Malaya Journal of Matematik, Vol. 7, No. 2, 213-222, 2019
https://doi.org/10.26637/MJM0702/0012

Analysing the fractional heat diffusion equation solution in comparison with the new fractional derivative by decomposition method

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Abstract
With the current raised issues on the new conformable fractional derivative having satisfied the Leibniz rule for derivatives which was proved not to be so for a differential operator to be fractional among others; we in the present article consider the fractional heat diffusion models featuring fractional order derivatives in both the Caputo's and the new conformable derivatives to further investigate this development by analyzing two solutions. A comparative analysis of the temperature distributions obtained in both cases will be established. The Laplace transform in conjunction with the well-known decomposition method by Adomian is employed. Finally, some graphical representations and tables for comparisons are provided together with comprehensive remarks.

Keywords
Caputo derivative; Conformable derivative; Heat diffusion models; Decomposition method, Laplace transform.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 45J05.

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Article History: Received **29** November **2018**; Accepted **09** January **2019**

Contents

1. Introduction

Heat diffusion models in one or n-dimensions frequently arise in many industrial processes that lead to much attention from researchers. A very good account can be found in the works of Carslaw and Jaeger [1] and Ozisik [2]. A typical nonhomogeneous transient heat diffusion equation is given by

$$u_t - \underline{\nabla}.(D\underline{\nabla}u) = v, \tag{1.1}$$

where D is the thermal diffusivity of the medium, which can be constant or depending on either of the variables (or both) and v is the source function. Also when $v = 0$, we obtain a homogeneous version of equation (1.1). Further, the thermal diffusivity D happens to be dependent upon temperature in case of nonhomogeneous medium; in such a case, we obtain a nonlinear heat equation version of equation (1.1) as considered in [1-2].

However, with the recent development in the field of fractional differential equations [3-6]; a lot of heat diffusion problems have been studies featuring fractional order derivatives in either time or space variables or both. Thus, a lot of techniques have been applied over a time to solve many models including both the analytical and numerical methods such that the novel series method for heat equation with non-integer order by Yan et al. [7], an approximate decomposition method solution for a non-integer diffusion-wave model [8] and the Adomian decomposition method [9-13]. Other methods include, the symmetry method for nonlinear heat equation by Ahmad et al. [14], the Aboodh decomposition method for nonlinear and time-fractional diffusion equations by Nurud-deen et al. [15-16], the q-homotopy analysis method [17], the double Laplace transform method for fractional heat equation by Anwar et al. [18] and lastly the Wiener-Hopf method [19-21] certain heat problems among others.

Furthermore, in the present article, the time fractional diffusion models in one and two-dimensions would be studied and analyzed. The time fractional derivative order would be based on the well-known Caputo's definition [4] and the recently devised conformable derivative [6]. We also aim to utilize the Laplace integral transform [22] in conjunction with the well-known decomposition method by Adomian [10] as the method of study, see [23-32, 34], also see [33] for the development of the conformable fractional derivative which we used its modification by [6] in this work. Lastly, a kind of comparision analysis between the obtained solutions featuring the two fractional derivatives under consideration would be presented.

We organize the paper as follows: Section 2 presents basics for the two fractional derivatives. In Section 3, we outline of the methods. Section 4 gives the numerical results. Section 5 discusses the obtained results while Section 6 concludes.

2. The Two Fractional Derivatives

The present section gives important concepts of the Caputo fractional derivative and conformable derivative that will later be considered in this paper.

2.1 Caputo's Derivative

Caputo defined a fractional derivative of a function $u(t)$ [3-4] as

$$D_t^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^{\infty}(t-s)^{-\alpha}u'(s)ds,\ (0<\alpha\leqslant 1),\quad (2.1)$$

with $\Gamma(.)$ the gamma function with the following integral represenation

$$(x-1)!=\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt.$$

Some useful properties of the Caputo derivative are given below:

1. $D_t^{\alpha}t^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}$,
2. $D_t^{\alpha}\left(cu(t)\right)=cD_t^{\alpha}u(t)$, c constant,
3. $D_t^{\alpha}c=0$,
4. $D_t^{\alpha}\left(cu(t)+kv(t)\right)=cD_t^{\alpha}\left(u(t)\right)+kD_t^{\alpha}\left(v(t)\right)$,
5. $D_t^{\alpha}\left(u(t)v(t)\right)=v(t)D_t^{\alpha}\left(u(t)\right)+u(t)D_t^{\alpha}\left(v(t)\right)$.

For more, see [1-2].

