



# A strong convergence theorem for $H(\cdot, \cdot) - \phi - \eta$ -accretive mapping using proximal point algorithms

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## Abstract

In this paper, we study an explicit iterative algorithm with resolvent technique using a more general  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator in uniformly convex Banach space. Using suitable conditions, we show that the corresponding iterative sequence converges strongly to a common point of two sets. It also becomes solution to the related variational inequality. The main result generalizes many such results.

## Keywords

$H(\cdot, \cdot) - \phi - \eta$ -Accretive operator, variational inequality, fixed point, weakly continuous duality mapping, contractive mapping, uniformly convex, resolvent, nonexpansive mapping.

## AMS Subject Classification

47H06, 47H09, 47H10, 47J25, 65J15.

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## 1. Introduction

The accretive operator exists in nonlinear models. In 1967, accretive operator was introduced first both by Browder [32] and Kato [33] independently. These nonlinear models which are evolution equations found in Heat, Wave and Schrodinger equations when modeled in the form of initial value problem,

$$s'(t) + As(t) = 0, \quad s(0) = s_0, \quad (1.1)$$

requires solution. The solution of the problem:

$$\text{Find } s \in X \text{ such that } As = 0 \quad (1.2)$$

are precisely the equilibrium points of the system (1.1) for an accretive operator  $A$  which is in the Banach space  $X$ . If  $s$  is not dependent on  $t$ , then  $\frac{ds}{dt} = 0$  and then (1.1) reduces

to (1.2) whose solution describes the equilibrium state or the stable state of the system described by (1.1). Considerable researches have been devoted to methods of solving (1.2) when  $A$  is accretive. A kind of accretive operator known as  $H$ -accretive operator was introduced by Fang and Huang [30] in 2004. He used resolvent operator technique (ROT) using  $H$ -accretive operator in order to solve variational inclusions in Banach spaces.

Next in 2007, Peng, Zhu and Zheng [31] introduced  $(H - \eta)$ -accretive operator defined in the Banach space. They proved the existence of solution and also its uniqueness. Also, a new iterative algorithm is proposed for a system of variational inclusions and the convergence of this with said operator in real smooth Banach space which is  $q$ -uniformly is proved.

Further, in 2008,  $H(\cdot, \cdot)$ -accretive operator was proposed by Zou and Huang [29] in Banach space. Using resolvent technique they proved the existence of the solution for the variational inclusions. They also showed the convergence of the iterative sequence which was generated by the algorithm for  $H(\cdot, \cdot)$ -accretive operator. Next in 2010, Wang and Ding [28] introduced  $(H(\cdot, \cdot), \eta)$ -accretive operators which is a new class of accretive operators. They studied a new class of set-valued variational inclusions containing  $(H(\cdot, \cdot), \eta)$ -accretive operators and constructed a new iterative algorithm for solving this class of set-valued variational inclusions. In 2015, the

notion of  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator was defined by Ahmad et al [26] in real uniformly smooth Banach space. Various iterative methods for the construction of zeros of accretive operators were studied by many authors (see, [1], [2], [3], [8], [9], [10],[11]). Among them, Rockafellar [11] is one who introduced one of the most powerful algorithm known as Rockafellar proximal point algorithm where, for any initial point  $x_0 \in X$ , a sequence  $\{x_n\}$  is being generated by

$$x_{n+1} = J_{s_n}(x_n + e_n), \forall n \geq 0.$$

where  $J_s = (I + sA)^{-1}$  for every  $s > 0$ , is called the resolvent operator of  $A$  and  $\{e_n\}$  is defined to be an error sequence defined in a Hilbert space  $X$ . However, prior to him, Bruck [3] in 1974, proposed the iterative algorithm  $x_{n+1} = J_{r_n}(u), \forall n \geq 0$  in a Hilbert space  $X$ , for any of the fixed point  $u \in X$ .

Later in 2006, Xu [16] and 2009, Song and Yang [14] respectively proved the strong convergence of a regularization method for Rockafellar proximal point algorithm as given below in a Hilbert space  $X$ , initially for any point  $z_0 \in X$ .

$$z_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)z_n + e_n), \text{ for all } n \geq 0, \quad (1.3)$$

where  $\{\alpha_n\} \subset (0, 1), \{e_n\} \subset E$  and  $\{r_n\} \subset (0, \infty)$ . On the other hand, in the same year 2009, Song [12] introduced a new iterative algorithm

$$y_{n+1} = \beta_n y_n + (1 - \beta_n)J_{r_n}(\alpha_n u + (1 - \alpha_n)y_n), \forall n \geq 0,$$

which finds zero of an accretive operator  $M$  defined in a reflexive Banach space  $X$  which is equipped by a uniformly Gâteaux differentiable norm. And for any initial point  $x_0 \in X$  each weakly compact convex subset of  $X$  satisfy the fixed point property for nonexpansive mappings. This result was extended in 2012 by Zhang and Song [17] in a uniformly convex Banach space  $X$  with a uniformly Gâteaux differentiable norm (or with a weakly sequentially continuous normalized duality mapping).

Further, in year 2013, Jung [5] extended the outcomes of Song [12], Zhang et al [17] to the viscosity algorithms along with imposing distinct conditions upon the parameter. In 2016, Jung [7], introduced the following algorithm in order to obtain a point which is common to both the set of zeros of accretive operator  $A$  and to the set of fixed points of non expansive mappings in a uniformly convex Banach space  $X$  having a uniformly Gâteaux differentiable norm:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1 - \alpha_n)S x_n), \forall n \geq 0, \quad (1.4)$$

where  $x_0 \in C$ , where  $C$  is a closed convex subset of  $X$ , and contractive mapping is  $f : C \rightarrow C$ , and  $\{\alpha_n\} \subset (0, 1); \{r_n\} \subset (0, \infty)$ . Although, on same year, Jung [20] extended his own earlier work and studied the iterative algorithm (1.6) having a weak contractive mapping and proved that the sequence which is originated by said algorithm has strong convergence

to a common point of  $A^{-1}0 \cap \text{Fix}(S)$  in a uniformly convex Banach space  $X$  which is also a solution of some variational inequality. This result improved and extended the corresponding results of ([5], [7], [12], [13] and [17]).

Later in 2006, Xu [16] and 2009, Song and Yang [14] respectively proved the strong convergence of a regularization method for Rockafellar proximal point algorithm as given below in a Hilbert space  $X$ , initially for any point  $z_0 \in X$ .

$$z_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)z_n + e_n), \text{ for all } n \geq 0, \quad (1.5)$$

where  $\{\alpha_n\} \subset (0, 1), \{e_n\} \subset E$  and  $\{r_n\} \subset (0, \infty)$ .

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which finds zero of an accretive operator  $M$  defined in a reflexive Banach space  $X$  which is equipped by a uniformly Gâteaux differentiable norm. And for any initial point  $x_0 \in X$  each weakly compact convex subset of  $X$  satisfy the fixed point property for nonexpansive mappings. This result was extended in 2012 by Zhang and Song [17] in a uniformly convex Banach space  $X$  with a uniformly Gâteaux differentiable norm (or with a weakly sequentially continuous normalized duality mapping).

Further, in year 2013, Jung [5] extended the outcomes of Song [12], Zhang et al [17] to the viscosity algorithms along with imposing distinct conditions upon the parameter. In 2016, Jung [7], introduced the following algorithm in order to obtain a point which is common to both the set of zeros of accretive operator  $A$  and to the set of fixed points of non expansive mappings in a uniformly convex Banach space  $X$  having a uniformly Gâteaux differentiable norm:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1 - \alpha_n)S x_n), \forall n \geq 0, \quad (1.6)$$

where  $x_0 \in C$ , where  $C$  is a closed convex subset of  $X$ , and contractive mapping is  $f : C \rightarrow C$ , and  $\{\alpha_n\} \subset (0, 1); \{r_n\} \subset (0, \infty)$ . Although, on same year, Jung [20] extended his own earlier work and studied the iterative algorithm (1.6) having a weak contractive mapping and proved that the sequence which is originated by said algorithm has strong convergence to a common point of  $A^{-1}0 \cap \text{Fix}(S)$  in a uniformly convex Banach space  $X$  which is also a solution of some variational inequality. This result improved and extended the corresponding results of ([5], [7], [12], [13] and [17]).

Now the object of this paper is to study the iteration algorithm (1.6) using  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator [26]. With the help of R.O.T, we prove that the sequence generated by said iterative algorithm converges strongly to a common point in  $\text{Fix}(\phi \circ M(\cdot, z)) \cap \text{Fix}(S)$ , for some fixed  $z \in X$  in a uniformly convex Banach space  $X$ . The importance of this operator lies with the fact that it also carries the single valued and Lipschitz continuity properties. In addition, we show that,



said common point is a solution of the variational inequality. Our main result generalizes, the results of Jung [5], [7], Song [12], Song et al [13], Zhang and Song [17], and Jung [20].

## 2. Preliminaries

The norm of a Banach space  $X$  is Known to be Gateaux differentiable only if

$$\lim_{u \rightarrow 0} \frac{\|a + ub\| - \|a\|}{u}$$

exists for each  $a, b$  belonging to unit sphere  $U_s = \{a \in X : \|a\| = 1\}$ . Such an  $X$  is known as a Smooth Banach space.  $X$  is called as uniformly convex if for every  $\epsilon \in [0, 2]$ , there is an existence of  $\delta_\epsilon > 0$  such that

$$\|u\| = \|v\| = 1 \text{ implies } \frac{\|u+v\|}{2} < 1 - \delta_\epsilon \text{ whenever } \|u-v\| \geq \epsilon$$

Let we have  $r > 1$  and  $\mathcal{N} > 0$  as fixed ,two real numbers. Then Banach space  $X$  is said to be uniformly convex iff there exists an strictly increasing continuous convex function  $g_1 : [0, \infty) \rightarrow [0, \infty)$  with  $g_1(0) = 0$  such that

$$\|\mu x + (1 - \mu)y\|^2 \leq \mu\|x\|^2 + (1 - \mu)\|y\|^2 - \mu(1 - \mu)g_1(\|x - y\|) \quad (2.1)$$

for every  $x, y \in B_{\mathcal{N}}(0) = \{z \in X : \|z\| \leq \mathcal{N}\}$ . For further detail, see Xu [15].

