{u}_{i,n-1}^{(1)}) \\ &- \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} \bar{u}_{i,n}^{(1)} + c_{i,n}^{(1)} \bar{u}_{i,n}^{(1)}] \\ =& d_{i,n}^{(1)} k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} \tilde{u}_{i,n} \\ &+ c_{i,n}^{(1)} \tilde{u}_{i,n} - [c_{i,n}^{(1)} \tilde{u}_{i,n} + f_{i,n}^{(1)}(\tilde{u}_{i,n}, \hat{v}_{i,n})] \\ &\text{(By iteration scheme)} \\ =& d_{i,n}^{(1)} k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} \tilde{u}_{i,n} \\ &- f_{i,n}^{(1)}(\tilde{u}_{i,n}, \hat{v}_{i,n}) \ge 0\end{aligned}$$

Since $\tilde{u}_{i,n}$ is an upper solution.
Thus we have

$$\begin{aligned}d_{i,n}^{(1)} k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} w_{i,n} + c_{i,n}^{(1)} w_{i,n} \\ \ge 0; (i,n) \in \Lambda_p \\ w_{i,n} = \tilde{u}_{i,n} - h_{i,n}^{(1)} \ge 0; (i,n) \in S_p \\ w_{i,0} = \tilde{u}_{i,0} - \Psi_i^{(1)} \ge 0; i \in \Omega_p\end{aligned}$$

By Positivity lemma 2.4,we get $w_{i,n} \ge 0$ implies that $\bar{u}_{i,n}^{(1)} \le \tilde{u}_{i,n}$.
Define $z_{i,n}^{(1)} = \bar{v}_{i,n}^{(1)} - \underline{v}_{i,n}^{(1)}$. Consider

$$\begin{aligned}L[z_{i,n}^{(1)}] =& L[\bar{v}_{i,n}^{(1)}] - L[\underline{v}_{i,n}^{(1)}] \\ L[z_{i,n}^{(1)}] =& d_{i,n}^{(2)} k_n^{-1}(\bar{v}_{i,n}^{(1)} - \bar{v}_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} \bar{v}_{i,n}^{(1)} \\ &+ c_{i,n}^{(2)} \bar{v}_{i,n}^{(1)} - [d_{i,n}^{(2)} k_n^{-1}(\underline{v}_{i,n}^{(1)} - \underline{v}_{i,n-1}^{(1)}) \\ &- \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} \underline{v}_{i,n}^{(1)} + c_{i,n}^{(2)} \underline{v}_{i,n}^{(1)}] \\ =& c_{i,n}^{(2)} \bar{v}_{i,n}^{(0)} + f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \bar{v}_{i,n}^{(0)}) - c_{i,n}^{(2)} \underline{v}_{i,n}^{(0)} \\ &- f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \text{(By iteration scheme)} \\ =& [c_{i,n}^{(2)}(\bar{v}_{i,n}^{(0)} - \underline{v}_{i,n}^{(0)}) + f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \bar{v}_{i,n}^{(0)}) \\ &- f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)})] + [f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \\ &- f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)})] \ge 0\end{aligned}$$

$f_{i,n}^{(2)}$ is Lipchitzian and quasimonotone nonincreasing

$$\begin{aligned}d_{i,n}^{(2)} k_n^{-1}(z_{i,n}^{(1)} - z_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} z_{i,n}^{(1)} + c_{i,n}^{(2)} z_{i,n}^{(1)} \ge 0; \\ z_{i,n}^{(1)} = \bar{v}_{i,n}^{(1)} - \underline{v}_{i,n}^{(1)} \ge h_{i,n}^{(2)} - \underline{v}_{i,n}^{(1)} \\ z_{i,0}^{(1)} = \bar{v}_{i,0}^{(1)} - \underline{v}_{i,0}^{(1)} \ge \Psi_i^{(2)} - \underline{v}_{i,0}^{(1)} \ge 0;\end{aligned}$$