2.1.1 Laplace Transform for Caputo's Derivative

The Laplace integral transform for the Caputo's definition of fractional derivative as given in equation (2.1) takes the following form [4]

$$\mathscr{L}\{u^{\alpha}(t)\}=s^{\alpha}\mathscr{L}\{u(t)\}-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{k}(0),\quad \alpha\in(n-1,n]. \quad (2.2)$$

2.1.2 Mittag-Leffler Function

The Mittag-Leffler function [4] that arises in fractional calculus is defined for complex t and $\alpha>0$ as

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}. \quad (2.3)$$

2.2 Conformable Derivative

For the conformable derivative, Jawad et al. [6] state the left and right definitions of the fractional conformable derivatives expressed as

$$_aD_t^{\alpha}(u(t))=(t-a)^{1-\alpha}u'(t), \quad (2.4)$$

and

$$_bD_t^{\alpha}(u(t))=(b-t)^{1-\alpha}u'(t), \quad (2.5)$$

where u is a differentiable function. Note also here that that the definitions in equations (2.4)-(2.5) are local derivatives, see [5-6] for more details and [33] for the original work.

3. Laplace Decomposition Methods

We consider the following initial-value problem of nonhomogeneous time-fractional partial differential equation of the form:

$$u_t^{\alpha}(x,t)=L\left(u(x,t)\right)+N\left(u(x,t)\right)+f(x,t),\ \alpha\in(n-1,n], \quad (3.1)$$

with

$$\begin{aligned}&D_0^k u(x,0)=g_k(x),\ \ (k=0,1,2,...n-1),\\ &D_0^n u(x,0)=0,\qquad n=[\alpha],\end{aligned} \quad (3.2)$$

where u_t^{α} stands for the derivative of u fractional order α; $f(x,t)$ is the source term, and L and N are the linear and nonlinear fractional differential operators, respectively. Thus, we consider cases of α using the above definitions as follows:

3.1 When α is in Caputo's derivative sense

Considering the α to be defined in the Caputo's fractional derivative sense, we give the following theorem

Theorem 1. The initial-value problem (3.1) with the condition (3.2) admits the following solution in Caputo's derivative sense:

$$\begin{cases} u_0(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\left(\mathscr{L}(f(x,t))\right)\right), & m=0, \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\mathscr{L}\left(L(u_m(x,t))+A_m\right)\right), & m \geq 0. \end{cases}$$

Proof.

Applying the Laplace transform in t to both sides of (3.1) with condition (3.1) gives

$$\mathscr{L}\{u(x,t)\} = \sum_{k=0}^{n-1} s^{-k-1} g_k(x) + \frac{1}{s^\alpha}\mathscr{L}\left(L(u(x,t))N(u(x,t)) + (f(x,t)\right). \tag{3.3}$$

Also employing the inversion formula of Laplace on (3.3), yields

$$u(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\left(\mathscr{L}(f(x,t))\right)\right) + \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\mathscr{L}\left(L(u(x,t)) + N(u(x,t))\right)\right). \tag{3.4}$$

We now represent the functions $u(x,t)$ and $N(u(x,t))$ by the series solution below:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \quad N(u(x,t)) = \sum_{m=0}^{\infty} A_m, \tag{3.5}$$

where A_m's are the Adomian polynomials, see [10]; we get

$$\sum_{m=0}^{\infty} u_m(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\left(\mathscr{L}(f(x,t))\right)\right) + \sum_{m=0}^{\infty} \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\mathscr{L}\left(L(u_m(x,t)) + A_m\right)\right). \tag{3.6}$$

Therefore, matching $u_0(x,t)$ with the terms from the non-homogeneous and initial condition, and the remaining $u_m(x,t)$ follow iteratively as shown below:

$$\begin{cases} u_0(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\left(\mathscr{L}(f(x,t))\right)\right), & m=0. \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\left(\frac{1}{s^\alpha}\mathscr{L}\left(L(u_m(x,t))+A_m\right)\right), & m \geq 0. \end{cases} \tag{3.7}$$

Thus, the approximate analytical solution of equations (3.1)-(3.2) is determined by the series

$$u(x,t) = \lim_{M\to\infty} \sum_{m=0}^{M} u_m(x,t). \tag{3.8}$$

3.2 When α is in Conformable derivative sense

Here, we consider the case of α to be defined in the conformable fractional derivative sense. Again, before presenting the method, it is good to note that we make use of the general conformable derivative definition in (2.4) for $\alpha \in (n-1,n]$, $(a=0)$.