**Definition 2.1.** ([4], [26]) Let  $X$  be a real Banach space and  $X^d$  be its topological dual. For  $q > 1$ , a mapping  $\mathcal{J}_q : X \rightarrow 2^{X^d}$  is said to be a generalized duality mapping, which is defined by

$$\mathcal{J}_q(y) = \{f \in X^d : \langle y, f \rangle = \|y\|^q, \|f\| = \|y\|^{q-1}\}, \forall y \in X. \quad (2.2)$$

In particular,  $\mathcal{J}_2$  is the usual normalized duality mapping on  $X$ . It is well known (see e.g. [4]) that

$$\mathcal{J}_q(x) = \|x\|^{q-2} \mathcal{J}_2(x), \forall x (\neq 0) \in X. \quad (2.3)$$

Recall that for a real Hilbert space  $X$ ,  $\mathcal{J}_2$  is the identity mapping. A gauge function is a continuous strictly increasing function  $\varphi$  on  $\mathbb{R}_+ := [0, \infty)$  such that  $\varphi(0) = 0$  and  $\lim_{r \rightarrow \infty} \varphi(r) = \infty$ . Let the mapping  $\mathcal{J}_q : X \rightarrow 2^{X^*}$  then, from the equation (2.2) and equation (2.3) we can write

$$\begin{aligned} \mathcal{J}_q(x) &= \|x\|^{q-2} \{f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \\ &= \|f\| = \varphi(\|x\|)\}, \forall x \in X \end{aligned}$$

as the duality mapping with gauge function  $\varphi$ . Particularly, the duality mapping with gauge function  $\varphi(s) = s$  denoted by  $\mathcal{J}_q$  is known to be the normalized duality mapping. Also,  $\mathcal{J}(-y) = -\mathcal{J}(y), \forall y \in X$ . Again we know that  $X$  is smooth

iff the normalized duality mapping  $J$  is single-valued.

It is known that a Banach space  $X$  has a weakly continuous duality mapping if there exists a gauge function such that the duality mapping  $\mathcal{J}_q$  is single-valued and continuous from the weak topology to the  $weak^*$  topology, i.e. for  $\{x_n\} \in X$  with  $x_n \rightharpoonup x, \mathcal{J}_q(x_n) \xrightarrow{*} J_q(x)$ . For example, every  $l^p$  space ( $1 < p < \infty$ ) has a weakly continuous duality mapping with gauge function  $\varphi(t) = t^{p-1}$  ([18], [19]). Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \forall t \geq 0.$$

Then for  $0 < \rho < 1, \varphi(\rho y) \leq \varphi(y)$ ,

$$\Phi(\rho t) = \int_0^{\rho t} \varphi(\tau) d\tau = \rho \int_0^t \varphi(\rho y) dy \leq \rho \int_0^t \varphi(y) dy = \rho \Phi(t).$$

Moreover,

$$\mathcal{J}_q(y) = \partial \Phi(\|y\|), \forall y \in X,$$

where  $\partial$  denotes the subdifferential in the sense of convex analysis, i.e.,

$$\partial \Phi(\|x\|) = \{x^* \in X^* : \Phi(\|y\|) \geq \Phi(\|x\|) + \langle x^*, y - x \rangle\}, \forall y \in X.$$

We now recall some definitions as below:

**Definition 2.2.** ([27], [28]) Suppose we consider  $Y$  as a real Banach space and let  $A, B, \phi : Y \rightarrow Y$  and  $H, \eta : Y \times Y \rightarrow X$  be single-valued functions. A multi-valued mapping  $M : Y \times Y \rightarrow 2^Y$  is known as

- (i) accretive if  $\langle x - y, J_q(u_1 - u_2) \rangle \geq 0, \forall u_1, u_2 \in X, x \in M(u), y \in M(v)$  and  $q > 1$ ;
- (ii)  $\eta$  accretive if  $\langle x - y, J_q(\eta(u_1, u_2)) \rangle \geq 0, \forall u_1, u_2 \in X, x \in M(u), y \in M(v)$  and  $q > 1$ ;
- (iii)  $m$ -accretive if  $M$  is accretive and  $(I + \rho M(\cdot, z))(X) = X \forall \rho > 0$  and for some fixed  $z \in X$ , where  $I$  denotes the identity mapping on  $X$ .
- (iv) generalized  $m$ -accretive if  $M$  is  $\eta$  accretive and  $(I + \rho M(\cdot, z))(X) = X \forall \rho > 0$  and for some fixed  $z \in X$  where  $I$  denotes the identity mapping on  $X$ .

**Definition 2.3.** [26] Let  $X$  be a real Banach space and let  $A, B : X \rightarrow X$  and  $H, \eta : X \times X \rightarrow X$  be the single-valued mappings,  $z \in X$ , is a fixed point of  $X$  then,

(i)  $A$  is said to be  $\eta$ -accretive if,

$$\langle Ax_1 - Ax_2, J_q(\eta(x_1, x_2)) \rangle \geq 0, \forall x_1, x_2 \in X, q > 1;$$

(ii)  $A$  is said to be strictly  $\eta$ -accretive, if  $A$  is  $\eta$ -accretive and the equality holds if and only if  $x_1 = x_2$ ;

(iii)  $H(A, \cdot)$  is said to be  $\alpha$ -strongly  $\eta$ -accretive with respect to  $A$  if, there exists a constant  $\alpha > 0$  such that

$$\begin{aligned} \langle H(Ax_1, u) - H(Ax_2, u), J_q(\eta(x_1, x_2)) \rangle &\geq \alpha \|x_1 - x_2\|^q, \\ \forall x_1, x_2, u \in X, q > 1; \end{aligned}$$



(iv)  $H(\cdot, B)$  is said to be  $\beta$ -relaxed  $\eta$ -accretive with respect to  $B$  if, there exists a constant  $\beta > 0$  such that

$$\begin{aligned} \langle H(u, Bx_1) - H(u, Bx_2), J_q(\eta(x_1, x_2)) \rangle \\ \geq (-\beta) \|x_1 - x_2\|^q, \\ \forall x_1, x_2, u \in X, q > 1; \end{aligned}$$

(v)  $H(\cdot, \cdot)$  is said to be  $r_1$ -Lipschitz continuous with respect to  $A$  if, there exists a constant  $r_1 > 0$  such that

$$\|H(Ax_1, u) - H(Ax_2, u)\| \leq r_1 \|x_1 - x_2\|, \forall x_1, x_2, u \in X;$$

(vi)  $\eta$  is said to be  $\tau$ -Lipschitz if,

$$\|\eta(x_1, x_2)\| \leq \tau \|x_1 - x_2\|, \forall x_1, x_2 \in X.$$

**Definition 2.4.** [26] Let  $X$  be a real Banach space. Let  $P, Q, \phi : X \rightarrow X$  and  $H, \eta : X \times X \rightarrow X$  be the single-valued mappings. A multi-valued mapping  $U : X \times X \rightarrow 2^X$  is called an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator with respect to mapping  $P$  and  $Q$  if, for some fixed  $z \in X$ ,  $\phi \circ U(\cdot, z)$  is  $\eta$ -accretive in the equation (i) of Definition 2.3 and

$$(H(P, Q) + \rho \phi \circ U(\cdot, z))(X) = X \text{ for all } \rho > 0.$$

**Definition 2.5.** [26] Let  $X$  be a real Banach space, let  $A, B, \phi : X \rightarrow X$  and  $H, \eta : X \times X \rightarrow X$  be the single-valued mappings. Let  $M : X \times X \rightarrow 2^X$  be an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator with respect to mappings  $A$  and  $B$ . The resolvent operator  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}$  for some fixed  $z \in X$  is defined by

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(x) = (H(A, B) + \rho \phi \circ M(\cdot, z))^{-1}(x).$$

First, prove the following proposition for the purpose of proving our main result.

**Proposition 2.6.** Let  $q > 1$ . Let  $X$  be a real Banach space and let  $A, B, \phi : X \rightarrow X$  and  $H, \eta : X \times X \rightarrow X$  be the single-valued mappings. Let  $H(A, B)$  be  $\alpha$ -strongly  $\eta$ -accretive with respect to  $A$ ,  $\beta$ -relaxed,  $\eta$ -accretive with respect to  $B$ ,  $\alpha > \beta$  and  $\eta$  is  $\tau$ -Lipschitz. Let  $M : X \times X \rightarrow 2^X$  be an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator with respect to mappings  $A$  and  $B$ . Then we have the following:

(i) The resolvent operator  $R_{M(\cdot, z)}^{H(\cdot, \cdot) - \phi - \eta} : X \rightarrow X$  is  $\frac{\tau^{q-1}}{\alpha - \beta}$ -Lipschitz, i.e.,

$$\begin{aligned} \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u) - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v)\| \leq \frac{\tau^{q-1}}{\alpha - \beta} \|u - v\|, \\ \forall u, v \in X \text{ and every fixed } z \in X. \end{aligned}$$

(ii) If  $H(A, B)$  is  $\frac{\alpha - \beta}{\tau^{q-1}}$ -Lipschitz, then  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)$  is nonexpansive.

(iii)  $u \Leftrightarrow 0 \in \phi \circ M(\cdot, z)(u)$ .

