MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **09(03)**(2021), 148–167.
http://doi.org/10.26637/mjm0903/009

Stability and convergence of new random approximation algorithms for random contractive-type operators in separable Hilbert spaces

IMO KALU AGWU*[1] AND DONATUS IKECHI IGBOKWE[2]

[1,2] *Department of Mathematics, Micheal Okpara University of Agriculture, Umudike, Umuahia Abia State, Nigeria.*

Received 25 February 2021; Accepted 17 June 2021

Abstract. In this paper, new iterative schemes called Jungck-DI-Noor random iterative scheme and Jungck-DI-SP random iterative scheme are introduced and studied. Also, stability and convergence results were obtained without necessarily imposing sum conditions on the countably finite family of the control sequences and injectivity condition on the operators, which makes our schemes to be more desirable in applications than the ones studied in this paper and several others currently in literature.

AMS Subject Classifications: 47H09, 47H10, 47J05, 65J15.

Keywords: Strong convergence, Jungck-DI-Noor random iterative scheme, Jungck-DI-SP random iterative scheme, Stability, Contractive-type operator, fixed point, separable Hilbert space.

Contents

1. Introduction

Let (Y, ρ) be a complete metric space and $\Gamma : Y \longrightarrow Y$ a selfmap of Y. Suppose that $F_\Gamma = \{q \in Y : \Gamma q = q\}$ is the set of fixed points of Γ.

Over the years, different iterative schemes have been succesfully employed in approximating fixed points (or common fixed point) of different contractive operators in different spaces (see for example, [1], [4] , [12] and [16] -[44] and the references therein for more details). In 1971, Kirk [20] introduced the following iterative scheme:

Let X be a normed linear space and $\Gamma : X \longrightarrow X$ be a self-map on X. For arbitrarily chosen $y_0 \in X$, define the sequence $\{y_n\}_{n=0}^{\infty}$ iteratively as follows:

$$y_{n+1} = \sum_{j=0}^{\ell} \alpha_j \Gamma^j y_n, \sum_{j=0}^{\ell} \alpha_j = 1, n \geq 0. \tag{1.1}$$

*Corresponding author. Email address: **agwuimo@gmail.com** (Imo Kalu Agwu), **igbokwedi@yahoo.com** (Donatus Ikechi Igbokwe)

Since its emergence, different researchers have modified and generalised (1.1) in different spaces, see for example, [11],[15] and [29] and the reference therein.

In [29], Olatinwo introduced the iterative schemes below: Let Y be a Banach space and $\Gamma : Y \longrightarrow Y$ be a selfmap of Y.

(i) For an arbitrary point $y_0 \in Y$, $\alpha_{n,t} \geq 0, \alpha_{n,0} \neq 0, \alpha_{n,t} \in [0,1]$ *and* ℓ as a fixed integer, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$y_{n+1} = \sum_{t=0}^{\ell} \alpha_{n,t}\Gamma^t y_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1, n \geq 0 \tag{1.2}$$

(ii) For an arbitrary point $y_0 \in Y$, $\ell \geq m, \alpha_{n,t}, \beta_{n,t} \geq 0 \ with \ \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,t}, \beta_{n,t} \in [0,1] \ and \ \ell, m$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \alpha_{n,0}y_n + \sum_{t=0}^{\ell} \alpha_{n,t}\Gamma^j z_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1; \\ z_n &= \sum_{t=0}^{m} \beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{\ell} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.3}$$

and called them Kirk-Mann and Kirk-Ishikawa algorithms, respectively.

Chugh and Kumar [12] introduced and studied the iterative scheme below: Let Y be a Banach space and $\Gamma : Y \longrightarrow Y$ be a selfmap of Y. For an arbitrary point $y_0 \in Y$ and for $\ell \geq m \geq p, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \in [0,1] \ and \ \ell, m, p$ as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0}y_n + \sum_{r=1}^{\ell} \gamma_{n,r}\Gamma^r z_n, \sum_{r=0}^{\ell} \gamma_{n,r} = 1; \\ z_n &= \alpha_{n,0}y_n + \sum_{s=1}^{m} \alpha_{n,s}\Gamma^s z_n, \sum_{s=0}^{m} \alpha_{n,s} = 1; \\ z_n &= \sum_{t=0}^{p} \beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{p} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.4}$$

In 1976, Jungck[19] introduced and studied the iterative scheme below: Let Z be a Banach space, Y an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ such that $\Gamma(Y) \subseteq S(Y)$. For arbitrary $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ as follows

$$Sx_{n+1} = \Gamma x_n, n = 1, 2, \cdots \tag{1.5}$$

The iterative sequence defined by (1.5) is called Jungck iterative scheme and becomes Picard iterative scheme if $S = I_d$ (identity mapping) and $Y = Z$. It is worthy to note that (1.5) has been studied and generalised by different authors in different nonlinear spaces. Interested readers should see [2], [23], [24], [27] and [41] for more details.

In [12], the following iterative scheme was introduced and studied as a generalisation of (1.4): Let Z be a Banach space, Y an arbitrary set and $S, \Gamma : Y \longrightarrow Z$ a nonself operator such that $\Gamma(Y) \subseteq S(Y)$. For arbitrary

$y_0 \in Y$, define the sequence $\{Sy_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} Sy_{n+1} &= \gamma_{n,0}Sy_n + \sum_{r=1}^{\ell}\gamma_{n,r}\Gamma^r z_n, \sum_{r=0}^{\ell}\gamma_{n,r} = 1; \\ Sz_n &= \alpha_{n,0}Sy_n + \sum_{s=1}^{m}\alpha_{n,s}\Gamma^s z_n, \sum_{s=0}^{m}\alpha_{n,s} = 1; \\ Sz_n &= \beta_{n,0}Sy_n + \sum_{t=1}^{p}\beta_{n,t}\Gamma^t y_n, \sum_{t=0}^{p}\beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.6}$$

where $\ell \geq m \geq p, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \in [0,1]$ *and* ℓ, m, p as fixed integers.

Remark 1.1. *Notably,* (1.6) *reduces to* (1.4) *if* $S = I_d$ *(identity).*

Following the introduction of random fixed point theorems by Prague school of probability in 1950, considerable efforts have been devoted toward developing this theory. This unwavering interest stem from the priceless stance of fixed point theorems in probabilistic functional analysis and probabilistic model along with their diverse applications. It is worthwhile mentioning that problems relating to measurability of solutions, probabilistic and statistical aspect of random solutions found their way in the current literature due to the introduction of randomness. Also, it is of interest to note that random fixed point theorems are stochastic generalization of classical fixed point theorems and are usually needed in the theory of random equations, random matrices, random differential equations, and different classes of random operators emanating in physical systems (see, for example, [10] for details). In 1976, a paper by Bharucha-Reid [6], which provided sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for random operators, prompted various mathematicians to construct varying degree of fixed point iteration procedures for approximating fixed point of nonlinear random operators. In [14] and [42], Hans and Spacek initiated the idea of random fixed point theorems for contraction self mappings, Subsequently, Itoh [7] extended the result to multivalued random operators. In [43], using mappings that satisfied inward or the Leray Schauder condition, Xu [43] generalised the results in [7] to the case of nonself random operators. Further results in this direction could be found in [10] and the refrence therein