Theorem 2. The initial-value problem (3.1) with the condition (3.2) admits the following solution in conformable derivative sense::

$$\begin{cases} u_0(x,t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^{[\alpha]}}\left(\frac{1}{t^{[\alpha]-\alpha}}\mathscr{L}(f(x,t))\right)\right), & m=0. \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\left(\frac{1}{s^{[\alpha]}}\mathscr{L}\left(\frac{1}{t^{[\alpha]-\alpha}}\left(L(u_m(x,t))+A_m\right)\right)\right), & m \geq 0. \end{cases}$$

Proof. First, we rewrite (3.1) using the generalized definition of equation (2.4) as

$$t^{[\alpha]-\alpha} u^{[\alpha]}(x,t) = L(u(x,t)) + N(u(x,t)) + f(x,t), \tag{3.9}$$

or

$$u^{[\alpha]}(x,t) = \frac{1}{t^{[\alpha]-\alpha}}\left(L(u(x,t)) + N(u(x,t)) + f(x,t)\right), \tag{3.10}$$

with $[\alpha]$ the least integer of α. As in above, the Laplace transform gives from (3.10) and condition (3.2) the following

$$\mathscr{L}\{u(x,t)\} = \sum_{k=0}^{[\alpha]-1} s^{-k-1} g_k(x) + \frac{1}{s^{[\alpha]}}\mathscr{L}\left(\frac{1}{t^{[\alpha]-\alpha}}\left(L(u(x,t)) + N(u(x,t)) + f(x,t)\right)\right). \tag{3.11}$$

Then taking the inverse Laplace transform of the above equation and proceeding as in above we get the recursive formular as follows:

$$\begin{cases} u_0(x,t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma(k+1)} g_k(x) + \mathscr{L}^{-1}\left(\frac{1}{s^{[\alpha]}}\left(\frac{1}{t^{[\alpha]-\alpha}}\mathscr{L}(f(x,t))\right)\right), & m=0. \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\left(\frac{1}{s^{[\alpha]}}\mathscr{L}\left(\frac{1}{t^{[\alpha]-\alpha}}\left(L(u_m(x,t))+A_m\right)\right)\right), & m \geq 0. \end{cases} \tag{3.12}$$

Also here, the approximate analytical solution of (3.1)-(3.2) is given by the following series

$$u(x,t) = \lim_{M\to\infty} \sum_{m=0}^{M} u_m(x,t). \tag{3.13}$$

4. Numerical Examples

In the present section, application of the presented methods will be established on two types of heat diffusion models with the fractional derivatives under consideration. A kind of comparison analysis of the two will be presented.

4.1 Example One

Consider the initial-value problem of 1-dimensional heat equation with fractional derivative in t

$$u_t^{\alpha} = u_{xx}, \quad 0 < \alpha \le 1, \quad 0 \le x, \le \pi, \; t > 0, \tag{4.1}$$

with

$$\begin{aligned} &u(x,0) = \sin(x),\\ &u(0,t) = u(\pi,t) = 0. \end{aligned} \tag{4.2}$$

When α is in Caputo's sense

We start by applying the Laplace transform to (4.1) and used conditions (4.2) to get

$$\mathscr{L}\{u(x,y,t)\} = \frac{1}{s}\sin(x) + \frac{1}{s^{\alpha}}\mathscr{L}\{u_{xx}\}. \tag{4.3}$$

The inversion of Laplace transform of (4.3) gives

$$u(x,t) = \sin(x) + \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{xx}\}. \tag{4.4}$$

We then replace the function $u(x,t)$ in (4.3) with the series

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t).$$

Thus, equation (4.3) becomes

$$\sum_{m=0}^{\infty} u_m(x,t) = \sin(x) + \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{\sum_{m=0}^{\infty} u_{m_{xx}}\}\}. \tag{4.5}$$

Then, we identify $u_0(x,t)$ with the initial condition term that originate from the initial condition; and the remaining $u_m(x,t)$ follow iteratively as shown below:

$$\begin{cases} u_0(x,t) = \sin(x), & m = 0,\\ u_{m+1}(x,t) = \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{m_{xx}}\}\}, & m \ge 0. \end{cases} \tag{4.6}$$

Some few terms of (4.6) are expressed below

$$u_0(x,t) = u(x,0) = \sin(x), \tag{4.7}$$

$$\begin{aligned} u_1(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{0_{xx}}\}\},\\ &= \frac{-t^{\alpha}}{\Gamma(\alpha+1)}\sin(x), \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_2(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{1_{xx}}\}\},\\ &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x), \end{aligned} \tag{4.9}$$

$$\begin{aligned} u_3(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{2_{xx}}\}\},\\ &= \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x), \end{aligned} \tag{4.10}$$

$$\vdots$$

Summing up, we get

$$u(x,t) = \lim_{M\to\infty} \sum_{m=0}^{M} u_m(x,t),$$

$$= \left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right)\sin(x), \tag{4.11}$$

with the following closed form

$$u(x,t) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\Gamma(\alpha m+1)}\sin(x) = E_{\alpha}(-t)\sin(x). \tag{4.12}$$

(See Fig. 1).

