On network criticality in robustness analysis of a network structure

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Abstract

Robustness of a network is the ability of the network to maintain its functionality when some vertices or edges are removed due to targeted attacks or random failures. This paper studies an interesting graph measure that we call the network criticality. The notion of network criticality is derived from the probabilistic definition of betweenness, which is defined based on random walks in a graph, as the main metric to quantify the survival value of a network with respect to changes in topology and dynamics. The objective of the paper is twofold. First, we discuss some known formulae of network criticality and derive its relation with some other graph measures. Second, we propose a new measure of network functionality based on network criticality.

Keywords

Robustness, Network criticality, Graph measures.

AMS Subject Classification

05C50, 05C82, 90B25

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1. Introduction

With the growing appreciation of complex network models of various systems like, water supply, energy supply, communication and transportation, the techniques of network analysis have come to play an important role in the representation and understanding of the structure of complex systems. In particular, different measures of importance and robustness have allowed to quantify aspects of networks which are responsible for their topological or dynamical properties.

A study of the effects of different catastrophic disasters like, floods or any natural calamity or the targeted attacks on networks, it is found that the transportation network is the most important [4]. As the restoration of other networks will be decided by movement of people and equipments to the sites where damage has occurred and in this situation the performance of transportation network is the key. So first one has to ensure the existence of a connected road network for the operation of the whole society. And second, we have to stop any disruption that makes partial or complete failure of the road network. As it might cause remarkable negative social and economic impact, even the disruption of the whole society. Thus, it is very important for a road network to maintain its function as much as possible after the disruption; and also to recover its function at earliest from the partial or complete failure in order to minimize the losses of money and casualties caused by the disruptions in the road network. The road network robustness concerns exactly with these two ideas.

Gribble [10] defined robustness as the ability of a network to operate correctly under a wide range of operational con-

ditions, and to fail gracefully outside of that range. Based on some of the indicators like the average speed, network throughput, the operation status of a road network is often evaluated for the network level performance. Thus the study of road network robustness can be simply understood as the analysis of the performance of the road network under the situations with considerable changes in its supply or/and demand compared with its normal or desired performance. But the concept of network robustness should not be confused with the concept of network reliability, which also analyzes the network performance under some changes in the operational conditions.

In computer science the concept of network robustness is often defined as the ability of a computer system to cope with the failures during run time. The robustness is a well studied topic in several types of large-scaled complex networks, such as communication networks [1], internet [19], metabolic networks [13], as well as general complex networks [5][16]. But for transportation networks, such as the road networks, robustness has gained very limited attention. Also it is hard to find a unified or widely accepted definition for road network robustness in the literature. The following definition summarizes the interpretations from several researchers, such as [2][10]: Road network robustness is the insusceptibility of a road network to disturbing incidents, and could be understood as the opposite of network vulnerability. In other words, road network robustness is the ability of a road network to continue to operate correctly across a wide range of operational conditions. And the most accepted definition of the network reliability is given by (Billington & Allan, 1992)[7] and (Wakabayashi & Iida, 1992) [11] as follows: Reliability is the probability of a road network performing its proposed service level adequately for the period of time intended under the operating conditions encountered. From these definitions of the two concepts given above, their objects are clearly described as the probability (for reliability) and the ability (for robustness) of road networks to perform properly. Besides this major difference, (Immers & Jansen, 2005)[12] later pointed out that (travel time) reliability is normally a user-oriented quality of the system, and robustness is one of the characteristics of the road system itself.

The rest of the paper is organized as follows. In the next section we present some known results related to network criticality and some other graph measures like tree-number. Section 3 discusses some relations between network criticality and the graph measures considered in Section 2. In Section 4, we propose a new measure of functionality of a network based on network criticality. Conclusions are presented in Section 5.

2. Preliminaries

An undirected graph $G = (V, E)$ consists of a finite set V of vertices (or nodes) and a finite set $E \subseteq V \times V$ of edges. If an edge $e = (u, v)$ connects two vertices u and v then vertices u and v are said to be incident with the edge e and adjacent to

each other. The set of all vertices which are adjacent to u is called the neighbourhood, $N(u)$ of u. The degree of a vertex, v is defined to be the number of edges having v as one of its end point. A directed graph or digraph G is a graph with some direction associated with each of its edges. A walk is defined as a sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A walk is called closed if the initial(starting) and terminal(terminating) vertices coincide and open otherwise. A trail is a walk without repeated edges and path is a walk without repeated vertices. A graph is said to be connected if there exists a path between every pair of its vertices.

An adjacency matrix A of a graph $G = (V, E)$, with $|V| = n$ is an $n \times n$ matrix, where $A_{ij} = 1$ if and only if $(i, j) \in E$ and $A_{ij} = 0$ otherwise. The adjacency matrix of any undirected graph is symmetric. Another matrix representation of graph is the Laplacian of a graph. The Laplacian, L of a graph G is an $n \times n$ matrix, defined by $L = D - A$, where D is a diagonal matrix whose entries correspond to the degree of the vertices in G and A is the adjacency matrix of G. Laplacian of any undirected graph is also symmetric. But unlike the adjacency matrix, the laplacian of a graph is always singular.