For $(i,n)\in\Lambda_p, \geq 0; (i,n)\in S_p$ and $i\in\Omega_p$ In view of positivity lemma 2.4,we get $z^{(1)}_{i,n}\geq 0$ implies that $\underline{v}^{(1)}_{i,n}\leq \bar{v}^{(1)}_{i,n}$ We conclude that

$$\hat{u}_{i,n}\leq \underline{u}^{(1)}_{i,n}\leq \bar{u}^{(1)}_{i,n}\leq \tilde{u}_{i,n} \text{ in } \bar{\Lambda}_p$$
$$\hat{v}_{i,n}\leq \underline{v}^{(1)}_{i,n}\leq \bar{v}^{(1)}_{i,n}\leq \tilde{v}_{i,n} \text{ in } \bar{\Lambda}_p$$

Thus result is true for $m=1$.
Assume that it is true for $m=l$

$$\underline{u}^{(l-1)}_{i,n}\leq \underline{u}^{(l)}_{i,n}\leq \bar{u}^{(l)}_{i,n}\leq \bar{u}^{(l-1)}_{i,n} \text{ in } \bar{\Lambda}_p$$
$$\underline{v}^{(l-1)}_{i,n}\leq \underline{v}^{(l)}_{i,n}\leq \bar{v}^{(l)}_{i,n}\leq \bar{v}^{(l-1)}_{i,n} \text{ in } \bar{\Lambda}_p$$

Now we prove that it is true for $m=l+1$.
Define $w^{(l)}_{i,n}=\bar{u}^{(l)}_{i,n}-\bar{u}^{(l+1)}_{i,n}$. Consider

$$\begin{aligned}
L[w^{(l)}_{i,n}] =& L[\bar{u}^{(l)}_{i,n}]-L[\bar{u}^{(l+1)}_{i,n}]\\
L[w^{(l)}_{i,n}] =& d^{(1)}_{i,n}k_n^{-1}(\bar{u}^{(l)}_{i,n}-\bar{u}^{(l)}_{i,n-1})\\
& -\sum_{\nu=1}^{p}D^{(1)}_{i,n}\triangle^{(\nu)}\bar{u}^{(l)}_{i,n}+c^{(1)}_{i,n}\bar{u}^{(l)}_{i,n}-\\
& [d^{(1)}_{i,n}k_n^{-1}(\bar{u}^{(l+1)}_{i,n}-\bar{u}^{(l+1)}_{i,n-1})\\
& -\sum_{\nu=1}^{p}D^{(1)}_{i,n}\triangle^{(\nu)}\bar{u}^{(l+1)}_{i,n}+c^{(1)}_{i,n}\bar{u}^{(l+1)}_{i,n}]\\
=& c^{(1)}_{i,n}\bar{u}^{(l-1)}_{i,n}+f^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n},\underline{v}^{(l-1)}_{i,n})\\
& -[c^{(1)}_{i,n}\underline{u}^{(l)}_{i,n}+f^{(1)}_{i,n}(\underline{u}^{(l)}_{i,n},\bar{v}^{(l)}_{i,n})]\\
& \text{(By iteration scheme)}\\
=& c^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n}-\underline{u}^{(l)}_{i,n})+f^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n},\underline{v}^{(l-1)}_{i,n})\\
& -f^{(1)}_{i,n}(\underline{u}^{(l)}_{i,n},\bar{v}^{(l)}_{i,n})\\
=& [c^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n}-\underline{u}^{(l)}_{i,n})+f^{(1)}_{i,n}(\bar{u}^{(k-1)}_{i,n},\underline{v}^{(l-1)}_{i,n})\\
& -f^{(1)}_{i,n}(\underline{u}^{(l)}_{i,n},\underline{v}^{(l-1)}_{i,n})]+\\
& [f^{(1)}_{i,n}(\underline{u}^{(l)}_{i,n},\underline{v}^{(l-1)}_{i,n})\\
& -f^{(1)}_{i,n}(\underline{u}^{(l)}_{i,n},\bar{v}^{(l)}_{i,n})]\\
\geq & 0
\end{aligned}$$