Definition 1.2. *Let* (Ω, Σ) *be a measurable space* $(\Omega -\ a\ set\ and\ \Sigma -\ sigma\ algebra)$, *$D$ a nonempty closed and convex subset of a real separable Banach space E and $\Gamma : \Omega \longrightarrow D$ a given mapping. Then,*

1. *Γ is said to be measurable if $\Gamma^{-1}(B \cap D) \in \Sigma$ for each Borel subset B of H;*
2. *$\Gamma : \Omega \times D \longrightarrow D$ is called random operator if $\Gamma(., \omega) : \Omega \longrightarrow D$ is measurable for every $\omega \in D$ and*
3. *Γ is siad to be continuous if for any given $\xi \in \Omega, \Gamma(\xi, .) : \Omega \times D \longrightarrow D$ is continuous.*

Definition 1.3. *Let* (Ω, Σ) *be a measurable space* $(\Omega -\ a\ set\ and\ \Sigma -\ sigma\ algebra)$, *$D$ a nonempty closed and convex subset of a real separable Banach space E and $\Gamma : \Omega \longrightarrow D$ a given mapping. A measurable function $g : \Omega \longrightarrow D$ is called a fixed point for the operator $\Gamma : \Omega \times D \longrightarrow D$ if $\Gamma(\xi, g(\xi)) = g(\xi)$ and it is referred to as a coincidence point for two random operators $S, \Gamma : \Omega \times D \longrightarrow D$ if $\Gamma(\xi, g(\xi)) = S(\xi, g(\xi)), \forall \xi \in \Omega$. The operators S, Γ are called random weakly compatible if they commute at the random coincidence point; i.e., if $\Gamma(\xi, g(\xi)) = S(\xi, g(\xi))$ for every $\xi \in \Omega$, then $\Gamma(S(\xi, g(\xi))) = S(\Gamma((\xi, g(\xi))))$ or $\Gamma(\xi, S(\xi, g(\xi))) = S(\xi, \Gamma(\xi, g(\xi)))$. The set of random common fixed points of the random mappings $S, \Gamma : \Omega \times D \longrightarrow D$ shall be denoted by $F(S, \Gamma) = \{g(\xi) \in D : S(\xi, g(\xi)) = \Gamma(\xi, g(\xi)) = g(\xi), \xi \in \Omega\}$.*

To approximate the fixed point of random mappings, different fixed point iterative schemes have been used by different authors (see, for example, [14], [40], [42], [43] and the reference therein).

Recently, Rashwan and Hammad [40] introduced the following random version of Jungck-Kirk-Noor iterative scheme defined in [24]: Let $\Gamma, S : \Omega \times Z \longrightarrow Y$ be two random mappings defined on a nonempty closed and convex subset D of a separable Banach space Y. Let $x_0 : \Omega \longrightarrow D$ be an arbitrary measurable mapping. For $\xi \in \Omega, n = 0, 1, 2, \cdots$, with $\Gamma(\xi, Z) \subseteq S(\xi, Z)$, then

$$\begin{cases} S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, z_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ S(\xi, z_n(\xi)) = \delta_{n,0}S(\xi, y_n(\xi)) + \sum_{j=1}^{\ell_2} \delta_{n,j}\Gamma^j(\xi, t_n(\xi)), \sum_{j=1}^{\ell_2} \delta_{n,j} = 1; \\ S(\xi, t_n(\xi)) = \sum_{k=0}^{\ell_3} \gamma_{n,k}\Gamma^k(\xi, y_n(\xi)), \sum_{k=1}^{\ell_3} \gamma_{n,k} = 1, \end{cases} \tag{1.7}$$

where $\ell_1, \ell_2 \; and \; \ell_3$ are fixed integers with $\ell_1 \geq \ell_2 \geq \ell_3, \alpha_{n,i} \geq 0, \alpha_{n,0} \neq 0, \delta_{n,j} \geq 0, \delta_{n,0} \neq 0$ and $\gamma_{n,k} \geq 0, \gamma_{n,0} \neq 0$ are measurable sequences in $[0, 1]$. They called (1.7) Jungck-Kirk-Noor random iterative scheme.

Remark 1.4. *If $\ell_3 = 0 \; and \; \ell_2 = \ell_3 = 0$ in* (1.7)*, then we have the following random iterative schemes:*

$$\begin{cases} S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, z_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1; \\ S(\xi, z_n(\xi)) = \delta_{n,0}S(\xi, y_n(\xi)) + \sum_{j=1}^{\ell_2} \delta_{n,j}\Gamma^j(\xi, y_n(\xi)) \end{cases} \tag{1.8}$$

and

$$S(\xi, y_{n+1}(\xi)) = \alpha_{n,0}S(\xi, y_n(\xi)) + \sum_{i=1}^{\ell_1} \alpha_{n,i}\Gamma^i(\xi, y_n(\xi)), \sum_{i=1}^{\ell_1} \alpha_{n,i} = 1, \tag{1.9}$$

respectively. (1.8) *and* (1.9) *are called Jungk-Kirk-Ishikawa and Jungck-Kirk-Man iterative schemes respectively.*

In real life applications, the workability of the various iterative schemes studied in this paper would be questionable if their stability is not guaranteed. In [32], Ostrowski initiated the notion of stability of iterative schemes and started investigation on this using Banach contractive conditions. Subsequently, different researchers have continued this investigation using more general contractive-type mappings than the one studied in [32]. Some recent works in this direction could be seen in [33], [34],[30],[28],[13],[32],[8],[31], [17],[11],[4] and the references therein.

Remark 1.5. *To obtain stability and convergence results in the papers studied using* (1.1)*,* (1.4)*,* (1.6)*,* (1.7)*,* (1.8)*,* (1.9) *and their variants required that the finite sum of the countably finite sequences of the measurable control parameters be unity (i.e., $\sum_{k=0}^{\ell} \gamma_{n,k} = 1, \sum_{i=0}^{m} \alpha_{n,i} = 1, \sum_{i=0}^{p} \delta_{n,j} = 1$, etc.). However, in real life applications, if $\ell, m \; and \; p$ are very large, it would be very difficult or almost impossible to generate a family of such measurable control parameters. Again, the computational cost of generating such a family of measurable control parameters (if possible) is quite enormous and also takes a very long process.*

In an attempt to overcome these challenges mentioned in Remark 1.3 for the case of a nonrandom operators, Agwu and Igbokwe introduced alternative iterative schemes in [1]. To the best of our knowledge, the problem of 'sum conditions' is still unresolved for the case of random iterative schemes. Consequently, the following question becomes necessary:

Question 1.1. *Is it possible to construct alternative random iterative schemes that would address the problems generated by the sum conditions $\Big(\sum_{k=0}^{\ell_3} \gamma_{n,k} = 1, \sum_{j=0}^{\ell_2} \delta_{n,j} = 1 \; and \; \sum_{i=0}^{\ell_1} \alpha_{n,i} = 1 \Big)$ imposed on the control parameters $\{\{\gamma_{n,k}\}_{n=1}^{\infty}\}_{k=1}^{\ell_3}, \{\{\alpha_{n,i}\}_{n=1}^{\infty}\}_{i=1}^{\ell_1}$ $and \; \{\{\delta_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{\ell_2}$, respectively while maintaining the convergence and stability results in [40]?*

Following the same argument as in [18] regarding the linear combination of the products of countably finite family of control parameters and the problems identified in each of the iterative schemes studied, the aim of this paper is to provide an affirmative answer to Question 1.1.