*Proof.* (i) Let  $u, v$  be any points in  $X$ . It follows from the definition of resolvent operator that

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u) = (H(A, B) + \rho \phi \circ M(\cdot, z))^{-1}(u),$$

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v) = (H(A, B) + \rho \phi \circ M(\cdot, z))^{-1}(v).$$

This implies that

$$\begin{aligned} u - H(A(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u)), B(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u))) \\ \in \rho \phi \circ M(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u), z), \end{aligned}$$

and

$$\begin{aligned} v - H(A(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v)), B(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v))) \\ \in \rho \phi \circ M(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v), z). \end{aligned}$$

Set  $Pu = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u)$ , and  $Pv = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v)$ . It follows from Definition 2.1 (i) that  $\phi \circ M(\cdot, z)$  is  $\eta$ -accretive. Hence

$$\begin{aligned} \langle u - H(A(Pu), B(Pu)) \\ - (v - H(A(Pv), B(Pv))), J_q(\eta(Pu, Pv)) \rangle \\ \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \langle u - v, J_q(\eta(Pu, Pv)) \rangle \geq \langle (H(A(Pu), B(Pu)) \\ - H(A(Pv), B(Pv))), J_q(\eta(Pu, Pv)) \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|u - v\| \|J_q(\eta(Pu, Pv))\|^{(q-1)} \\ \geq \langle u - v, J_q(\eta(Pu, Pv)) \rangle \\ \geq \langle H(A(Pu), B(P(u))) - H(A(Pv), B(P(v))) \\ , J_q(\eta(Pu, Pv)) \rangle \\ = \langle H(A(Pu), B(P(u))) - H(A(Pv), B(P(u))) \\ , J_q(\eta(Pu, Pv)) \rangle \\ + \langle H(A(Pv), B(P(u))) - H(A(Pv), B(P(v))) \\ , J_q(\eta(Pu, Pv)) \rangle \\ \geq \alpha \|Pu - Pv\|^q - \beta \|Pu - Pv\|^q \\ = (\alpha - \beta) \|Pu - Pv\|^q. \end{aligned}$$

Thus using  $\eta$  is  $\tau$ -Lipschitz, we have

$$(\alpha - \beta) \|Pu - Pv\|^q \leq \|u - v\| (\tau \|Pu - Pv\|)^{(q-1)}$$

Thus,

$$\|(Pu - Pv)\| \leq \frac{\tau^{q-1}}{\alpha - \beta} \|u - v\|.$$

Therefore,

$$\|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(u) - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(v)\| \leq \frac{\tau^{q-1}}{\alpha - \beta} \|u - v\|.$$

(ii) It is obvious that,

$$\begin{aligned} \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(H(A, B)x) - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(H(A, B)y)\| \\ \leq \frac{\tau^{q-1}}{\alpha - \beta} \|H(A, B)x - H(A, B)y\| \\ \leq \|x - y\| \end{aligned}$$



(iii)

$$\begin{aligned} u &= R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(u) \\ \Leftrightarrow u &= (H(A, B) + \rho\phi \circ M(\cdot, z))^{-1} H(A, B)(u) \\ \Leftrightarrow H(A, B)(u) &\in (H(A, B) + \rho\phi \circ M(\cdot, z))(u) \\ \Leftrightarrow 0 &\in \phi \circ M(\cdot, z)(u). \end{aligned}$$

□

This complete the proof. We shall use the following lemmas in order to prove our main result.

**Lemma 2.7.** ([18], [19], [20]) *Let  $X$  be a real Banach space and let  $\phi$  be a continuous strictly increasing function on  $\mathbb{R}^+$  such that  $\phi(0) = 0$  and  $\lim_{r \rightarrow \infty} \phi(r) = \infty$ . Define*

$$\Phi(t) = \int_0^t \phi(\tau) d\tau, \forall t \in \mathbb{R}^+$$

Then the inequality holds:

$$\Phi(\|u + v\|) \leq \Phi(\|u\|) + \langle v, j_\phi(u + v) \rangle, \forall u, v \in X,$$

where  $j_\phi(u + v) \in \mathcal{J}_\phi(u + v)$ . In particular, if  $X$  is smooth, then one has

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, \mathcal{J}(u + v) \rangle, \forall u, v \in X.$$

**Lemma 2.8.** ([18], [19], [20]) *(Principle of Demiclosedness). Let  $R$  be a Banach space which is reflexive and have a weakly continuous duality mapping  $\mathcal{J}_\phi$  with  $\phi$  as a gauge function let  $C$  be a convex subset of  $R$  and let  $R$  be nonempty and closed, and let  $Q : C \rightarrow C$  be a non expansive mapping. In such case, the mapping  $I - Q$  is said to be demiclosed on  $C$ , where the identity mapping is denoted by  $I$ ; that is,  $x_n \rightarrow x$  in  $R$  also  $(I - Q)x_n \rightarrow y$  implies that  $x \in C$  and also  $(I - Q)x = y$ .*

**Lemma 2.9.** ([20], [21], [22]) *Let  $\{w_n\}$  be a sequence of nonnegative real numbers satisfying*

$$w_{n+1} \leq (1 - \delta_n)w_n + \delta_n\mu_n + \mu_n, \forall n \geq 0,$$

where  $\{\delta_n\}, \{\lambda_n\}$  and  $\{\mu_n\}$  satisfy the following conditions:

- (i)  $\{\delta_n\} \subset [0, 1]$  and  $\sum_{n=0}^\infty \delta_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \lambda_n \leq 0$  or  $\sum_{n=0}^\infty \delta_n |\lambda_n| < \infty$ ;
- (iii)  $\mu_n \geq 0 (n \geq 0), \sum_{n=0}^\infty \delta_n < \infty$ .

Then  $\lim_{n \rightarrow \infty} w_n = 0$ .

**Definition 2.10.** *A mapping  $g : Y \rightarrow Y$  is called as contractive on  $X$  if there exists  $m \in (0, 1)$  such that  $\|g(x) - g(y)\| \leq m\|x - y\|, \forall x, y \in X$ .*

We know that a mapping  $h : C \rightarrow C$  is called as weakly contractive [24] if

$$\|h(u) - h(v)\| \leq \|u - v\| - \psi(\|u - v\|), \forall u, v \in C.$$

where  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  is a continuous and strictly increasing function so that  $\psi$  is positive on  $(0, \infty)$  and  $\psi(0) = 0$ . Specially, if  $\psi(t) = (1 - m)t$  for  $t \in [0, +\infty)$ , where  $m \in (0, 1)$ , then the weakly contractive mapping  $h$  is a contraction with constant  $m$ .

Rhodes [23] (see also [24]) obtained the following result for the weakly contractive mapping.

**Lemma 2.11.** ([20] [23]) *Let  $(M, d)$  be a complete metric space and  $h$  be a weakly contractive mapping on  $M$ . Then  $h$  has a unique fixed point  $p$  in  $M$ .*

**Lemma 2.12.** ([25], [20]) *Let  $\{w_n\}$  and  $\{\mu_n\}$  be two sequences of nonnegative real numbers and let  $\{\delta_n\}$  be a sequence of positive numbers satisfying the conditions:*

- (i)  $\sum_{n=0}^\infty \delta_n = \infty$ ;
- (ii)  $\lim_{n \rightarrow \infty} \frac{\mu_n}{\delta_n} = 0$ .

Let the recursive inequality

$$w_{n+1} \leq w_n - \delta_n \psi(w_n) + \delta_n \mu_n, n \geq 0;$$

be given, where  $\psi(t)$  is a continuous and strict increasing function on  $[0, \infty)$  with  $\psi(0) = 0$ . Then  $\lim_{n \rightarrow \infty} w_n = 0$ .

**Lemma 2.13.** *Let  $X$  be a real Banach space and let  $A, B, \phi : X \rightarrow X$  and  $H, \eta : X \times X \rightarrow X$  be the single-valued mappings. Let  $M : X \times X \rightarrow 2^X$  be an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator with respect to mappings  $A$  and  $B$  if for some fixed  $z \in X$ , where  $\rho > 0, t > 0$ . Then we have*

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x = R_{M(\cdot, z), t}^{H(\cdot, \cdot) - \phi - \eta} \left( \frac{t}{\rho} x + \left(1 - \frac{t}{\rho}\right) R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x \right),$$

*Proof.* To prove the lemma, Let  $t > 0$  and  $\rho > 0$  and let  $x \in D_\rho = R(I + \rho M(\cdot, z))$ . Then there is  $(x_0, u_0) \in G(M(\cdot, z))$  (i.e.,  $u_0 \in M(\cdot, z)x_0$ ) such that  $x = x_0 + \rho u_0$ . It follows that

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x = x_0$$

and

$$\frac{t}{\rho} x + \left(1 - \frac{t}{\rho}\right) R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x = \frac{t}{\rho} (x_0 + \rho u_0) + \left(1 - \frac{t}{\rho}\right) x_0 = x_0 + t u_0.$$

Hence

$$\frac{t}{\rho} x + \left(1 - \frac{t}{\rho}\right) R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x \in R(I + t M(\cdot, z)) = D_t$$

and

$$\begin{aligned} R_{M(\cdot, z), t}^{H(\cdot, \cdot) - \phi - \eta} \left( \frac{t}{\rho} x + \left(1 - \frac{t}{\rho}\right) R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x \right) \\ &= R_{M(\cdot, z), t}^{H(\cdot, \cdot) - \phi - \eta} (x_0 + t u_0) \\ &= x_0 \\ &= R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x. \end{aligned}$$

□



Let  $X$  be a real Banach space with the norm  $\|\cdot\|$ , and let  $X^*$  be its dual. When  $\{x_n\}$  is a sequence in  $X$ , then  $x_n \rightarrow x, x_n \rightharpoonup x$  and  $x_n \overset{*}{\rightharpoonup} x$  will denote strong convergence, weak convergence and *weak\** convergence respectively of the sequence  $\{x_n\}$  to  $x$ .