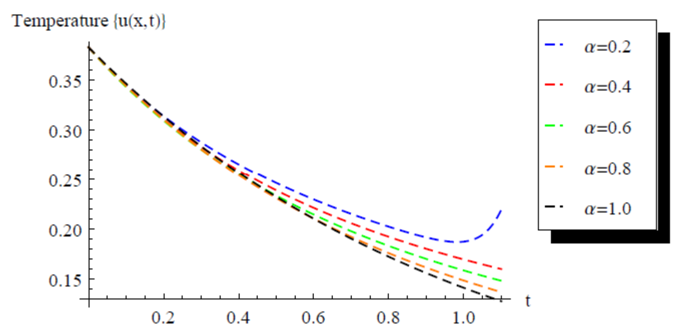

Fig. 1.: Temperature plots for equation (4.12) at various α with $x = \frac{\pi}{6}$, $M = 20$.

When α is in Conformable sense

As presented in the method, we determine the solution recursively given by:

$$\begin{cases} u_0(x,t) = \sin(x), & m = 0,\\ u_{m+1}(x,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}u_{m_{xx}}\}\}, & m \ge 0. \end{cases} \tag{4.13}$$

Some few terms of (4.13) are expressed below

$$u_0(x,t) = u(x,0) = \sin(x), \tag{4.14}$$

$$\begin{aligned} u_1(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}u_{0_{xx}}\}\}, \\ &= \frac{-t^{\alpha}}{\alpha}\sin(x), \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_2(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}u_{1_{xx}}\}\}, \\ &= \frac{t^{\alpha+1}}{\alpha(\alpha+1)}\sin(x), \end{aligned} \tag{4.16}$$

$$\begin{aligned} u_3(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}u_{2_{xx}}\}\}, \\ &= \frac{-t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}\sin(x), \\ &\vdots \end{aligned} \tag{4.17}$$

On summing up gives

$$u(x,t) = \lim_{M\to\infty}\sum_{m=0}^{M} u_m(x,t),$$

$$= \left(1 - \frac{t^{\alpha}}{\alpha} + \frac{t^{\alpha+1}}{\alpha(\alpha+1)} - \frac{t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} + \ldots\right)\sin(x), \tag{4.18}$$

with the following closed form

$$u(x,t) = \left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m t^{\alpha+m-1}}{\prod_{i=0}^{m-1}(\alpha+i)}\right)\sin(x). \tag{4.19}$$

(See Fig. 2).

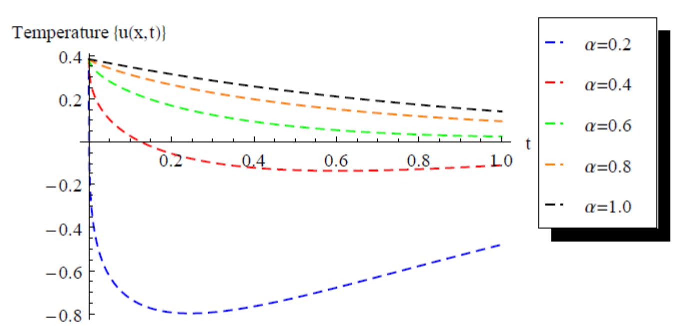

Fig. 2.: Temperature plots for equation (4.19) at various α with $x = \frac{\pi}{6}$, $M = 20$.

4.2 Example Two

Consider the initial-value problem of 1-dimensional nonlinear heat diffusion equation with time fractional derivative in t

$$u_t^{\alpha} + uu_x = u_{xx}, \quad 0 < \alpha \le 1, \quad 0 \le x, \le L, \;\; L > 0, \;\; t > 0, \tag{4.20}$$

with

$$u(x,0) = x. \tag{4.21}$$

When α is in Caputo's sense

As discussed, we give the solution iteratively below:

$$\begin{cases} u_0(x,t) = x, & m = 0, \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{m_{xx}-A_m}\}\}, & m \ge 0, \end{cases} \tag{4.22}$$

where A_m's represent the nonlinear term uu_x given by the polynomials [10] and of the form

$$\begin{aligned} A_0 &= u_0 u_{0_x}, \\ A_1 &= u_{0_x}u_1 + u_0 u_{1_x}, \\ A_2 &= u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0, \\ &\vdots \end{aligned} \tag{4.23}$$