2. Preliminary

The following definitions, lemmas and propositions will be needed to prove our main results.

Definition 2.1. *(see [32]) Let (Y, d) be a metric space and let $\Gamma : Y \longrightarrow Y$ be a self-map of Y. Let $\{x_n\}_{n=0}^{\infty} \subseteq Y$ be a sequence generated by an iteration scheme*

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ is the initial approximation and g is some function. Suppeose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point q of Γ. Let $\{t_n\}_{n=0}^{\infty} \subseteq Y$ be an arbitrary sequence and set $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ Then, the iteration scheme (2.1) *is called Γ-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$.*

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained in the following manner: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Now, $x_1 = g(\Gamma, x_0)$ because of rounding in the function Γ, and a new value t_1 (approximately equal to x_1) might be calculated to give t_2, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

Definition 2.2. *(see, e.g., [40]) For two random operators $S, \Gamma : \Omega \times D \longrightarrow E$ with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and C is a nonempty closed and convex subset of a separable Banach space E, there exist real numbers $\eta \in [0, 1], \delta \in [0, 1)$ and a monotone increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi(0) = 0$ and $\forall x, y \in C$, we get*

$$\|\Gamma(\xi, x) - \Gamma(\xi, y)\| \leq \frac{\phi(\|S(\xi, x) - \Gamma(\xi, x)\|) + \delta\|S(\xi, x) - S(\xi, y)\|}{1 + \eta\|S(\xi, x) - \Gamma(\xi, x)\|} \tag{2.2}$$

Lemma 2.3. *Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\tau_n \to 0$ as $n \to \infty$. For $0 \leq \delta < 1$, let $\{w_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $w_{n+1} \leq \delta w_n + \tau_n, n = 0, 1, 2, \cdots$ Then, $w_n \to 0$ as $n \to \infty$.*

Lemma 2.4. *(see, e.g., [40]) Let $(E, \|, \|)$ be a normed linear space and S, Γ random commuting mappings on an arbitrary set D with values in E satisfying* (2.2) *such that $\forall x, y \in D, \xi \in \Omega$,*

$$\begin{cases} \Gamma(\xi, D) \subseteq S(\xi, D); \\ \|S(\xi, S(\xi, x)) - \Gamma(\xi, S(\xi, x))\| \leq \|S(\xi, x) - \Gamma(\xi, x)\| \\ \|S(\xi, S(\xi, x)) - S(\xi, S(\xi, x))\| \leq \|S(\xi, x) - S(\xi, y)\| \end{cases} \tag{2.3}$$

Consider $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, a sublinear monotone increasing function such that $\phi(0 = 0)$ and $\phi(u) = (1 - \delta)u, \forall \delta \in [0, 1), u \in \mathbb{R}^+$. Then, $\forall i \in \mathbb{N}$ and $\forall x, y \in D$, we get

$$\|\Gamma^i(\xi, x) - \Gamma^i(\xi, y)\| \leq \frac{\sum_{j=1}^{i} \binom{i}{j} \nu^{i-1} \phi^j(\|S(\xi, x) - \Gamma(\xi, x)\|) + \nu^i\|S(\xi, x) - S(\xi, y)\|}{1 + \eta^i\|S(\xi, x) - \Gamma(\xi, x)\|} \tag{2.4}$$

Proposition 2.5. *(see,e.g., [18]) Let $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and $\mathbb{N}$ is any integer with $k + 1 \leq N$. Then, the following holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) + \prod_{j=k}^{N} (1 - \alpha_j) = 1. \tag{2.5}$$

Proposition 2.6. *(see,e.g., [18]) Let t, u and v be arbitrary elements of a real Hilbert space H. Let k be any fixed nonnegetive integer and $N \in \mathbb{N}$ be such that $k + 1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0, 1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) v_{i-1} + \prod_{j=k}^{N} (1 - \alpha_j) v.$$

Then,

$$\|y-u\|^2 = \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2$$
$$-\alpha_k\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big]$$
$$-(1-\alpha_k)\Big[\sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2$$
$$+\alpha_N\prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big],$$

where $w_k = \sum_{i=k+1}^{N}\alpha_i\prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{i-1}(1-\alpha_j)v, k=1,2,\cdots,N$ *and* $w_n=(1-c_n)v.$

3. Main Results I

Let $\Gamma, S:\Omega\times D\longrightarrow H$ be two random mappings defined on a nonempty closed convex subset of a separable Hilbert space, H. Let $x_0:\Omega\longrightarrow C$ be an arbitrary measurable mapping. For $\xi\in\Omega, n=1,2,\cdots$, with $\Gamma(\xi,D)\subseteq S(xi,D)$, then

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,x_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B;\\ S(\xi,z_n(\xi)) = \delta_{n,1}S(\xi,x_n(\xi)) + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi,x_n(\xi)) + C, n\geq 0,1,2,.., \end{cases} \tag{3.1}$$

and

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,y_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,z_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B;\\ S(\xi,z_n(\xi)) = \delta_{n,1}S(\xi,x_n(\xi)) + \sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi,x_n(\xi)) + C, n\geq 0,1,2,.., \end{cases} \tag{3.2}$$

where $A = \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi,y_n(\xi)), B = \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Gamma^{\ell_2}(\xi,z_n(\xi)), C = \prod_{c=1}^{\ell_3}(1-\delta_{n,s})\Gamma^{\ell_3}(\xi,x_n(\xi)), \{\{\delta_{n,s}\}_{n=0}^{\infty}\}_{s=1}^{a}, \{\{\gamma_{n,t}\}_{n=0}^{\infty}\}_{t=1}^{b}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{c}$ are countable finite family of measurable real sequences in $[0,1]$ and $\ell_1,\ell_2,\ell_3\in\mathbb{N}$. We shall call the iterative schemes defined by (3.1) and (3.2) the Jungck-DI-Noor random iterative scheme and Jungck-DI-SP random iterative scheme, respectively.