If  $M$  is an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator which satisfies the range condition, then we can define, for each  $\rho > 0$  and for each fixed  $z \in X$ , a mapping  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} : R(I + \rho M(\cdot, z)) \rightarrow D(M(\cdot, z))$  defined by  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} = (H(A, B) + \rho \phi \circ M(\cdot, z))^{-1}$ , which is called the resolvent of  $M$ .

By the proposition (2.6), we know that  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)$  is nonexpansive and

$$Fix(\phi \circ M(\cdot, z))^{-1} 0 = Fix(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}) = \{x \in D(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}) :$$

$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x = x\}$  for all  $\rho > 0$  and for each fixed  $z \in X$ . Moreover, for  $\rho > 0, t > 0$  and  $x \in X$ , by lemma 2.13

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x = (R_{M(\cdot, z), t}^{H(\cdot, \cdot) - \phi - \eta}) \left( \frac{t}{\rho} x + \left(1 - \frac{t}{\rho}\right) R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} x \right), \tag{2.4}$$

for each fixed  $z \in X$ ,

which is known as the Resolvent Identity (see [18], [19]).

### 3. The Iterative algorithm

Here, let us have the iterative algorithm of Jung [20] for  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator. For this purpose let us recall the following:

Let  $X$  be a real Banach space, let  $C \neq \emptyset$  be closed convex subset of  $X$ , let  $M \subset X \times X$  be an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator in  $X$ , and for each fixed  $z \in X$  such that  $Fix(\phi \circ M(\cdot, z))^{-1} 0 \neq \emptyset$  and  $\overline{D(M)} \subset C \cap \cap_{\rho > 0} R(I + \rho M(\cdot, z))$ , and let  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}$  be the resolvent of  $M$  for each  $\rho > 0$ . Let  $S : C \rightarrow C$  be a nonexpansive mapping with  $Fix(S) \cap Fix(\phi \circ M(\cdot, z))^{-1} 0 \neq \emptyset$ , and let  $f : C \rightarrow C$  be a contractive mapping with a constant  $k \in (0, 1)$ . Then, following is the algorithm that generates a net  $\{x_t\}_{t \in (0, 1)}$  in an implicit way:

$$x_t = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} (t f x_t + (1 - t) S x_t). \tag{3.1}$$

In the theorem 3.2, we prove the strong convergence of  $\{x_t\}$  as  $t \rightarrow 0$  to a point  $d$  in  $Fix(\phi \circ M(\cdot, z))^{-1} 0 \cap Fix(S)$  which is a solution of the following variational inequality:

$$\langle (I - f)d, \mathcal{J}_q(d - p) \rangle \leq 0, \forall p \in Fix(\phi \circ M(\cdot, z))^{-1} 0 \cap Fix(S) \tag{3.2}$$

We also consider the algorithm that generates a sequence in an explicit way as given:

$$x_{n+1} = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (\alpha_n f x_n + (1 - \alpha_n) S x_n), \forall n \geq 0, \tag{3.3}$$

where  $\{\alpha_n\}, \{\beta_n\} \subset (0, 1), \{\rho_n\} \subset (0, \infty)$  and  $x_0 \in C$  is an initial guess, arbitrarily. We establish the strong convergence

of this sequence to a point  $d$  in  $Fix(\phi \circ M(\cdot, z))^{-1} 0 \cap Fix(S)$ , where  $d$  is also a solution of the variational inequality (3.2).

#### Strongly convergent algorithm

Here, for  $h \in (0, 1)$ , let we have a mapping  $Q_t : C \rightarrow C$  defined as

$$Q_t u = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} (h f u + (1 - h) S u), \forall u \in C.$$

Clearly, here we have that  $Q_h$  is contractive having a constant  $1 - (1 - k)h$ . Thus, we have

$$\begin{aligned} \|Q_h u - Q_h v\| &\leq h \|f u - f v\| + \|(1 - h) S u - (1 - h) S v\| \\ &\leq h \|u - v\| + (1 - h) \|u - v\| \\ &= (1 - (1 - k)h) \|u - v\|. \end{aligned}$$

Therefore  $Q_h$  has a fixed point which is unique, denoted by  $x_h$ , that solves the fixed point problem (3.1) uniquely.

Next, we prove the following proposition in the light of [20] and [7].

**Proposition 3.1.** *Let  $X$  be a real Banach space which is uniformly convex. Let a convex nonempty closed subset of  $X$  be denoted by  $C$ . And suppose  $M \subset X \times X$  be an  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator in  $X$  and for each fixed  $z \in X$  such that  $Fix(\phi \circ M(\cdot, z))^{-1} 0 \neq \emptyset$  and  $\overline{D(M)} \subset C \cap \cap_{\rho > 0} R(I + \rho M(\cdot, z))$ , and let  $(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta})$  be the resolvent operator of  $M$  for each  $\rho > 0$ . Let  $S : C \rightarrow C$  denotes a nonexpansive mapping having  $Fix(S) \cap Fix(\phi \circ M(\cdot, z))^{-1} 0 \neq \emptyset$  and let  $f : C \rightarrow C$  be a contractive mapping having a constant  $k \in (0, 1)$ . Let the net  $\{x_t\}$  be defined via (3.1), and let  $\{y_t\}$  also denotes a net that is defined as  $y_t = t f x_t + (1 - t) S x_t$  for  $t \in (0, 1)$ . Then we have the following:*

1.  $\{x_t\}$  and  $\{y_t\}$  are bounded for  $t \in (0, 1)$ ;
2.  $x_t$  defines a continuous path from  $(0, 1)$  in  $C$  and so does  $y_t$ ;
3.  $\lim_{t \rightarrow 0} \|y_t - S x_t\| = 0$ ;
4.  $\lim_{t \rightarrow 0} \|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} (y_t)\| = 0$ ;
5.  $\lim_{t \rightarrow 0} \|x_t - y_t\| = 0$ ;
6.  $\lim_{t \rightarrow 0} \|y_t - S y_t\| = 0$ .

*Proof.* 1. Let  $p \in Fix(S) \cap Fix(\phi \circ M(\cdot, z))^{-1} 0$ . Taking  $p = S p = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} p$ . By (3.1) and Proposition 2.6,



we have

$$\begin{aligned} & \|x_t - p\| \\ &= \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(H(A, B))(tfx_t + (1-t)Sx_t) \\ &\quad - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}H(A, B)(p)\| \\ &= \|Sy_t - Sp\| \\ &\leq \|y_t - p\| \\ &= \|t(fx_t - fp) + t(fp - p) + (1-t)(Sx_t - Sp)\| \\ &\leq tk\|x_t - p\| + t\|fp - p\| + (1-t)\|x_t - p\|. \end{aligned}$$

Thus,

$$\|x_t - p\| \leq \frac{\|fp - p\|}{1-k} \text{ and } \|y_t - p\| \leq \frac{\|fp - p\|}{1-k}.$$

Hence  $\{x_t\}$  and  $\{y_t\}$  are bounded and consequently  $\{fx_t\}$ ,  $\{Sx_t\}$ ,  $\{R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}x_t\}$ ,  $\{Sy_t\}$  and  $\{R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\}$  are bounded.

2. Let  $t, t_0 \in (0, 1)$ . Then,

$$\begin{aligned} & \|x_t - x_{t_0}\| \\ &= \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}(H(A, B))(tfx_t + (1-t)Sx_t) \\ &\quad - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}H(A, B)(t_0fx_{t_0} + (1-t_0)Sx_{t_0})\| \\ &\leq \|(t-t_0)fx_t + t_0(fx_t - fx_{t_0}) - (t-t_0)Sx_t \\ &\quad + (1-t_0)Sx_t - (1-t_0)R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}x_{t_0}\| \\ &\leq |t-t_0|\|fx_t\| + t_0k\|x_t - x_{t_0}\| + |t-t_0|\|Sx_t\| \\ &\quad + (1-t_0)\|x_t - x_{t_0}\|. \end{aligned}$$

It therefore follows that

$$\|x_t - x_{t_0}\| \leq \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|.$$

This means that  $x_t$  is locally Lipschitzian and thus it is continuous. Also we have

$$\|y_t - y_{t_0}\| \leq \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|,$$

and therefore,  $y_t$  becomes continuous path.