and so on. Solution (4.20) with conditions (4.21) takes the following form

$$u_0(x,t) = u(x,0) = x, \tag{4.24}$$

$$\begin{aligned} u_1(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{0_{xx}-A_0}\}\}, \\ &= \frac{-t^{\alpha}}{\Gamma(\alpha+1)}x, \end{aligned} \tag{4.25}$$

$$\begin{aligned} u_2(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{1_{xx}-A_1}\}\}, \\ &= \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}x, \end{aligned} \tag{4.26}$$

$$\begin{aligned} u_3(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{2_{xx}-A_2}\}\}, \\ &= \frac{-6t^{3\alpha}}{\Gamma(3\alpha+1)}x, \\ &\vdots \end{aligned} \tag{4.27}$$

Summing the above terms yields

$$\begin{aligned} u(x,t) &= \lim_{M\to\infty}\sum_{m=0}^{M} u_m(x,t), \\ &= x - \frac{t^{\alpha}x}{\Gamma(\alpha+1)} + 2\frac{t^{2\alpha}x}{\Gamma(2\alpha+1)} - 6\frac{t^{3\alpha}x}{\Gamma(3\alpha+1)} + \ldots, \end{aligned}$$

$$(4.28)$$

with the following closed form

$$u(x,t) = \left(1 + \sum_{m=1}^{\infty} m! \frac{(-1)^m t^{m\alpha}}{\Gamma(\alpha m+1)}\right) x. \tag{4.29}$$

(See Fig. 3).

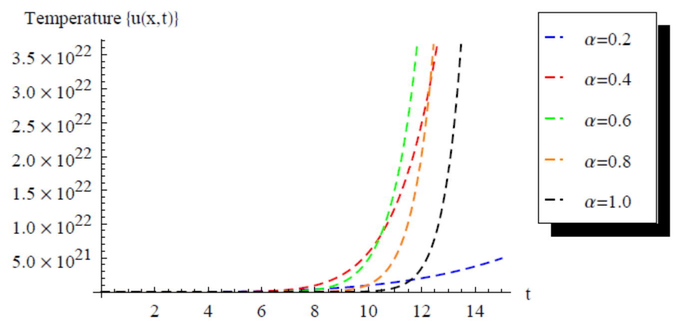

Fig. 3.: Temperature plots for equation (4.29) at various α with $x = 0.5,\ M = 20$.

When α is in Conformable sense

We obtain the solution recursively given in Conformable sense as follows:

$$\begin{cases} u_0(x,t) = x, & m = 0, \\ u_{m+1}(x,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{m_{xx}} - A_m)\}\}, & m \geq 0, \end{cases} \tag{4.30}$$

with A_m' given in (4.23) and expressed below

$$u_0(x,t) = u(x,0) = x, \tag{4.31}$$

$$\begin{aligned} u_1(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{0_{xx}} - A_0)\}\}, \\ &= \frac{-t^{\alpha}}{\alpha}x, \end{aligned} \tag{4.32}$$

$$\begin{aligned} u_2(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{1_{xx}} - A_1)\}\}, \\ &= \frac{t^{2\alpha}}{\alpha^2}x, \end{aligned} \tag{4.33}$$

$$\begin{aligned} u_3(x,t) &= \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{2_{xx}} - A_2)\}\}, \\ &= \frac{-t^{3\alpha}}{\alpha^3}x, \end{aligned} \tag{4.34}$$

$$\vdots$$

On summing we get obtain

$$\begin{aligned} u(x,t) &= \lim_{M\to\infty} \sum_{m=0}^{M} u_m(x,t) \\ &= \left(1 - \frac{t^{\alpha}}{\alpha} + \left(\frac{t^{\alpha}}{\alpha}\right)^2 - \left(\frac{t^{\alpha}}{\alpha}\right)^3 + \ldots\right) x, \end{aligned} \tag{4.35}$$

corresponding closed form

$$u(x,t) = \sum_{m=0}^{\infty} \left(-\frac{t^{\alpha}}{\alpha}\right)^m x = \frac{x}{1 + \frac{t^{\alpha}}{\alpha}}. \tag{4.36}$$

(See Fig. 4).