Remark 3.1. $1(a)$ *If* $\ell_3=0$ *in* (3.1)*, we obtain the following remarkable iterative schemes:*

$$\begin{cases} S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A;\\ S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,x_n(\xi)) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + B, n\geq 0,1,2,.., \end{cases} \tag{3.3}$$

(b) *if* $\ell_2=\ell_3=0$ *in* (3.1)*, we have the following important algorithm:*

$$S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + A, \tag{3.4}$$

where A *and* B *are as defined above. The iterative schemes defined by* (3.3) *and* (3.4) *are called Jungck-DI-ishikawa and Jungck-DI-Mann random iterative schemes respectively.*

2. *If Ω is a singleton in* (3.1) *and* (3.2)*, we obtain the nonrandom version of* (3.1) *and* (3.2)*, respectively.*

3. *If S is an identity mapping in* (3.1) *and* (3.2)*, we get the following iterative algorithms:*

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}x_n(\xi) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B; \\ z_n(\xi) = \sum_{s=1}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi, x_n(\xi)) + C \\ , n \geq 0, 1, 2, .., \end{cases} \tag{3.5}$$

and

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}y_n(\xi) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}z_n(\xi) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B; \\ z_n(\xi) = \sum_{s=1}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi, x_n(\xi)) + C \\ , n \geq 0, 1, 2, .., \end{cases} \tag{3.6}$$

where $A, B, C, \{\{\delta_{n,s}\}_{n=0}^{\infty}\}_{s=1}^{a}, \{\{\gamma_{n,t}\}_{n=0}^{\infty}\}_{t=1}^{b}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{i=1}^{c}$ are and ℓ_1, ℓ_2, ℓ_3 are as defined in (3.1)*. We shall call the iterative schemes defined by* (3.5) *and* (3.6) *the the modified DI-Noor random iterative scheme and the modified DI-SP random iterative scheme, respectively.*

4(a). *If $\ell_3 = 0$ in* (3.5)*, we obtain the following remarkable iterative schemes:*

$$\begin{cases} x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A; \\ y_n(\xi) = \gamma_{n,1}x_n(\xi) + \sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi, z_n(\xi)) + B, n \geq 0, 1, 2, .., \end{cases} \tag{3.7}$$

(b) *if $\ell_2 = \ell_3 = 0$ in* (3.5)*, we have the following important algorithm:*

$$x_{n+1}(\xi) = \alpha_{n,1}x_n(\xi) + \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, y_n(\xi)) + A, \tag{3.8}$$

Theorem 3.2. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators S, Γ, S^i, Γ^i(i.e., $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.1) *converges to $q(\xi)$, then the random Jungck-DI-Noor iterative scheme is S, Γ-stable.*

Proof. Let $q(\xi) : \Omega \longrightarrow D$ be a measurable mapping and $z(\xi) : \Omega \longrightarrow D$ a random coincidence point of the random operators S, Γ, S^i, Γ^i(*i.e.*, $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). Let $\{S(\xi, t_n(\xi))\}_{n=0}^{\infty} \subset H$ and

$$\begin{aligned} \epsilon_n = {} & \|S(\xi, t_{n+1}(\xi)) - \alpha_{n,1}S(\xi, t_n(\xi)) - \sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi, g_n(\xi)) \\ & - \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Gamma^{\ell_1}(\xi, g_n(\xi))\|, \end{aligned} \tag{3.9}$$

where, for every $\xi \in \Omega$,

$$S(\xi, g_n(\xi)) = \gamma_{n,1} S(\xi, t_n(\xi)) + \sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1} (1 - \gamma_{n,b}) \Gamma^{t-1}(\xi, f_n(\xi)) + \prod_{b=1}^{\ell_2} (1 - \gamma_{n,b}) \Gamma^{\ell_2}(\xi, f_n(\xi)), \tag{3.10}$$

and

$$S(\xi, f_n(\xi)) = \delta_{n,1} S(\xi, t_n(\xi)) + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1} (1 - \delta_{n,c}) \Gamma^{s-1}(\xi, t_n(\xi)) + \prod_{c=1}^{\ell_3} (1 - \delta_{n,c}) \Gamma^{\ell_3}(\xi, t_n(\xi)). \tag{3.11}$$

Let $\epsilon_n \to 0 \; as \; n \to \infty$, then by lemma 2.2 and Proposition 2.4, with $S(\xi, t_n(\xi)) = t, \Gamma^{i-1}(\xi, g_n(\xi)) = v_{j-1}, \Gamma^{\ell_1}(\xi, g_n(\xi)) = v \; and \; k = 1$, we get the following estimates:

$$\begin{aligned}
\|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 &= \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi) - \Big[\alpha_{n,1} S(\xi, t_n(\xi)) \\
&\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) \\
&\quad - S(\xi, t_n(\xi))\Big]\|^2 \\
&\leq \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 + \| - \Big[\alpha_{n,1} S(\xi, t_n(\xi)) \\
&\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) \\
&\quad - S(\xi, t_n(\xi))\Big]\|^2 \\
&= \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 + \epsilon_n
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon_n \\
&\quad +\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&\quad +\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,g_n(\xi))-q(\xi)\|^2 \\
&\quad +\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\|\Gamma^{\ell_1}(\xi,g_n(\xi))-q(\xi)\|^2 \qquad (3.12)
\end{aligned}$$

But,

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq H, \qquad (3.13)$$

where

$$H=\frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{i-1}\phi^j(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^i\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|}$$

(3.13) implies

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq\frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{i-1}\phi^j(0)+\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^i\|0\|}$$

Since $\phi^i(0)=0$, it follows from the last inequality above that

$$\|\Gamma^{i-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\|\leq\nu^i\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\| \qquad (3.14)$$

(3.12) and (3.14)

$$\begin{aligned}
\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2 &\leq \epsilon_n \\
&\quad +\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&\quad +\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|^2 \\
&\quad +\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,z(\xi))-S(\xi,g_n(\xi))\|^2 \qquad (3.15)
\end{aligned}$$

Also, using (3.10) and Proposition 2.4, with $S(\xi,t_n(\xi))=t,\Gamma^{i-1}(\xi,f_n(\xi))=v_{j-1},\Gamma^{\ell_2}(\xi,f_n(\xi))=v\ and\ k=1$, we obtain the following estimaes:

$$\begin{aligned}
&\|S(\xi,g_n(\xi))-q(\xi)\| \\
&\qquad =\|\gamma_{n,1}S(\xi,t_n(\xi)) \\
&\qquad\quad +\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,f_n(\xi)) \\
&\qquad\quad +\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Gamma^{\ell_2}(\xi,f_n(\xi))-q(\xi)\|^2 \\
&\qquad \leq\|\gamma_{n,1}S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&\qquad\quad +\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{t-1}(\xi,f_n(\xi))-q(\xi)\|^2 \\
&\qquad\quad +\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\|\Gamma^{\ell_2}(\xi,f_n(\xi))-q(\xi)\|^2 \qquad (3.16)
\end{aligned}$$

Since $\phi(0)=0$, it follows from Lemma 2.2 that

$$\|\Gamma^{t-1}(\xi,g_n(\xi))-\Gamma^{t-1}(\xi,z(\xi))\|\leq H^{\star}, \tag{3.17}$$

where

$$H^{\star}=\frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|}$$

(3.17) implies

$$\begin{aligned}\|\Gamma^{t-1}(\xi,g_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^{j}(0)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t}\|0\|}\\ &=\nu^{t}\|S(\xi,z(\xi))-S(\xi,f_n(\xi))\| \end{aligned}\tag{3.18}$$