3. By the boundedness of  $\{fx_t\}$  and  $\{R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}x_t\}$  in condition (1) of this proposition, we have

$$\begin{aligned} \|y_t - Sx_t\| &= \|tfx_t + (1-t)Sx_t - Sx_t\| \\ &\leq t\|fx_t - Sx_t\| \rightarrow 0 \text{ when } t \rightarrow 0. \end{aligned}$$

4. Suppose  $p \in \text{Fix}(S) \cap \text{Fix}(\phi \circ M(\cdot, z))^{-1}0$ . Putting  $x = y_t$  and  $t = \frac{1}{2}$  in Resolvent Identity (2.4), we have

$$R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t = R_{M(\cdot, z), \frac{\rho}{2}}^{H(\cdot, \cdot) - \phi - \eta}\left(\frac{1}{2}y_t + \frac{1}{2}R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\right).$$

Then we have

$$\begin{aligned} & \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p\| \\ &= \|R_{M(\cdot, z), \frac{\rho}{2}}^{H(\cdot, \cdot) - \phi - \eta}\left(\frac{1}{2}y_t + \frac{1}{2}R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\right) - p\| \\ &\leq \left\|\frac{1}{2}(y_t - p) + \frac{1}{2}(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p)\right\|. \end{aligned}$$

By using the inequality (2.1) ( $q = 2, \lambda = \frac{1}{2}$ ) we get that

$$\begin{aligned} & \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p\|^2 \\ &= \|R_{M(\cdot, z), \frac{\rho}{2}}^{H(\cdot, \cdot) - \phi - \eta}\left(\frac{1}{2}y_t + \frac{1}{2}R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\right) - p\|^2 \\ &\leq \left\|\frac{1}{2}(y_t - p) + \frac{1}{2}(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p)\right\|^2 \\ &\leq \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) \\ &\leq \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|y_t - p\|^2 - \frac{1}{4}g(\|y_t \\ &\quad - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) \tag{3.4} \\ &= \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|). \end{aligned}$$

Thus, from equation(3.1), and the convexity of the real function  $\psi(t) = t^2 (t \in (-\infty, \infty))$  and from (3.4) we get,

$$\begin{aligned} \|x_t - p\|^2 &= \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t - p\|^2 \\ &\leq \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) \\ &= \|t(fx_t - p) + (1-t)(Sx_t - p)\|^2 \\ &\quad - \frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) \\ &\leq t\|(fx_t - p)\|^2 + (1-t)\|x_t - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|). \end{aligned}$$

Hence

$$\frac{1}{4}g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) \leq t(\|(fx_t - p)\|^2 - \|x_t - p\|^2).$$

By the property of boundedness of  $\{fx_t\}$  and  $\{x_t\}$ , letting  $t \rightarrow 0$  we get

$$\lim_{t \rightarrow 0} g(\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}y_t\|) = 0.$$



Thus, from the property of the function  $g$  in the inequality (2.1) it follows that

$$\lim_{t \rightarrow 0} (\|y_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_t\|) = 0.$$

5. Using the condition (4) we obtain

$$\begin{aligned} & \|x_t - y_t\| \\ & \leq \|x_t - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_t\| + \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_t - y_t\| \\ & = \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_t - y_t\| \rightarrow 0 (t \rightarrow 0). \end{aligned}$$

6. By the condition (3) and (5), we have,

$$\begin{aligned} \|y_t - S y_t\| & \leq \|y_t - S x_t\| + \|S x_t - S y_t\| \\ & \leq \|y_t - S x_t\| + \|x_t - y_t\| \rightarrow 0 (t \rightarrow 0). \end{aligned}$$

□

We have therefore established the strong convergence of  $\{x_t\}$  as  $t \rightarrow 0$ . It ensures that the solutions of the variational inequality (3.2) exists.

Next, we prove the following theorem:

**Theorem 3.2.** *Let  $X$  be a uniformly convex and real Banach space with a continuous weakly duality mapping  $\mathfrak{J}_q$  having gauge function  $\phi$ , suppose  $C$  is a nonempty set such that  $C$  is convex subset of  $X$  which is closed, let  $M \subset X \times X$  be a  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator in  $X$ , and for each fixed  $z \in X$  such that  $(\phi \circ M(\cdot, z))^{-1} \neq \emptyset$  and  $\overline{D(M)} \subset C \subset \cap_{\rho > 0} R(I + \rho M(\cdot, z))$ , and let  $R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta}$  is the resolvent operator of  $M$  for each  $\rho > 0$ . Suppose  $S : C \rightarrow C$  is a nonexpansive mapping having  $Fix(S) \cap Fix(\phi \circ M(\cdot, z))^{-1} \neq \emptyset$  and let  $f : C \rightarrow C$  be a contractive mapping with a constant  $k \in (0, 1)$ . Let  $\{x_t\}$  be a net defined by equation(3.1), and let  $\{y_t\}$  be a net defined as  $y_t = t f x_t + (1 - t) S x_t$  for  $t \in (0, 1)$ . Then the nets  $\{x_t\}$  and  $\{y_t\}$  converges strongly to point  $d$  of  $Fix(\phi \circ M(\cdot, z))^{-1} \cap Fix(S)$  as  $t \rightarrow 0$ , which provides solution to variational inequality(3.2).*

*Proof.* From the definition of the weak continuity of duality mapping  $\mathfrak{J}_q$  it implies that  $X$  is smooth. By using condition (1) in Proposition(3.1), we get that  $\{x_t\}$  and  $\{y_t\}$  are bounded sequence. Let  $t_n \rightarrow 0$ . Put  $x_n := x_{t_n}$  and  $y_n := y_{t_n}$ . The space  $X$  being reflexive, we can assume that  $y_n \rightarrow d$  for some  $d \in C$ . Since  $\mathfrak{J}_q$  is weakly continuous,  $\|y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \rightarrow 0$  and  $\|y_n - S y_n\| \rightarrow 0$  by condition (4) and (6) in Proposition(3.1), respectively. Thus by the lemma 2.8 we have,  $d = S d = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} d$ , and consequently  $d \in Fix(\phi \circ M(\cdot, z))^{-1} \cap Fix(S)$ .

Next, we show that  $\{x_t\}$  and  $\{y_t\}$  converge strongly to a point in  $Fix(\phi \circ M(\cdot, z))^{-1} \cap Fix(S)$  when, it is bounded for  $t \rightarrow 0$ .

Suppose  $\{t_n\}$  is a sequence defined in  $(0, 1)$  so that  $t_n \rightarrow 0$  and  $x_{t_n} \rightarrow d$  as  $n \rightarrow \infty$ . Also, using condition (5) of the Proposition (3.1),  $y_{t_n} \rightarrow d$  as  $n \rightarrow \infty$ . Then, as argued above, we have,  $d \in Fix(\phi \circ M(\cdot, z))^{-1} \cap Fix(S)$ .

We show that  $x_{t_n} \rightarrow d$ . Using the lemma 2.7, we have,

$$\begin{aligned} \Phi(\|x_{t_n} - d\|) & \leq \Phi(\|y_{t_n} - d\|) \\ & = \Phi(\|t_n(f x_{t_n} - f d) + (1 - t_n)(S x_{t_n} - d) \\ & \quad + t_n(f d - d)\|) \\ & \leq \Phi(t_n k \|x_{t_n} - d\| + (1 - t_n) \|x_{t_n} - d\|) \\ & \quad + t_n \langle f d - d, \mathfrak{J}_q(y_{t_n} - d) \rangle \\ & = \Phi(1 - t_n + t_n k) \|x_{t_n} - d\| + t_n \langle f d - d \\ & \quad , \mathfrak{J}_q(y_{t_n} - d) \rangle \\ & = \Phi(1 - (1 - k)t_n) \|x_{t_n} - d\| + t_n \langle f d - d \\ & \quad , \mathfrak{J}_q(y_{t_n} - d) \rangle \\ & \leq (1 - (1 - k)t_n) \Phi \|x_{t_n} - d\| + t_n \langle f d - d \\ & \quad , \mathfrak{J}_q(y_{t_n} - d) \rangle. \end{aligned}$$

This implies that

$$\Phi(\|x_{t_n} - d\|) \leq \frac{1}{1 - k} \langle f d - d, \mathfrak{J}_q(y_{t_n} - d) \rangle.$$

As,  $y_{t_n} \rightarrow d$  implies  $\mathfrak{J}_q(y_{t_n} - d) \rightarrow 0$ , we may infer from the last inequality that,

$$\Phi(\|x_{t_n} - d\|) \rightarrow 0.$$

Consequently,  $x_{t_n} \rightarrow d$  and  $y_{t_n} \rightarrow d$  by the condition (5) of Proposition(3.1).

Next we show that the entire net  $\{x_t\}$  and  $\{y_t\}$  converges strongly to  $d$ . For such purpose, let us consider the two sequences  $\{t_n\}$  and  $\{s_n\}$  belonging to  $(0, 1)$  such that

$$x_{t_n} \rightarrow d, y_{t_n} \rightarrow d \text{ and } x_{s_n} \rightarrow \bar{d}, y_{s_n} \rightarrow \bar{d}.$$

Thus, we need to prove that  $d = \bar{d}$ . Infact, for  $p \in Fix(\phi \circ M(\cdot, z))^{-1} \cap Fix(S)$ , we can see that

$$\begin{aligned} & \langle y_t - S x_t, \mathfrak{J}_q(x_t - p) \rangle \\ & = \langle y_t - x_t, \mathfrak{J}_q(x_t - p) \rangle + \langle x_t - p + p - S x_t \\ & \quad , \mathfrak{J}_q(x_t - p) \rangle \\ & \geq \langle y_t - x_t, \mathfrak{J}_q(x_t - p) \rangle + \Phi(\|x_t - p\|) \\ & \quad - \langle S x_t - p, \mathfrak{J}_q(x_t - p) \rangle \\ & \geq \langle y_t - x_t, \mathfrak{J}_q(x_t - p) \rangle + \Phi(\|x_t - p\|) \\ & \quad - \|x_t - p\| \|\mathfrak{J}_q(x_t - p)\| \\ & \geq \langle y_t - x_t, \mathfrak{J}_q(x_t - p) \rangle + \Phi(\|x_t - p\|) \\ & \quad - \Phi(\|x_t - p\|), \\ & = \langle y_t - x_t, \mathfrak{J}_q(x_t - p) \rangle. \end{aligned}$$

On the other side, since,

$$y_t - S x_t = -\frac{t}{1 - t} (y_t - f x_t).$$





we get for  $t \in (0, 1)$  and for  $p \in F(S) \cap \text{Fix}(\phi \circ M(\cdot, z))^{-1}0$ ,

$$\begin{aligned} \langle y_t - fx_t, \mathfrak{J}_q(x_t - p) \rangle &\leq \frac{1-t}{t} \langle x_t - y_t, \mathfrak{J}_q(x_t - p) \rangle \\ &\leq (1 - \frac{1}{t}) \|x_t - y_t\| \|\mathfrak{J}_q(x_t - p)\| \\ &\leq \|x_t - y_t\| \|\mathfrak{J}_q(x_t - p)\|. \end{aligned} \tag{3.5}$$