$f^{(1)}_{i,n}$ is Lipchitzian and quasimonotone nonincreasing.
We have

$$\begin{aligned}
d^{(1)}_{i,n}k_n^{-1}(w^{(l)}_{i,n}-w^{(l)}_{i,n-1})-\sum_{\nu=1}^{p}D^{(1)}_{i,n}\triangle^{(\nu)}w^{(l)}_{i,n}&\\
+c^{(1)}_{i,n}w^{(l)}_{i,n}&\geq 0;\\
w^{(l)}_{i,n}=\bar{u}^{(l)}_{i,n}-\underline{u}^{(l)}_{i,n}\geq h^{(1)}_{i,n}-\underline{u}^{(l)}_{i,n}&\geq 0;\\
w^{(l)}_{i,0}=\bar{u}^{(l)}_{i,0}-\underline{u}^{(l)}_{i,0}\geq \Psi^{(1)}_{i}-\underline{u}^{(l)}_{i,0}&\geq 0;
\end{aligned}$$

For $(i,n)\in\Lambda_p$, $(i,n)\in S_p$ and $i\in\Omega_p$ By lemma 2.4,we get $w^{(l)}_{i,n}\geq 0$ implies that $\bar{u}^{(l+1)}_{i,n}\leq \bar{u}^{(l)}_{i,n}$.On similar lines we can prove that $\underline{u}^{(l)}_{i,n}\leq \underline{u}^{(l+1)}_{i,n}$ and $\underline{u}^{(l+1)}_{i,n}\leq \bar{u}^{(l+1)}_{i,n}$.
Define $z^{(l)}_{i,n}=\bar{v}^{(l)}_{i,n}-\bar{v}^{(l+1)}_{i,n}$. Consider

$$\begin{aligned}
L[z^{(l)}_{i,n}] =& L[\bar{v}^{(l)}_{i,n}]-L[\bar{v}^{(l+1)}_{i,n}]\\
L[z^{(l)}_{i,n}] =& d^{(2)}_{i,n}k_n^{-1}(\bar{v}^{(l)}_{i,n}-\bar{v}^{(l)}_{i,n-1})-\sum_{\nu=1}^{p}D^{(2)}_{i,n}\triangle^{(\nu)}\bar{v}^{(l)}_{i,n}\\
& +c^{(2)}_{i,n}\bar{v}^{(l)}_{i,n}-[d^{(2)}_{i,n}k_n^{-1}(\bar{v}^{(l+1)}_{i,n}-\bar{v}^{(l+1)}_{i,n-1})\\
& -\sum_{\nu=1}^{p}D^{(2)}_{i,n}\triangle^{(\nu)}\bar{v}^{(l+1)}_{i,n}+c^{(2)}_{i,n}\bar{v}^{(l+1)}_{i,n}]\geq 0
\end{aligned}$$

it follows from iteration scheme,Lipchitzian and quasimonotone nonincreasing property of $f^{(2)}_{i,n}$. We have

$$\begin{aligned}
d^{(2)}_{i,n}k_n^{-1}(z^{(l)}_{i,n}-z^{(l)}_{i,n-1})-\sum_{\nu=1}^{p}D^{(2)}_{i,n}\triangle^{(\nu)}z^{(l)}_{i,n}+c^{(2)}_{i,n}z^{(l)}_{i,n}&\geq 0;\\
z^{(l)}_{i,n}=\bar{v}^{(l)}_{i,n}-\bar{v}^{(l+1)}_{i,n}\geq h^{(2)}_{i,n}-\bar{v}^{(l+1)}_{i,n}&\geq 0;\\
z^{(l)}_{i,0}=\bar{v}^{(1)}_{i,0}-\bar{v}^{(l+1)}_{i,0}\geq \Psi^{(2)}_{i}-\bar{v}^{(l+1)}_{i,0}&\geq 0;
\end{aligned}$$