Again, using (3.11) and Proposition 2.4, with

$$S(\xi,t_n(\xi))=t,\Gamma^{i-1}(\xi,t_n(\xi))=v_{j-1},\Gamma^{\ell_2}(\xi,t_n(\xi))=v \text{ and } k=1,$$

we obtain the following estimaes:

$$\begin{aligned}\|S(\xi,f_n(\xi))-q(\xi)\|^2 &= \|\delta_{n,1}S(\xi,t_n(\xi))+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,t_n(\xi))\\ &\quad+\prod_{c=1}^{\ell_3}(1-\delta_{n,c})\Gamma^{\ell_3}(\xi,t_n(\xi))-q(\xi)\|^2\\ &\leq \delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\|\Gamma^{s-1}(\xi,t_n(\xi))-q(\xi)\|^2\\ &\quad+\prod_{c=1}^{\ell_3}(1-\delta_{n,c})\|\Gamma^{\ell_3}(\xi,t_n(\xi))-q(\xi)\|^2 \end{aligned}\tag{3.19}$$

Since $z(\xi)$ is the coincidence point of $S,\Gamma,\phi(0)=0$ and

$$\|\Gamma^{s-1}(\xi,g_n(\xi))-\Gamma^{s-1}(\xi,z(\xi))\|\leq W^{\star},$$

where

$$W^{\star}=\frac{\sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{s}\|S(\xi,z(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{s-1}(\xi,t_n(\xi))-\Gamma^{s-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{s}\binom{s}{j}\nu^{s-j}\phi^{j}(0)+\nu^{s}\|S(\xi,z(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s}\|0\|}\\ &=\nu^{s}\|S(\xi,z(\xi))-S(\xi,t_n(\xi))\|. \end{aligned}\tag{3.20}$$

Since (3.16) and (3.18) imply

$$\begin{aligned}\|S(\xi,g_n(\xi))-q(\xi)\| &\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\ &\quad+\Big(\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t})^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^{t})^2\Big)\\ &\quad\times\|S(\xi,z(\xi))-S(\xi,f_n(\xi))\|^2 \end{aligned}\tag{3.21}$$

and (3.19) and (3.20) imply

$$\|S(\xi, f_n(\xi)) - q(\xi)\|^2 \leq \Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big) \times \|S(\xi, z(\xi)) - S(\xi, t_n(\xi))\|^2,$$

we have (using (3.15)) that

$$\begin{aligned}\|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 \leq \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\}\\ &\times\|S(\xi, z(\xi)) - S(\xi, t_n(\xi))\|^2 + \epsilon_n \qquad (3.22)\end{aligned}$$

Let

$$\begin{aligned}\delta_n = \Big\{&\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\},\end{aligned}$$

so that from Proposition 2.3 and the fact that $\nu^i \in [0,1)$, we obtain

$$\begin{aligned}\delta_n &= \Big\{\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2 + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\quad\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2 + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\quad\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2 + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\Big]\Big\},\\ &< \Big\{\alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a}) + \prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Big)\\ &\quad\times\Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b}) + \prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Big)\\ &\quad\times\Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c}) + \prod_{c=1}^{\ell_3}(1-\delta_{n,c})\Big)\Big]\Big\} = 1 \qquad (3.23)\end{aligned}$$

Using Lemma 2.1, we obtain from (3.22) and (3.23) that $S(\xi, t_n(\xi)) \to q(\xi)\ as\ n \to \infty$.

Conversely, let $S(\xi, t_n(\xi)) \to 0\ as\ n \to \infty$. Then, we show that $\epsilon_n \to 0\ as\ n \to \infty$. Now, by using (3.9), (3.22), Proposition 2.4 and Lemma 2.2, we estimate as follows:

$$\begin{aligned}
\epsilon_n &= \|S(\xi, t_{n+1}(\xi)) - q(\xi) - \Big[\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\Big]\|^2 \\
&\leq \|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 + \|\alpha_{n,1} S(\xi, t_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \Gamma^{i-1}(\xi, g_n(\xi)) \\
&\quad + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 \\
&\leq \|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 + \alpha_{n,1} \|S(\xi, t_n(\xi)) - q(\xi)\|^2 \\
&\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) \|\Gamma^{i-1}(\xi, g_n(\xi)) - q(\xi)\|^2 + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) \|\Gamma^{\ell_1}(\xi, g_n(\xi)) - q(\xi)\|^2 \\
&\leq \|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 + \alpha_{n,1} \|S(\xi, t_n(\xi)) - q(\xi)\|^2 \\
&\quad + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) (\nu^i)^2 + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) (\nu^i)^2 \Big) \|S(\xi, g_n(\xi)) - q(\xi)\|^2 \\
&\leq \|S(\xi, t_{n+1}(\xi)) - q(\xi)\|^2 + \Bigg\{ \alpha_{n,1} + \Big(\sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1-\alpha_{n,a}) (\nu^i)^2 + \prod_{a=1}^{\ell_1} (1-\alpha_{n,a}) (\nu^i)^2 \Big) \\
&\quad \times \Big[\gamma_{n,1} + \Big(\sum_{t=2}^{\ell_2} \gamma_{n,t} \prod_{b=1}^{t-1} (1-\gamma_{n,b}) (\nu^t)^2 + \prod_{b=1}^{\ell_2} (1-\gamma_{n,b}) (\nu^t)^2 \Big) \\
&\quad \times \Big(\delta_{n,1} + \sum_{s=2}^{\ell_3} \delta_{n,s} \prod_{c=1}^{s-1} (1-\delta_{n,c}) (\nu^s)^2 + \prod_{c=1}^{\ell_3} (1-\delta_{n,c}) (\nu^s)^2 \Big) \Big] \Bigg\} \|S(\xi, z(\xi)) - S(\xi, t_n(\xi))\|^2
\end{aligned}$$

Observe that the right hand side of the last inequality tends to 0 as $n \to \infty$, hence $\epsilon_n \to 0$ as $n \to \infty$. The completes the proof.

If $\ell_3 = 0$ *and* $\ell_2 = \ell_3 = 0$, then Theorem 3.1 yields the following corollaries: ■

Corollary 3.3. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators S, Γ, S^i, Γ^i(i.e., $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.3) *converges to $q(\xi)$, then the random Jungck-DI-Ishikawa iterative scheme is S, Γ-stable.*

Corollary 3.4. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators S, Γ, S^i, Γ^i(i.e., $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi)$). For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.4) *converges to $q(\xi)$, then the random Jungck-DI-Mann iterative scheme is S, Γ-stable.*

If S is an identity in (3.1), (3.3) and (3.4), we obtain the following corollaries:

Corollary 3.5. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{x_n(\xi)\}_{n=0}^{\infty}$ generated by* (3.5) *converges to $q(\xi)$, then the random DI-Noor iterative scheme is S, Γ-stable.*

Corollary 3.6. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.7) *converges to $q(\xi)$, then the random DI-Ishikawa iterative scheme is S, Γ-stable.*

Corollary 3.7. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.8) *converges to $q(\xi)$, then the random DI-Mann iterative scheme is S, Γ-stable.*

Theorem 3.8. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.2) *converges to $q(\xi)$, then the random Jungck-DI-SP iterative scheme is S, Γ-stable.*

Proof. Using similar argument as in Theorem 3.1, the proof of Theorem 3.4 follows immediately. ■