In particular, we obtain

$$\langle y_{t_n} - fx_{t_n}, \mathfrak{J}_q(x_{t_n} - p) \rangle \leq \|x_{t_n} - y_{t_n}\| \|\mathfrak{J}_q(x_{t_n} - p)\|.$$

and

$$\langle y_{s_n} - fx_{s_n}, \mathfrak{J}_q(x_{s_n} - p) \rangle \leq \|x_{s_n} - y_{s_n}\| \|\mathfrak{J}_q(x_{s_n} - p)\|.$$

Now, taking  $n \rightarrow \infty$  in above inequalities and using the condition (5) of Proposition(3.1), we have,

$$\langle d - fd, \mathfrak{J}_q(d - p) \rangle \leq 0, \text{ and } \langle \bar{d} - f\bar{d}, \mathfrak{J}_q(\bar{d} - p) \rangle \leq 0.$$

In particular, we have

$$\langle d - fd, \mathfrak{J}_q(d - \bar{d}) \rangle \leq 0, \text{ and } \langle \bar{d} - f\bar{d}, \mathfrak{J}_q(\bar{d} - d) \rangle \leq 0.$$

Summing the above inequalities give

$$\begin{aligned} \|d - \bar{d}\| \|\mathfrak{J}_q(d - \bar{d})\| &= \langle d - \bar{d}, \mathfrak{J}_q(d - \bar{d}) \rangle \\ &\leq \langle fd - f\bar{d}, \mathfrak{J}_q(d - \bar{d}) \rangle \\ &\leq k \|d - \bar{d}\| \|\mathfrak{J}_q(d - \bar{d})\|. \end{aligned}$$

This implies that  $(1 - k)\|d - \bar{d}\| \|\mathfrak{J}_q(d - \bar{d})\| \leq 0$ . Hence  $d = \bar{d}$  and  $\{x_t\}$  and  $\{y_t\}$  converges strongly to  $d$ .

Now, we prove that  $d$  is the solution of the variational inequality(3.2)which is unique. Since,  $x_t, y_t \rightarrow d$ , then using the condition (5) of Proposition (3.1) and  $fx_t \rightarrow fd$  as  $t \rightarrow 0$  and taking  $t \rightarrow 0$  in(3.5), we have,

$$\langle (I - f)d, \mathfrak{J}_q(d - p) \rangle \leq 0, \forall p \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S).$$

Thus it shows that point  $d$  gives the solution of the variational inequality 3.2. If  $\bar{d} \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S)$  is the other solution of the same variational inequality problem(3.2), thus

$$\langle (I - f)\bar{d}, \mathfrak{J}_q(\bar{d} - d) \rangle \leq 0. \tag{3.6}$$

Interchanging  $\bar{d}$  and  $d$ , we obtain

$$\langle (I - f)d, \mathfrak{J}_q(d - \bar{d}) \rangle \leq 0. \tag{3.7}$$

Summing the above two inequalities we get,

$$(1 - k)\|\bar{d} - d\| \|\mathfrak{J}_q(\bar{d} - d)\| \leq 0.$$

This means,  $d = \bar{d}$ . Hence  $d$  is the unique solution of the variational inequality problem(3.2). This completes the proof. Next, with help of Theorem 3.2, we establish the strong convergence of the sequence generated by the explicit algorithm (3.3) to the point  $d \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S)$ . It also gives a solution of the variational inequality(3.2). For this purpose, we use ROT.  $\square$

**Theorem 3.3.** Let  $X$  be a real uniformly convex Banach space. And let  $\mathcal{J}_q$  be a weakly continuous duality mapping with gauge function  $\phi$ . Let  $C \neq \emptyset$  be a closed convex subset of  $X$ . Let  $M \subset X \times X$  be an  $H - (\cdot, \cdot) - \phi - \eta$ -accretive mapping in  $X$  and for each fixed  $z \in X$  such that  $\text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \neq \emptyset$  and  $\bar{D}(M) \subset C \subset \cap_{\rho > 0} R(I + \rho M(\cdot, z))$ , and let  $R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta}$  be the resolvent of  $M$  for each  $\rho_n > 0$ . Let  $\rho > 0$  be any given positive number, and let  $S : C \rightarrow C$  be a nonexpansive mapping with  $\text{Fix}(S) \cap \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \neq \emptyset$ . Let  $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$  and  $\{\rho_n\} \subset (0, \infty)$  satisfy the conditions:

(A1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;

(A2)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;

(A3)  $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n, \sum_{n=0}^{\infty} \sigma_n < \infty$  (the perturbed control condition);

(A4)  $\lim_{n \rightarrow \infty} \rho_n = \rho$  and  $\rho_n \geq \varepsilon > 0$  for  $n \geq 0$  and  $\sum_{n=0}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ .

Suppose  $f : C \rightarrow C$  be any contractive mapping having a constant  $k \in (0, 1)$  and  $x_0 = x \in C$  be an arbitrary chosen point. Suppose  $\{x_n\}$  is a sequence obtained by

$$x_{n+1} = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} (\alpha_n f x_n + (1 - \alpha_n) S x_n), \forall n \geq 0. \tag{3.8}$$

and let  $\{y_n\}$  be a sequence defined by  $y_n = \alpha_n f x_n + (1 - \alpha_n) R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} x_n$ . Then  $\{x_n\}$  and  $\{y_n\}$  converge strongly to  $d \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S)$ , where  $d$  is the solution which is unique, of the variational inequality problem(3.2).

*Proof.* It is to be noted that using the Theorem 3.2, there exists a unique solution  $d$  of the variational inequality

$$\langle (I - f)d, \mathcal{J}_q(d - p) \rangle \leq 0, \forall p \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S),$$

where  $d = \lim_{t \rightarrow 0} x_t = \lim_{t \rightarrow 0} y_t$  with  $x_t$  and  $y_t$  being defined by

$x_t = R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} (t f x_t + (1 - t) S x_t)$  and  $y_t = t f x_t + (1 - t) S x_t$  for  $0 < t < 1$ , respectively. We shall prove the theorem using the following steps.

**Step I.** To prove  $\{x_n\}$  is bounded.

Let  $p \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S)$ . From  $\text{Fix}(\phi \circ M(\cdot, z))^{-1}0 = \text{Fix}(R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta})$  for each  $\rho > 0$ , we know  $p = S_p = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} p$ . Thus, we have

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n) - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(p)\| \\ &\leq \|y_n - p\| \\ &= \|\alpha_n (f x_n - p) + (1 - \alpha_n) (S x_n - S p)\| \\ &\leq \alpha_n \|f x_n - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \alpha_n (\|f x_n - f p\| + \|f p - p\|) + (1 - \alpha_n) \|x_n - p\| \\ &\leq \alpha_n k \|x_n - p\| + \alpha_n \|f p - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{1}{1 - k} \|f(p) - p\| \right\}. \end{aligned}$$



By using induction, we get

$$\begin{aligned} \|x_n - p\| &\leq \max \left\{ \|x_0 - p\|, \frac{1}{1-k} \|fp - p\| \right\} \text{ and} \\ \|y_n - p\| &\leq \max \left\{ \|x_0 - p\|, \frac{1}{1-k} \|fp - p\| \right\}, \forall n \geq 0. \end{aligned}$$

Hence, we have  $\{x_n\}$  is bounded, and so the sequence

$$\{y_n\}, \{R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} x_n\}, \{Sx_n\}, \{R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\}, \{Sy_n\}$$

and also  $\{f(x_n)\}$ .

Also, this follows from condition (A1) that

$$\|y_n - Sx_n\| = \alpha_n \|f(x_n) - Sx_n\| \rightarrow 0 (n \rightarrow \infty). \quad (3.9)$$

**Step II.**  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

Using the resolvent identity(2.4) we have,

$$\begin{aligned} &\|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - R_{M(\cdot, z), \rho_{n-1}}^{H(\cdot, \cdot) - \phi - \eta} y_{n-1}\| \\ &= \|R_{M(\cdot, z), \rho_{n-1}}^{H(\cdot, \cdot) - \phi - \eta} H(A, B) \left( \frac{\rho_{n-1}}{\rho_n} y_n \right. \end{aligned} \quad (3.10)$$

$$\left. + \left(1 - \frac{\rho_{n-1}}{\rho_n}\right) R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n \right) - R_{M(\cdot, z), \rho_{n-1}}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_{n-1})\| \quad (3.11)$$

$$\begin{aligned} &\leq \left\| \frac{\rho_{n-1}}{\rho_n} y_n + \left(1 - \frac{\rho_{n-1}}{\rho_n}\right) R_{M(\cdot, z), \rho_{n-1}}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_{n-1} \right\| \\ &\quad (3.12) \end{aligned}$$

$$\begin{aligned} &\leq \|(y_n - y_{n-1}) + \frac{\rho_{n-1}}{\rho_n} (y_n - y_{n-1}) + (y_n - y_{n-1}) \\ &+ [(1 - \frac{\rho_{n-1}}{\rho_n}) R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_{n-1} (1 - \frac{\rho_{n-1}}{\rho_n})]\| \\ &\leq \|y_n - y_{n-1}\| + \left|1 - \frac{\rho_{n-1}}{\rho_n}\right| \|y_n - y_{n-1}\| \quad (3.13) \end{aligned}$$

$$\begin{aligned} &+ \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right| N_1, \end{aligned}$$

where  $N_1 = \sup_{n \geq 0} \{ \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_{n-1}\| + \|y_n - y_{n-1}\| \}$ .