For $(i,n)\in\Lambda_p$, $(i,n)\in S_p$ and $i\in\Omega_p$. In view of positivity lemma 2.4,we get $w^{(l)}_{i,n}\geq 0$ implies that $\bar{v}^{(l+1)}_{i,n}\leq \bar{v}^{(l)}_{i,n}$. Note that using similar arguments we can prove $\underline{v}^{(l)}_{i,n}\leq \underline{v}^{(l+1)}_{i,n}$ as well as $\underline{v}^{(l+1)}_{i,n}\leq \bar{v}^{(l+1)}_{i,n}$.Thus from principle of mathematical induction result is true for all m

4. Application

In this section, we show that the monotone iterative technique is successfully applied to prove existence - comparison result for the solution of the discrete time degenerate Dirichlet IBVP (2.4)-(2.6)

Theorem 4.1. *(Existence -Comparison Theorem)Suppose that (i) $(\tilde{u}_{i,n},\tilde{v}_{i,n})$ and $(\hat{u}_{i,n},\hat{v}_{i,n})$ are ordered upper and lower solutions of discrete time degenerate Dirichlet IBVP (2.4)-(2.6)*
(ii)the function $(f^{(1)}_{i,n},f^{(2)}_{i,n})$ is quasimonotone nonincreasing in $S_{i,n}$,
(iii)the function $(f^{(1)}_{i,n},f^{(2)}_{i,n})$ satisfies the one sided Lipschitz condition (2.8)

Then the sequence $\{\bar{u}^{(m)}_{i,n},\bar{v}^{(m)}_{i,n}\}$ converges monotonically from above to maximal solution $\{\bar{u}_{i,n},\bar{v}_{i,n}\}$ and the sequence $\{\underline{u}^{(m)}_{i,n},\underline{v}^{(m)}_{i,n}\}$ converges monotonically from below to minimal solution $\{\underline{u}_{i,n},\underline{v}_{i,n}\}$ of discrete Dirichlet IBVP (2.4)-(2.6). More over maximal solution $\{\bar{u}_{i,n},\bar{v}_{i,n}\}$ and minimal solution $\{\underline{u}_{i,n},\underline{v}_{i,n}\}$ satisfy the relation,

$$(\hat{u}_{i,n},\hat{v}_{i,n})\leq(\underline{u}_{i,n},\underline{v}_{i,n})\leq(\bar{u}_{i,n},\bar{v}_{i,n})\leq(\tilde{u}_{i,n},\tilde{v}_{i,n}) \text{ in } \bar{\Lambda}_p \quad (4.1)$$

Proof. From Lemma 3.1, we conclude that the sequence $\{\bar{u}^{(m)}_{i,n},\bar{v}^{(m)}_{i,n}\}$ is monotone nonincreasing and bounded from below hence it is convergent to some limit function. Also the sequence $\{\underline{u}^{(m)}_{i,n},\underline{v}^{(m)}_{i,n}\}$ is monotone nondecreasing and is bounded from above hence it is convergent to some limit function.So,

$$\lim_{m\to\infty} \bar{u}_{i,n}^{(m)} = \bar{u}_{i,n} \ ; \ \lim_{m\to\infty} \bar{v}_{i,n}^{(m)} = \bar{v}_{i,n}$$

and

$$\lim_{m\to\infty} \underline{u}_{i,n}^{(m)} = \underline{u}_{i,n} \ ; \ \lim_{m\to\infty} \underline{v}_{i,n}^{(m)} = \underline{v}_{i,n}$$

exist and called maximal and minimal solutions respectively of the discrete Dirichlet IBVP (2.4)-(2.6) and they satisfy the relation

$$(\hat{u}_{i,n}, \hat{v}_{i,n}) \leq (\underline{u}_{i,n}, \underline{v}_{i,n}) \leq (\bar{u}_{i,n}, \bar{v}_{i,n}) \leq (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \text{ in } \Lambda_p \quad (4.2)$$

This completes the proof.

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