Again, if S is an identity in (3.2), we obtain the following corollary from Theorem 3.7:

Corollary 3.9. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, if the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by* (3.6) *converges to $q(\xi)$, then the random DI-SP iterative scheme is S, Γ-stable.*

4. Main Result II

Theorem 4.1. *Let H be a real separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting operators for an arbitrary set D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random Jungck-DI-SP iterative scheme generated by* (3.2)*. Then,*

(i) *q is the unique common fixed point of Γ^{i-1} and $S^{i-1} (i = 2, 3, \cdots)$ if $D = H$ and Γ, S commute at q (i.e., Γ, S are weakly comprtible);*

(ii) *the Jungck-DI-SP iteration scheme converges strongly to $q(\xi) \in \Gamma(\xi)$.*

Proof. Assume that $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ has a pointwise limit ($i.e.$, $\lim_{n\to\infty} S(\xi, x_n(\xi)) = q(\xi), \forall \xi \in \Omega$). Since H is a separable Hilbert space, it follows that $S(\xi, g(\xi)) = q(\xi)$ is a measurable mapping for any random operator $S : \Omega \times K \longrightarrow K$ and any measurable mapping $g : \Omega \longrightarrow K$. Thus, the sequence $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ generated by the random Jungck-DI-SP iterative scheme (3.2) is a sequence of measurable mappings. Also, since K is convex and $q(\xi)$ is measurable, then $q : \omega \longrightarrow K$ (being limit of measurable mapping) is as well measurable.

Now, we show that $S, \Gamma, S^i\ and\ \Gamma^i$ have a unique coincidence point $z(\xi)$. Let $K(S, \Gamma, S^i, \Gamma^i)$ be the set of all coincidence points of $S, \Gamma, S^i\ and\ \Gamma^i$; and suppose there exists another coincidence point $q' \in K(S, \Gamma, S^i, \Gamma^i)\ with\ q' \neq q$. Then, we can find $z^\star(\xi) \neq z(\xi)$ such that $S(\xi, z^\star(\xi)) = \Gamma(\xi, z^\star(\xi)) = S^i(\xi, z^\star(\xi)) = S^i(\xi, z^\star(\xi)) = q'(\xi)$. Using (2.4) and the fact that $\phi(0) = 0$, we get

$$\|q(\xi) - q'(\xi)\| = \|\Gamma^{i-1}(\xi, z(\xi)) - \Gamma^{i-1}(\xi, z^\star(\xi))\| \leq Q^\star, \tag{4.1}$$

where

$$Q^\star = \frac{\sum_{j=1}^{i} \binom{s}{j} \nu^{i-j} \phi^j(\|S(\xi, z(\xi)) - \Gamma(\xi, z(\xi))\|) + \nu^i \|S(\xi, z(\xi)) - S(\xi, z^\star(\xi))\|}{1 + \eta^i \|S(\xi, z(\xi)) - \Gamma(\xi, z(\xi))\|}.$$

From (4.1), we obtain

$$\begin{aligned} \|q(\xi) - q'(\xi)\| &\leq \frac{\sum_{j=1}^{i} \binom{i}{j} \nu^{i-j} \phi^j(0) + \nu^i \|S(\xi, z_1(\xi)) - S(\xi, z_2(\xi))\|}{1 + \eta^i \|0\|} \\ &= \nu^s \|S(\xi, z(\xi)) - S(\xi, z^\star(\xi))\| = \nu^i \|q(\xi) - q'(\xi)\|., \end{aligned}$$

which yields $(1 - \nu^i)\|q(\xi) - q'(\xi)\| \leq 0$. Since $\nu^i \in [0, 1)$ and the norm is a nonnegative function, it follows that $q(\xi) = q'(\xi)$, which is a contradiction to our earlier assumption that $q(\xi) \neq q'(\xi)$. Hence, $S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = S^i(\xi, z(\xi)) = q(\xi)$. Therefore, $q(\xi)$ is unique. Further, since $\Gamma(\xi)\ and\ S(\xi)$ are weakly compartible, we have $\Gamma(\xi, S(\xi, z(\xi))) = S(\xi, \Gamma(\xi, z(\xi)))\ and\ \Gamma^i(\xi, S(\xi, z(\xi))) = S^i(\xi, \Gamma^i(\xi, z(\xi)))$. Hence, $\Gamma(\xi, q(\xi)) = S(\xi, q(\xi)) = \Gamma^i(\xi, q(\xi)) = S^i(\xi, q(\xi))$ so that $q(\xi)$ is the coimcidence point of $\Gamma, S, \Gamma^i\ and\ S^i$. Also, since the coincidence point is unique, we get $q(\xi) = z(\xi)$. Thus, $\Gamma(\xi, z(\xi)) = S(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = S^i(\xi, z(\xi)) = q(\xi)$.

Next, we show that $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ converges to $q(\xi)$. Using (3.2), lemma 2.2 and Proposition 2.4, with $S(\xi, y_n(\xi)) = t, \Gamma^{i-1}(\xi, y_n(\xi)) = v_{j-1}, \Gamma^{\ell_1}(\xi, y_n(\xi)) = v\ and\ k = 1$, we get the following estimates:

$$\begin{aligned} \|S(\xi, x_{n+1}(\xi)) - q(\xi)\|^2 &= \|\alpha_{n,1} S(\xi, y_n(\xi)) + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \Gamma^{i-1}(\xi, y_n(\xi)) \\ &\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \Gamma^{\ell_1}(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\leq \alpha_{n,1} \|S(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\quad + \sum_{i=2}^{\ell_1} \alpha_{n,i} \prod_{a=1}^{i-1} (1 - \alpha_{n,a}) \|\Gamma^{i-1}(\xi, y_n(\xi)) - q(\xi)\|^2 \\ &\quad + \prod_{a=1}^{\ell_1} (1 - \alpha_{n,a}) \|\Gamma^{\ell_1}(\xi, y_n(\xi)) - q(\xi)\|^2. \end{aligned} \tag{4.2}$$

Since $z(\xi)$ is the coincidence point of $S, \Gamma, \phi(0) = 0$ and

$$\|\Gamma^{i-1}(\xi, y_n(\xi)) - \Gamma^{i-1}(\xi, z(\xi))\| \leq P^\star,$$

where

$$P^{\star}=\frac{\sum_{j=1}^{s}\binom{i}{j}\nu^{i-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{i}\|S(\xi,z(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{i-1}(\xi,y_n(\xi))-\Gamma^{i-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{s-j}\phi^{j}(0)+\nu^{s}\|S(\xi,z(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i}\|0\|}\\ &=\nu^{i}\|S(\xi,z(\xi))-S(\xi,y_n(\xi))\|. \end{aligned} \tag{4.3}$$

(4.2) and (4.3) imply

$$\begin{aligned}\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2 \leq &\Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i})^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^{i})^2\Big)\\ &\times\|S(\xi,y_n(\xi))-q(\xi)\|^2. \end{aligned} \tag{4.4}$$