By (3.10), we have

$$\|x_{n+1} - x_n\| \quad (3.14)$$

$$\begin{aligned} &= \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n) \\ &- R_{M(\cdot, z), \rho_{n-1}}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_{n-1})\| \\ &\leq \|y_n - y_{n-1}\| + \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right| N_1 \\ &= \|(1 - \alpha_n)(Sx_n - Sx_{n-1}) + \alpha_n(fx_n - fx_{n-1}) \\ &+ (\alpha_n - \alpha_{n-1})(fx_{n-1} - Sx_{n-1})\| + \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right| N_1 \end{aligned} \quad (3.15)$$

$$\begin{aligned} &\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + k\alpha_n \|x_n - x_{n-1}\| \quad (3.16) \\ &+ |\alpha_n - \alpha_{n-1}| N_2 + \left|1 - \frac{\rho_{n-1}}{\rho_n}\right| N_1 \\ &\leq (1 - (1 - k)\alpha_n) \|x_n - x_{n-1}\| \\ &+ |\alpha_n - \alpha_{n-1}| N_2 + \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right| N_1, \end{aligned}$$

where  $N_2 = \sup\{\|f(x_n) - Sx_n\| : n \geq 0\}$ . Thus, using the condition (A3) we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (1 - k)\alpha_n) \|x_n - x_{n-1}\| \\ &+ N_2(\alpha_n + \sigma_{n-1}) + N_1 \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right|. \end{aligned}$$

Next, by taking  $s_{n+1} = \|x_{n+1} - x_n\|$ ,  $\lambda_n = (1 - k)\alpha_n$ ,  $\lambda_n \delta_n = N_2 \alpha_n$  and

$$\gamma_n = N_1 \left| \frac{\rho_n - \rho_{n-1}}{\varepsilon} \right| + N_2 \sigma_{n-1},$$

in the above equation, we have,

$$s_{n+1} \leq (1 - \lambda_n) s_n + \lambda_n \delta_n + \gamma_n.$$

Hence, by the conditions (A1), (A2), (A3), (A4) and the lemma 2.9, we can have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

**Step III.**

$$\lim_{n \rightarrow \infty} \|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\| = 0.$$

From the Resolvent Identity(2.4), we have

$$R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n = R_{M(\cdot, z), \frac{\rho_n}{2}}^{H(\cdot, \cdot) - \phi - \eta} \left( \frac{1}{2} y_n + \frac{1}{2} R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n \right).$$

Then we have

$$\begin{aligned} &\|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - p\| \\ &= \|R_{M(\cdot, z), \frac{\rho_n}{2}}^{H(\cdot, \cdot) - \phi - \eta} \left( \frac{1}{2} y_n + \frac{1}{2} R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - p \right)\| \\ &\leq \left\| \frac{1}{2} (y_n - p) + \frac{1}{2} (R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - p) \right\|. \end{aligned}$$



Further, by the inequality(2.1) ( $\lambda = \frac{1}{2}$ ), we have

$$\begin{aligned} & \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - p\|^2 \\ &= \|R_{M(\cdot, z), \frac{\rho_n}{2}}^{H(\cdot, \cdot) - \phi - \eta} (\frac{1}{2}y_n + \frac{1}{2}(R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n) - p)\|^2 \\ &\leq \|(\frac{1}{2}y_n + \frac{1}{2}R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n) - (\frac{p}{2} + \frac{p}{2}))\|^2 \\ &\leq \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n)\|). \tag{3.17} \\ &\leq \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|y_n - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n)\|). \\ &= \|y_n - p\|^2 - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n)\|). \end{aligned}$$

Thus, using the convex property of the real function

$$\psi(t) = t^2 (t \in (-\infty, \infty))$$

and by the inequality 3.17, we have for  $p \in \text{Fix}(\phi \circ M(\cdot, z))^{-1}0 \cap \text{Fix}(S)$ ,

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ &= \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n) - p\|^2 \\ &\leq \|y_n - p\|^2 - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n)\|) \\ &\leq \|\alpha_n f(x_n) + (1 - \alpha_n)Sx_n - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) \\ &\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|Sx_n - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) \\ &\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\ &\quad - \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) \end{aligned}$$

□

and hence

$$\begin{aligned} & \frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) - \alpha_n(\|f(x_n) - p\|^2 \\ & \quad - \|x_n - p\|^2) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \end{aligned}$$

Here, arises two cases:

Case 1. When  $\frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) \leq \alpha_n(\|f(x_n) - p\|^2 - \|x_n - p\|^2)$ . Then by the boundedness of  $\{f(x_n)\}$  and  $\{x_n\}$  and the condition (A1), we have,

$$\lim_{n \rightarrow \infty} g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) = 0.$$

Case 2. When  $\frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) > \alpha_n(\|f(x_n) - p\|^2 - \|x_n - p\|^2)$  Then, we obtain

$$\begin{aligned} & \sum_{n=0}^{\mathbb{N}} [\frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) - \alpha_n(\|f(x_n) - p\|^2 \\ & \quad - \|x_n - p\|^2)] \\ & \leq \|x_0 - p\|^2 - \|x_{\mathbb{N}} - p\|^2 \leq \|x_0 - p\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} [\frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) - \alpha_n(\|f(x_n) - p\|^2 \\ & \quad - \|x_n - p\|^2)] < \infty. \end{aligned}$$

So,

$$\begin{aligned} & \lim_{n \rightarrow \infty} [\frac{1}{4}g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) - \alpha_n(\|f(x_n) - p\|^2 \\ & \quad - \|x_n - p\|^2)] = 0. \end{aligned}$$

Thus by the condition (A1), we have

$$\lim_{n \rightarrow \infty} g(\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) = 0.$$

Consequently, using the property of the function g from (2.1), it follows that  $\lim_{n \rightarrow \infty} (\|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\|) = 0$ .

**Step IV.** Here, We prove that  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

From Step II and Step III, we have

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq \|x_n - x_{n+1}\| + \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_n\| \\ &\rightarrow 0, (n \rightarrow \infty). \end{aligned}$$

By the equation(3.9), we have,

$$\begin{aligned} \|y_n - S y_n\| &\leq \|y_n - S x_n\| + \|S x_n - S y_n\| \\ &\leq \|y_n - S x_n\| + \|x_n - y_n\| \rightarrow 0, (n \rightarrow \infty). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - S x_n\| &\leq \|x_n - y_n\| + \|y_n - S y_n\| + \|S y_n - S x_n\| \\ &\leq 2\|x_n - y_n\| + \|y_n - S y_n\| \rightarrow 0, (n \rightarrow \infty). \end{aligned}$$

Note

$$\begin{aligned} & \|x_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} x_n\| \\ & \leq \|x_n - y_n\| + \|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (y_n)\| \\ & \quad + \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n) \\ & \quad - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(x_n)\| \\ & \leq 2\|x_n - y_n\| + \|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ & \rightarrow 0, (n \rightarrow \infty). \end{aligned}$$



We now claim that  $\lim_{n \rightarrow \infty} \|y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| = 0$  for  $\rho = \lim_{n \rightarrow \infty} \rho_n$ .

Using (2.4), known as the resolvent identity and boundedness of  $\{R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\}$  we can show

$$\begin{aligned} & \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n) - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n)\| \\ &= \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B) \left( \frac{\rho}{\rho_n} y_n + \left(1 - \frac{\rho}{\rho_n}\right) R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n \right) \\ &\quad - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n)\| \\ &\leq \left\| \left( \frac{\rho}{\rho_n} y_n + \left(1 - \frac{\rho}{\rho_n}\right) R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - y_n \right) \right\| \quad (3.18) \\ &\leq \left| 1 - \frac{\rho}{\rho_n} \right| \|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Hence, by Step 3 and inequality(3.18), we have

$$\begin{aligned} & \|y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ &\leq \|y_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ &\quad + \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \|x_n - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} x_n\| \\ &\leq \|x_n - y_n\| + \|y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ &\quad + \|R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(y_n) \\ &\quad - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(x_n)\| \\ &\leq 2\|x_n - y_n\| + \|y_n - R_{M(\cdot, z), \rho}^{H(\cdot, \cdot) - \phi - \eta} y_n\| \\ &\rightarrow 0, \quad (n \rightarrow \infty). \text{ as } n \rightarrow \infty. \end{aligned}$$

**Step V.**  $\limsup_{n \rightarrow \infty} \langle (I - f)d, \mathcal{J}_q(d - y_n) \rangle \leq 0$ .

Suppose there be a subsequence  $\{y_{n_j}\}$  of  $\{y_n\}$  such that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle (I - f)d, \mathcal{J}_q(d - y_n) \rangle \\ &= \limsup_{j \rightarrow \infty} \langle (I - f)d, \mathcal{J}_q(d - y_{n_j}) \rangle \end{aligned}$$

and  $y_{n_j} \rightarrow z$  exists for some  $z \in X$ . Then, by using Step III, Step IV and the lemma 2.8, we have  $z \in \text{Fix}(\phi \circ M(\cdot, z))^{-1} 0 \cap \text{Fix}(S)$ .