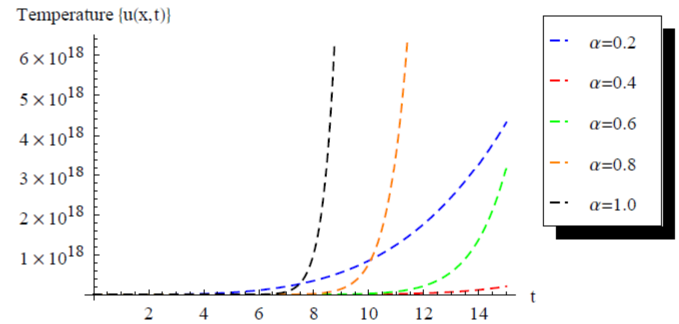

Fig. 4.: Temperature plots for equation (4.36) at various α with $x = 0.5,\ M = 20$.

4.3 Example Three

Consider the initial-boundary-value problem of 2-dimensional heat conduction equation equation with time fractional derivative in t

$$u_t^{\alpha}(x,t) = u_{xx}(x,t) + u_{yy}(x,t),\ 0 < \alpha \leq 1,\ \ 0 \leq x, y \leq \pi,\ t > 0, \tag{4.37}$$

with conditions

$$\begin{aligned} u(x,y,0) &= \sin(x)\sin(y), \\ u(0,y,t) &= u(x,0,t) = 0, \\ u(\pi,y,t) &= u(x,\pi,t) = 0. \end{aligned} \tag{4.38}$$

When α is in Caputo's sense

Proceeding as before, we obtain the solution recursively given by:

$$\begin{cases} u_0(x,y,t) = \sin(x)\sin(y), & m = 0, \\ u_{m+1}(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s^{\alpha}}\mathscr{L}\{u_{m_{xx}} + u_{m_{yy}}\}\}, & m \geq 0. \end{cases} \tag{4.39}$$

with certain terms expressed below

$$u_0(x,y,t) = u(x,y,0) = \sin(x)\sin(y), \tag{4.40}$$

$$u_1(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s^\alpha}\mathscr{L}\{u_{0_{xx}} + u_{0_{yy}}\}\}, = \frac{-2t^\alpha}{\Gamma(\alpha+1)}\sin(x)\sin(y), \tag{4.41}$$

$$u_2(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s^\alpha}\mathscr{L}\{u_{1_{xx}} + u_{1_{yy}}\}\}, = \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x)\sin(y), \tag{4.42}$$

$$u_3(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s^\alpha}\mathscr{L}\{u_{2_{xx}} + u_{2_{yy}}\}\}, = \frac{-8t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x)\sin(y), \tag{4.43}$$

$$\vdots$$

Summing gives

$$u(x,y,t) = \lim_{M\to\infty}\sum_{m=0}^{M} u_m(x,t) = \left(1 - \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right) \times \sin(x)\sin(y), \tag{4.44}$$

with the following closed from

$$u(x,y,t) = \sum_{m=0}^{\infty}\frac{(-2t)^m}{\Gamma(\alpha m+1)}\sin(x)\sin(y), = E_\alpha(-2t)\sin(x)\sin(y). \tag{4.45}$$

(See Fig. 5).

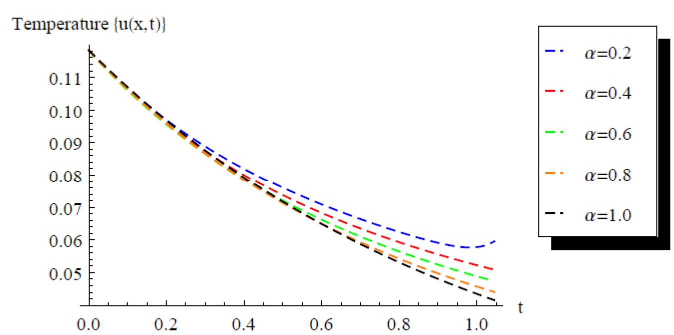

Fig. 5.: Temperature plots for equation (4.45) at various α with $x = \frac{9\pi}{4}$, $y = \frac{8\pi}{9}$, $M = 20$.

When α is in Conformable sense

We also obtain the solution recursively given in conformable sense as follows:

$$\begin{cases} u_0(x,y,t) = \sin(x)\sin(y), & m = 0, \\ u_{m+1}(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{m_{xx}} + u_{m_{yy}})\}\}, & m \geq 0. \end{cases} \tag{4.46}$$