Again, from (3.2), lemma 2.2 and Proposition 2.4, with $S(\xi,z_n(\xi))=t, \Gamma^{i-1}(\xi,z_n(\xi))=v_{j-1}, \Gamma^{\ell_1}(\xi,z_n(\xi))=v \; and \; k=1$, we get the following estimates:

$$\begin{aligned}\|S(\xi,y_n(\xi))-q(\xi)\|^2 &= \|\gamma_{n,1}S(\xi,z_n(\xi))+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi))\\ &\quad+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Gamma^{\ell_2}(\xi,z_n(\xi))-q(\xi)\|^2\\ &\leq \gamma_{n,1}\|S(\xi,z_n(\xi))-q(\xi)\|^2\\ &\quad+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{i-1}(\xi,z_n(\xi))-q(\xi)\|^2\\ &\quad+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\|\Gamma^{\ell_2}(\xi,z_n(\xi))-q(\xi)\|^2. \end{aligned} \tag{4.5}$$

Since $z(\xi)$ is the coincidence point of $S,\Gamma,\phi(0)=0$ and

$$\|\Gamma^{t-1}(\xi,z_n(\xi))-\Gamma^{t-1}(\xi,z(\xi))\|\leq P^{\star\star},$$

where

$$P^{\star\star}=\frac{\sum_{j=1}^{t}\binom{t}{j}\nu^{t-j}\phi^{j}(\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t}\|S(\xi,z(\xi))-\Gamma(\xi,z(\xi))\|},$$

it follows that

$$\begin{aligned}\|\Gamma^{t-1}(\xi,z_n(\xi))-\Gamma^{t-1}(\xi,z(\xi))\| &\leq \frac{\sum_{j=1}^{i}\binom{i}{j}\nu^{t-j}\phi^{j}(0)+\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t}\|0\|}\\ &=\nu^{t}\|S(\xi,z(\xi))-S(\xi,z_n(\xi))\|. \end{aligned} \tag{4.6}$$

(4.5) and (4.6) imply that

$$\begin{aligned}\|S(\xi,y_n(\xi))-q(\xi)\|^2 \leq &\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t})^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^{t})^2\Big)\\ &\times\|S(\xi,z_n(\xi))-q(\xi)\|^2. \end{aligned} \tag{4.7}$$

Further, using (3.2) and similar argument as above, we obtain

$$\|S(\xi,z_n(\xi))-q(\xi)\|^2 \leq \Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big) \times\|S(\xi,x_n(\xi))-q(\xi)\|^2. \tag{4.8}$$

(4.4), (4.7) and (4.9) imply

$$\begin{aligned}\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2 \leq{}& \Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\\ &\times\|S(\xi,x_n(\xi))-q(\xi)\|^2.\end{aligned} \tag{4.9}$$

Let

$$\begin{aligned}\delta_n^\star ={}& \Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\end{aligned} \tag{4.10}$$

Since $\nu^i \in [0,1)$, w obtain from (4.10) and Proposition 2.3 that

$$\begin{aligned}\delta_n^\star ={}& \Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})(\nu^i)^2\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})(\nu^t)^2\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})(\nu^s)^2+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})(\nu^s)^2\Big)\\ <{}& \Big(\alpha_{n,1}+\sum_{i=2}^{\ell_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})+\prod_{a=1}^{\ell_1}(1-\alpha_{n,a})\Big)\\ &\times\Big(\gamma_{n,1}+\sum_{t=2}^{\ell_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})+\prod_{b=1}^{\ell_2}(1-\gamma_{n,b})\Big)\\ &\times\Big(\delta_{n,1}+\sum_{s=2}^{\ell_3}\delta_{n,s}\prod_{b=1}^{s-1}(1-\delta_{n,c})+\prod_{b=1}^{\ell_3}(1-\delta_{n,c})\Big)=1\end{aligned} \tag{4.11}$$

From (4.9), (4.11) and Lemma 2.1, we get that $S(\xi,x_n(\xi))\to q(\xi)\ as\ n\to\infty$. The proof is completed. ■

If s is an identity in (3.1), then the following corollary from Theorem 4.1:

Corollary 4.2. *Let H be a real separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting operators for an arbitrary set D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random DI-SP iterative scheme generated by* (3.6)*. Then,*

(i) *q is the unique common fixed point of Γ^{i-1} and $S^{i-1} (i = 2, 3, \cdots)$ if $D = H$ and Γ, S commute at q (i.e., Γ, S are weakly comprtible);*

(ii) *the DI-SP iteration scheme converges strongly to $q(\xi) \in \Gamma(\xi)$.*

Theorem 4.3. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random Jungck-DI-Noor iterative scheme generated by* (3.1)*. Then,*

(i) *$\Gamma(\xi)$ defined by* (2.4) *has a unique fixed point q;*

(ii) *the Jungck-DI-SP iteration scheme converges strongly to $q\xi \in \Gamma(\xi)$.*

Proof. Using similar argument as in Theorem 4.1, the proof of Theorem 4.2 follows immediately. ■

Also, if S is an identity in (3.1), we obtain the following corollary from Theorem 4.3:

Corollary 4.4. *Let H be a separable Hilbert space, $\Gamma, S : D \longrightarrow H$ random commuting uperators defined on D with $\Gamma(\xi, D) \subseteq S(\xi, D)$ and $S(\xi, D)$ a complete subspace of H satisfying* (2.4)*, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a sublinear monotone increasing function with $\phi(0) = 0$. Asumme that $z(\xi)$ is the random coincidence point of the random operators $S, \Gamma, S^i, \Gamma^i (i.e., S(\xi, z(\xi)) = \Gamma(\xi, z(\xi)) = S^i(\xi, z(\xi)) = \Gamma^i(\xi, z(\xi)) = q(\xi))$. For arbitrary $x_0(\xi) \in H$, let $\{S(\xi, x_n(\xi))\}_{n=0}^{\infty}$ be the random DI-Noor iterative scheme generated by* (3.5)*. Then,*

(i) *$\Gamma(\xi)$ defined by* (2.4) *has a unique fixed point q;*

(ii) *the DI-SP iteration scheme converges strongly to $q\xi \in \Gamma(\xi)$.*

Remark 4.5. *The following areas are still open:*

(i) to reconstruct, approximate the fixed points and the stability results of some existing random iterative schemes in the current literature, other than the ones under study, for finite family of certain class of contractive-type map;

(ii) to compare convergent rates of the iterative schemes defined by (3.1) *and* (3.2) *with those of* (1.7)*;*

(iii) to prove Proposition [2.3 and 2.4] in more general spaces so as to extend the results in this paper to such spaces.

5. Conclusion

An affirmative answer has been provided for Question 1.1. The results obtained in this paper improve the corresponding results in [10], [19], [40] and several others currrently announced in literature.

References

[1] I. K. AGWU AND D. I. IGBOKWE, *New iteration algorithm for equilibrium problems and fixed point problems of two finite families of asymptotically demicontractive multivalued mappings* (in press.)

[2] H. AKEWE AND A. MOGBADEMU, Common fixed point of Jungck-Kirk-type iteration for nonself operators in normed linear spaces, *Fasciculi Mathemathici*, **2016**(2016), 29–41.