Hence, from variational inequality(3.2) we have,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle (I - f)d, \mathcal{J}_q(d - y_n) \rangle \\ &= \limsup_{j \rightarrow \infty} \langle (I - f)d, \mathcal{J}_q(d - y_{n_j}) \rangle \\ &= \langle (I - f)d, \mathcal{J}_q(d - z) \rangle \leq 0. \end{aligned}$$

**Step VI.**  $\lim_{n \rightarrow \infty} \|x_n - d\| = 0$ .

Applying lemma 2.7, we obtain

$$\Phi(\|x_{n+1} - d\|) \quad (3.19)$$

$$\begin{aligned} & \leq \Phi(\|y_n - d\|) \\ &= \Phi(\|\alpha_n(fx_n - d) + (1 - \alpha_n)(Sx_n - d)\|) \\ &\leq \Phi(\alpha_n\langle fd - d, \mathcal{J}_q(y_n - d) \rangle + (1 - \alpha_n)\langle Sx_n - d, \mathcal{J}_q(y_n - d) \rangle) \quad (3.20) \end{aligned}$$

$$+ \alpha_n\langle fd - d, \mathcal{J}_q(y_n - d) \rangle \quad (3.21)$$

$$\leq \Phi(k\alpha_n\|x_n - d\| + (1 - \alpha_n)\|x_n - d\|) \quad (3.22)$$

$$+ \alpha_n\langle fd - d, \mathcal{J}_q(y_n - d) \rangle$$

$$\leq (1 - (1 - k)\alpha_n)\Phi(\|x_n - d\|) \quad (3.23)$$

$$+ \alpha_n\langle fd - d, \mathcal{J}_q(y_n - d) \rangle.$$

Putting value  $\lambda_n = (1 - k)\alpha_n$  and  $\delta_n = \frac{1}{1 - k} \langle (I - f)d, \mathcal{J}_q(d - y_n) \rangle$ . From the conditions (A1), (A2) and Step 8, it is clear that  $\lambda_n \rightarrow 0$ ,  $\sum_{n=0}^{\infty} \lambda_n = \infty$  and  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ . Hence, the inequality(3.19) reduces to

$$\Phi(\|x_{n+1} - d\|) \leq (1 - \lambda_n)\Phi(\|x_n - d\|) + \lambda_n \delta_n,$$

from the lemma 2.9 with  $\gamma_n = 0$  and so, we conclude that  $\lim_{n \rightarrow \infty} \Phi(\|x_n - d\|) = 0$ . Consequently,  $\lim_{n \rightarrow \infty} x_n = d$ . Using the Step 4, we can have  $\lim_{n \rightarrow \infty} y_n = d$ .

**Corollary 3.4.** Let  $X, C, M, R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta}, S, f$  and  $\rho > 0$ , for each fixed  $z \in X$  are as given in Theorem 3.3. Let  $\{\alpha_n\} \in (0, 1)$  and  $\{\rho_n\} \subset (0, \infty)$  which satisfies the conditions (A1) - (A4) of Theorem 3.3. Let  $x_0 = x \in C$  be chosen arbitrarily, and let  $\{x_n\}$  be an iterative sequence generated by

$$x_{n+1} = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (\alpha_n f x_n + (1 - \alpha_n) S x_n + e_n), \quad \forall n \geq 0,$$

where  $\{e_n\} \subset X$  satisfies  $\sum_{n=0}^{\infty} \|e_n\| < \infty$  or  $\lim_{n \rightarrow \infty} \frac{e_n}{\alpha_n} = 0$ , and let  $\{y_n\}$  be a sequence defined by  $y_n = \alpha_n f x_n + (1 - \alpha_n) S x_n + e_n$ . Then  $\{x_n\}$  and  $\{y_n\}$  converge strongly to  $d \in F(S) \cap \text{Fix}(\phi \circ M(\cdot, z))^{-1} 0$ , where  $d$  is the unique solution of the variational inequality(3.2).

*Proof.* Suppose  $z_{n+1} = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} (\alpha_n f z_n + (1 - \alpha_n) S z_n)$  for  $n \geq 0$ . So, now by using the Theorem 3.3,  $\{z_n\}$  converges to  $d$  strongly which belongs to  $\text{Fix}(\phi \circ M(\cdot, z))^{-1} 0 \cap \text{Fix}(S)$ . Here,  $d$  is the common element of unique solution of the variational inequality problem(3.2) also, and so we have,

$$\begin{aligned} & \|x_{n+1} - z_{n+1}\| \\ &\leq \|y_n - z_{n+1}\| \\ &\leq \|\alpha_n f x_n + (1 - \alpha_n) S x_n - (\alpha_n f z_n \\ &\quad + (1 - \alpha_n) S z_n) + e_n\| \\ &\leq \alpha_n \|f x_n - f z_n\| + (1 - \alpha_n) \|S x_n - S z_n\| + \|e_n\| \\ &\leq (1 - (1 - k)\alpha_n) \|x_n - z_n\| + \|e_n\|. \end{aligned}$$

By using Lemma 2.9, we have  $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ , and hence the desired result follows.  $\square$



Therefore, using [6], we shall prove the following theorem considering an iterative method with the weakly contractive mapping:

**Theorem 3.5.** Assume that  $X, C, M, R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta}, S, f$  and for each fixed  $z \in X, \rho > 0$  are as given in Theorem 3.3. Let  $\{\alpha_n\} \in (0, 1)$  and  $\{\rho_n\} \subset (0, \infty)$  satisfies the conditions (A1) - (A4) of the Theorem 3.3. Suppose  $g : C \rightarrow C$  be a mapping which is weakly contractive and having the function  $\psi$ . Suppose  $x_0 = x$  is an element of  $C$  arbitrarily, and assume  $\{x_n\}$  is an iterative sequence induced by

$$x_{n+1} = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta}(\alpha_n g x_n + (1 - \alpha_n) S x_n), \forall n \geq 0.$$

and  $\{y_n\}$  be another sequence defined by  $y_n = \alpha_n g x_n + (1 - \alpha_n) S x_n$ . Then  $\{x_n\}$  and  $\{y_n\}$  converges strongly to common element  $d \in F(S) \cap \text{Fix}(\phi \circ M(\cdot, z))^{-1} 0$ .

*Proof.* : Since the Banach Space  $X$  is smooth, so we have a sunny nonexpansive retraction denoted as  $\mathbb{Q}$  from  $C$  onto  $\text{Fix}(\phi \circ M(\cdot, z))^{-1} 0 \cap \text{Fix}(S)$ . Thus  $\mathbb{Q}g$  becomes weakly contractive from  $C$  into itself. Thus, for every  $u, v \in C$ ,

$$\|\mathbb{Q}gu - \mathbb{Q}gv\| \leq \|gu - gv\| \leq \|u - v\| - \psi(\|u - v\|).$$

Then Lemma 2.11 ensures the existence of  $x^* \in C$  which is unique and also an element of  $\text{Fix}(\phi \circ M(\cdot, z))^{-1} 0 \cap \text{Fix}(S)$  such that  $x^* = \mathbb{Q}g x^*$ .

Now, here an iterative sequence is defined as follows:

$$w_{n+1} = R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta}(\alpha_n g x^* + (1 - \alpha_n) S w_n) \quad n \geq 0. \quad (3.24)$$

Let  $\{w_n\}$  be an iterated sequence being generated by the equation (3.24). Thus from the Theorem 3.3 that has a constant  $f = g x^*$  ensures that  $\{w_n\}$  has strong convergence to  $\mathbb{Q}g x^* = x^*$  as  $n \rightarrow \infty$ . Now, for  $n > 0$ ,

$$\begin{aligned} & \|x_{n+1} - w_{n+1}\| \\ &= \|R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(\alpha_n g x_n + (1 - \alpha_n) S x_n) \\ &\quad - R_{M(\cdot, z), \rho_n}^{H(\cdot, \cdot) - \phi - \eta} H(A, B)(\alpha_n g x^* + (1 + \alpha_n) S w_n)\| \\ &\leq \alpha_n (\|g x_n - g x^*\|) + (1 - \alpha_n) \|x_n - w_n\| \\ &\leq \alpha_n (\|g x_n - g w_n\|) + \|g w_n - g x^*\| \\ &(1 - \alpha_n) \|x_n - w_n\| \\ &\leq \alpha_n (\|x_n - w_n\|) - \psi(\|x_n - w_n\|) \\ &\quad + \|w_n - x^*\| - \psi(\|w_n - x^*\|) (1 - \alpha_n) \|x_n - w_n\| \\ &\leq (\|x_n - w_n\|) - \alpha_n \psi(\|x_n - w_n\|) \\ &\quad + \alpha_n \|w_n - x^*\|. \end{aligned}$$

Thus, following inequality is obtained for  $s_n = \|x_n - w_n\|$  :

$$s_{n+1} \leq s_n - \alpha_n \psi(s_n) + \alpha_n \|w_n - x^*\|.$$

Since  $\lim_{n \rightarrow \infty} \|w_n - x^*\| = 0$ , then using the condition (A2) and lemma 2.12, it follows that  $\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0$ . Hence,

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| \leq \lim_{n \rightarrow \infty} (\|x_n - w_n\| + \|w_n - x^*\|) = 0.$$

By using the Step IV from the proof of the Theorem 3.3, we can also have  $\lim_{n \rightarrow \infty} y_n = d$ . This completes the proof.  $\square$

## 4. Conclusion

We therefore conclude to say that  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator are more general to establish the convergence of explicit iterative algorithm using the resolvent operator technique in uniformly convex Banach space. Also those could be the solution of certain variation inequality problem.

**Remark 4.1.** 1. We improve the result of Jung[5] for real uniformly convex Banach space using  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator by an iteration method of [7].

2. We improve the result of Jung[7] having real uniformly convex Banach space with weakly continuous duality mapping in place of reflective Banach space that has Gateaux differentiable norm using  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator.
3. We improve the result of Song [12] and utilize real uniformly convex Banach space with weakly continuous duality mapping in place of reflexive Banach space that has  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator for an iteration method of [7] also using contractive mapping with R.O.T.
4. We improve the result of Song[13] for real uniformly convex Banach space using  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator by an iteration method of [7].
5. We improve the result of Zang and song [17] for real uniformly convex Banach space using  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator by an iteration method of [7] involving R.O.T. and also using contractive mapping.
6. We extend the result of Jang [20], Theorem 3.2, Theorem 3.3 and Theorem 3.5 improve Theorem 3.2, Theorem 3.3 and Theorem 3.5 of Jang 2016 [20] by  $H(\cdot, \cdot) - \phi - \eta$ -accretive operator .

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