Also from (4.46), we get

$$u_0(x,y,t) = u(x,y,0) = \sin(x)\sin(y), \tag{4.47}$$

$$u_1(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{\frac{1}{t^{1-\alpha}}(u_{0_{xx}} + u_{0_{yy}})\}\}, = \frac{-2t^\alpha}{\alpha}\sin(x)\sin(y), \tag{4.48}$$

$$u_2(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{u_{1_{xx}} + u_{1_{yy}}\}\}, = \frac{4t^{\alpha+1}}{\alpha(\alpha+1)}\sin(x)\sin(y), \tag{4.49}$$

$$u_3(x,y,t) = \mathscr{L}^{-1}\{\frac{1}{s}\mathscr{L}\{u_{2_{xx}} + u_{2_{yy}}\}\}, = \frac{-8t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}\sin(x)\sin(y), \tag{4.50}$$

$$\vdots$$

which sums up

$$u(x,y,t) = \lim_{M\to\infty}\sum_{m=0}^{M} u_m(x,t), = \left(1 - \frac{2t^\alpha}{\alpha} + \frac{4t^{\alpha+1}}{\alpha(\alpha+1)} - \frac{8t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} + \dots\right) \times \sin(x)\sin(y), \tag{4.51}$$

and has the following closed form

$$u(x,y,t) = \left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m 2^m t^{\alpha+m-1}}{\prod_{i=0}^{m-1}(\alpha+i)}\right)\sin(x)\sin(y). \tag{4.52}$$

(See Fig. 6).

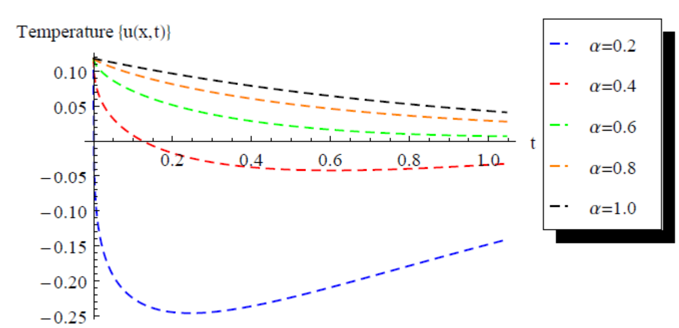

Fig. 6.: Temperature plots for equation (4.52) at various α with $x = \frac{9\pi}{4}$, $y = \frac{8\pi}{9}$, $M = 20$.

5. Comparision of results

In this section, we aim to make comparisions between the Caputo's and fractional conformable derivatives solutions of the problems presented in Examples 1-3. The temperature distributions in each case would be analazyed by studying the effect of varying α ranging from 0 to 1. The comparision tables are presented in Tables 1-3 with corresponding graphical illustrations depicted in Figs. 7-9.

Example One

In example one, it was observed from Fig. 1. that the temperature distribution in the case of Caputo's derivative enhances as the value of α decreases from 1 to 0; while opposite trend is observed in the case of fractional conformable derivative as shown in Fig. 2. However it is worth noting that the two are the same at $\alpha = 1$ and also the temperature distribution is realized when $\alpha = 0$ in case of Caputo's fractional derivative while it diverges in comformable sense. For this see Table 1. and Fig. 7.

Table 1. Comparison of Caputo's and fractional conformable derivative solutions of Example One at $M = 10,\ t = 1,\ x = \frac{\pi}{6}$.

α	Caputo	Conformable	Absolute Difference
0.0	0.500000	-	-
0.1	0.487485	-1.562520	2.050000
0.2	0.348997	-0.626532	0.975529
0.3	0.262153	-0.306019	0.568172
0.4	0.228376	-0.1406430	0.369020
0.5	0.215029	-0.0380794	0.253108
0.6	0.206832	0.0325988	0.174233
0.7	0.199825	0.0847376	0.115087
0.8	0.193476	0.1250690	0.0684072
0.9	0.188033	0.1573720	0.0306615
1.0	0.183940	0.1839400	0.0000000

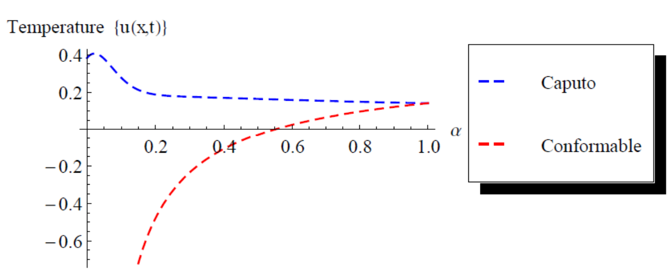

Fig. 7.: Temperature plots for comparison of solutions of Example One.

Example Two

In example two, it was observed from both Fig. 3. and 4. that the temperature distribution in the case of Caputo's and fractional conformable derivative decays as the value of α increases from 0 to 1. However it is worth noting here also that the two are the same at $\alpha = 1$ and the temperature distribution is also realized when $\alpha = 0$ in case of Caputo's fractional derivative while it diverges in comformable sense. See Table 2. and Fig. 8.