[3] H. AKEWE AND H. OLAOLUWA, On the convergence of modified iteration process for generalise contractive-like operators, *Bull. Math. Anal. Appl.*, **4(3)**(2012), 78–86.

[4] H. AKEWE, G. A. OKEEKE AND A. OLAYIWOLA, Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators. *Fixed Point Theory Appl.*, **2014**(2014), 45.

[5] H. AKEWE, *Approximation of fixed and common fixed points of generalised contractive-like operators*, PhD Thesis, University of Lagos, Nigeria, (2010).

[6] A.T. BHARUCHA-REID, Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.*, **82**(1976), 641–657.

[7] S. ITOH, Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.,* **67**(1979), 261–273.

[8] V. BERINDE, On the stability of some fixed point problems. *Bull. Stint. Univ. Bala Mare, Ser. B Fasc. Mat-inform. XVIII(1)*, **14**(2002), 7–14.

[9] V. BERINDE, Iterative approximation of fixed points. *Editorial Efemerede, Bala Mare.* (2012).

[10] A. D. BOSEDE, H. AKEWE, A. S. WUSU AND O. F. BAKRE, Random hybrid iterative algorithms of Jungck-type and common random fixed point theorems with stability results, *Int'l J. of Research and Innovation in AppL. Sci.,IV(IX)*, (2019), 2454–6494.

[11] R. CHUGH AND V. KUMMAR, Stability of hybrid fixed point iterative algorithm of Kirk-Noor-type in nonlinear spaces for self and nonself operators, *Intl. J. Contemp. Math. Sci.*, **7(24)**(2012), 1165–1184.

[12] R. CHUGH AND V. KUMMAR, Strong convergence of SP iterative scheme for quasi-contractive operators. *Intl. J. Comput. Appl.*, **31(5)**(2011), 21–27.

[13] A. M. HARDER AND T. L. HICKS, Stability results for fixed point iterative procedures. *Math. Jpn*, **33(5)**(1988), 693–706.

[14] O. HANS, Reduzierende zufallige transformationen, *Czechoslov. Math. J.,* **7**(1957), 154–158.

[15] N. HUSSAIN, R. CHUGH, V. KUMMAR AND A. RAFIG, On the convergence of Kirk-type iterative schemes. *J. Appl. Math.*, 2012 (2012), Article ID 526503, 22 pages.

[16] S. ISHIKAWA, Fixed points by a new iteration methods. *Proc. Am. Math. Soc.*, **44**(1974), 147–150.

[17] C.O. IMORU AND M.O. OLATINWO, On the stability of Picard's and Mann's iteration. *Carpath. J. Math.*, **19**(2003), 155–160.

[18] F. O. ISIOGUGU, C. IZUCHUKWU AND C. C. OKEKE, New iteration scheme for approximating a common fixed point of a finite family of mappings. *Hindawi J. Math.*, **2020**(2020), Article ID 3287968.

[19] G. JUNGCK, Commuting mappings and fixed points, *Amer. Math. Monthly,* **83(4)**(1976), 261–263.

[20] W. A. KICK, On successive approximations for nonexpansive mappings in Banach spaces, *Glasg. Math. J.*, **12**(1971), 6–9.

[21] W. R. MANN, Mean value method in iteration, *Proc. Am. Math. Soc.*, **44**(2000), 506–510.

[22] M. A. NOOR, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251**(2000), 217–229.

[23] M. O. OLATINWO, A generalization of some convergence results using a Jungck-Noor three-step iteration process in arbitrary Banach space, *Fasciculi Mathemathici*, **40**(2008), 37–43.

[24] J. O. OLALERU AND H. AKEWE, On the convergence of Jungck-type iterative schemes for generalized contractive-like operators, *Fasciculi Mathemathici*, **45**(2010), 87–98.

[25] M. O. OLATINWO, Stability results for Jungck-kirk-Mann and Jungck-kirk hybrid iterative algorithms, *Anal. Theory Appl.*, **29**(2013), 12–20.

[26] M. O. OLATINWO, Convergence results for Jungck-type iterative process in convex metric spaces,, *Acta Univ. Palacki Olomue, Fac. Rer. Nat. Math.,* **51**(2012), 79–87.

[27] M. O. OLATINWO, Some stability and strong convergence results for the Jungck-Ishikawa iteration process, *Creative Math. and Inf.*, **17**(2008), 33–42.

[28] M. O. OLAERU, H. Akewe, An extension of Gregus fixed point theorem, *Fixed Point Theory Appl.*, **2007**(2007), Article ID 78628.

[29] M. O. OLUTINWO, Some stability results for two hybrid fixed point iterative algorithms in normed linear space. *Mat. vesn*, **61(4)**(2009), 247–256.

[30] M. O. OSILIKE AND A. UDOEMENE, A short proof of stability resultsfor fixed point iteration procedures for a class of contractive-type mappings, *Indian J. Pure Appl. Math.*, **30**(1999), 1229–1234.

[31] M. O. OSILIKE, Stability results for lshikawa fixed point iteration procedure, *Indian J. Pure Appl. Math.*, **26(10)**(1996), 937–941.

[32] A. M. OSTROWSKI, The round off stability of iterations, *Z. Angew Math. Mech.*, **47**(1967), 77–81.

[33] B. E. RHOADE, Fixed point theorems and stability results for fixed point iteration procedures. *Indian J. Pure Appl. Math.*, **24(11)**(1993), 691–03.

[34] B. E. RHOADE, Fixed point theorems and stability results for fixed point iteration procedures. *Indian J. Pure Appl. Math.*, **21**(1990), 1–9.

[35] A. RATIQ, A convergence theprem for Mann's iteration procedure, *Appl. Math. E-Note*, **6**(2006), 289–293.

[36] B. E. RHOADE, A comparison of various definitions of contractive mappings, *Trans. Am. Math. Soc.*, **266**(1977), 257–290.

[37] B. E. RHOADE, Comments on two fixed point iteration methods, *Trans. Am. Math. Soc.*, **56**(1976), 741–750.

[38] B. E. RHOADE, Fixed point iteration using infinite matrices, *Trans. Am. Math. Soc.*, **196**(1974), 161–176.

[39] A. RATIQ, On the convergence of the three step iteration process in the class of quasi-contractive operators, *Acta. Math. Acad. Paedagag Nayhazi*, **22**(2006), 300–309.

[40] R. A. RASHWAN AND H. A. HAMMAD, Stability and strong convergence results for random Jungck-Kirk-Noor iterative scheme, *Fasciculi Mathemathici,* (2017), 167–182.

[41] S. L. SINGH, C. BHATNAGAR AND S.N. MISHRA, Stability of Jungck-type iteration procedures, *Int. J. Math. Math. Sci.,* **19**(2005), 3035–3043.

[42] A. SPACEK, Zufallige Gleichungen, Czechoslovak, *Math. J.*, **5**(1955), 462–466.

[43] H. K. XU, Some random fixed point theorems for condensing and nonexpansive operators, *Proc. Ame. Math. Soc.,* **110(2)**(1990), 395–400.

[44] T. ZAMFIRESCU, Fixed point theorems in metric spaces, *Arch. Math.*, **23**(1972), 292–298.