Table 2. Comparison of Caputo's and fractional coformable derivative solutions of Example Two at $M = 20,\ t = 0.5,\ \ x = 0.5$.

α	Caputo	Conformable	Absolute Difference
0.0	1.15866×10^{18}	-	-
0.1	1.43653×10^{17}	1.129×10^{19}	1.11463×10^{19}
0.2	2.94124×10^{15}	2.42347×10^{12}	2.93882×10^{15}
0.3	2.38508×10^{13}	1.63626×10^{8}	2.38506×10^{13}
0.4	1.02194×10^{11}	116269.	1.02194×10^{11}
0.5	2.67164×10^{8}	300.13	2.67164×10^{8}
0.6	463318.	1.9867	463316.
0.7	561.961	0.28394	561.677
0.8	0.812698	0.291323	0.521374
0.9	0.326371	0.313401	0.0129695
1.0	0.333333	0.333333	0.000000

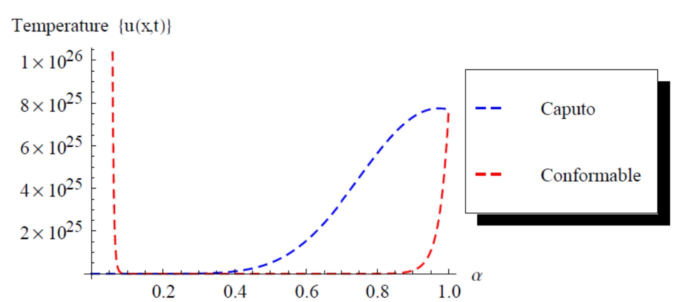

Fig. 8.: Temperature plots for comparison of solutions of Example Two.

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Example Three

Example three has the same interpretation with example one. One can quickly see that $\sin(y)$ is the only different; and upon fixing y varibale we arrived at example one's solution. Thus, see Table 3. and Fig. 9 respectively.

Table 3. Comparison of Caputo's and fractional conformable derivative solutions of Example Three at $M = 20,\ t = 0.5,\ x = \frac{9\pi}{4},\ y = \frac{8\pi}{9}$.

α	Caputo	Conformable	Absolute Difference
0.0	0.241845	-	-
0.1	0.175102	-1.61978	1.79488
0.2	0.118217	-0.706868	0.825084
0.3	0.110547	-0.391489	0.502036
0.4	0.106912	-0.227834	0.334747
0.5	0.103409	-0.126223	0.229632
0.6	0.0999611	-0.0564657	0.156427
0.7	0.0966441	-0.00544053	0.102085
0.8	0.0935815	0.033528	0.0600535
0.9	0.0909496	0.0642243	0.0267253
1.0	0.0889697	0.0889697	1.38778×10^{17}

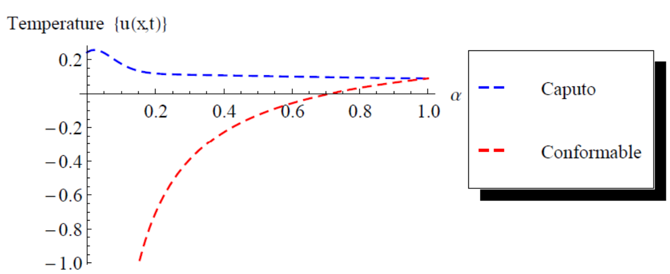

Fig. 9.: Temperature plots for comparison of solutions of Example Three.

6. Conclusion

In conclusion, the one and two-dimensional heat diffusion models involving fractional order derivative in time have been analyzed. The fractional orders considered include the Caputo's and the new fractional conformable derivatives. The Laplace integral transform method in conjunction with the decomposition method by Adomian is utilized as a tool for this study. We gathered that the temperature distribution in the case of Caputo's derivative enhances as the value of α decreases from 1 to 0; while opposite trend is observed in the case of the conformable derivative in the numerical examples 1 and 3. However it is worth noting that the two are the same at $\alpha = 1$ and also the temperature distribution is realized when $\alpha = 0$ in case of Caputo's fractional derivative while it diverges in conformable sense in all the three examples. However, looking at the comparision plots in Figs 7-9, we deduced that the Caputo's fractional derivative is indeed of practical interest in heating models since it turned out to be positive for all values of $\alpha \in (0,1]$ unlike the conformable with negative values in examples 1 and 3 and unpredicted profile pace in example 2 by looking at the temperature profiles. For the rigorous aspect of whether the conformable fractional derivative is indeed a fractional or not, see [35-37] and the references therein.

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ISSN(P):2319 − 3786
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ISSN(O):2321 − 5666
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