2.1 Betweenness centrality

A well-known centrality measure is the betweenness centrality [9]. The betweenness centrality of a node u, η_u is the number of shortest path going through u. Mathematically it is defined as:

$$\eta_u = \sum_{s \neq u \in V} \sum_{t \neq u \in V} \frac{\sigma_{st}(u)}{\sigma_{st}}, \quad (2.1)$$

where, σ_{st} is the number of shortest paths from vertex s to t, and $\sigma_{st}(u)$ is the number of shortest paths from s to t that pass through u. Betweenness centrality identifies nodes that make the most traffic flow of the network. An important node will lie on a large number of paths between other nodes in the network. From this node we can control the traffic flow of the network. In general the high degree nodes have high betweenness centrality because many of the shortest paths may pass through them. However a high betweenness centrality node need not always be a high degree node.

Similarly one can define the betweenness of a link (edge) (i, j), η_{ij}, is the number of shortest path going through (i, j). Mathematically,

$$\eta_{ij} = \sum_{s \in V} \sum_{t \in V} \frac{\sigma_{st}((i, j))}{\sigma_{st}}, \quad (2.2)$$

where, σ_{st} is the number of shortest paths from vertex s to t, and $\sigma_{st}((i, j))$ is the number of shortest paths from s to t that pass through (i, j).

2.2 The tree-number

The number of spanning trees of a graph G is known as the tree-number of G [6]. It is denoted by $\kappa(G)$. In 1964, *Temper-*

ley obtained the following result, which gives a formula for the tree-number of a graph G with n vertices.

Theorem 2.1. *(Temperley) The tree-number of a graph G with n vertices is given by the formula*

$$\kappa(G) = n^{-2} det(J + L),$$

where L is the laplacian of the graph G and J is the $n \times n$ matrix with all entries equal to 1.

Using this theorem one can obtain the following result that gives a formula for $\kappa(G)$ in terms of the laplacian spectrum of the graph. The formula is as follows:

$$\kappa(G) = \frac{\mu_1 \mu_2 \ldots \mu_{n-1}}{n}, \qquad (2.3)$$

where $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ is the laplacian spectrum of graph G with n vertices.

2.3 The effective graph resistance

The effective graph resistance is another popular graph measure that attracts many researchers specially people working in circuit theory. Klein and Randić [14] have proved that it can be written as a function of the non-zero Laplacian eigenvalues, which is as follows.

Theorem 2.2. *The effective graph resistance R satisfies*

$$R = n \sum_{i=1}^{n-1} \frac{1}{\mu_i},$$

where $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ is the laplacian spectrum of graph G with n vertices.

Some results and properties of effective graph resistance may be found in [8].

2.4 Network criticality

This measure of robustness was first proposed by Leon-Garcia and Tizghadam [17] and its applications are well studied in [15][18]. This measure is a probabilistic measure for traffic engineering (specifically routing and resource allocation) in backbone networks, where the transport is the main service and robustness to the unexpected changes in network parameters is required. It reflects the effect of environmental changes such as traffic variation and capacity changes.

A random-walk on a network is defined with transition probability matrix $[pr(l) = pr(i \rightarrow j)]$, where the elements are functions of link weights and denote the probability of transitioning from node i to neighbour node j along link $l = (i, j)$. In the generic random-walks, the transition probability from node i to its neighbouring node j is proportional to the weight of link $l = (i, j)$ (i.e. high weight due to low travel time):

$$pr(l) = \frac{W_l}{\sum_{e \in A^o(i)} W_e}, \qquad (2.4)$$

where $A^o(i)$ denotes the set of outgoing edges attached to node i, and W_l, W_e are the weights on the edges l, e respectively. Using this probability transition matrix, one can obtain a betweenness matrix [17]. Betweenness of link $l = (i, j)$ is equal to $b(l) = \tau W_l$. Similarly, the criticality of a link l would be $\eta(l) = \frac{b(l)}{W_l} = \tau$. Here τ is independent of the node/link position. In fact τ is a global quantity of the network. This τ is termed as 'network criticality'. An alternative formula for τ, which is in terms of the Moore-Penrose inverse of Laplacian matrix L, is as follows [8]:

$$\tau = 2n\,trace(L^+), \qquad (2.5)$$

where n is the number of nodes and L^+ is the Moore-Penrose inverse of Laplacian matrix L of the graph.

A smaller value of τ means a higher level of robustness. Indeed τ is the survival value that we need to model the robustness because it can be used to quantify the resistance of a network to the unwanted changes in network topology or traffic demands, the less the network criticality, the less the sensitivity to the changes in topology and traffic. It is also known that τ is a strictly convex function of graph weights. Further, τ is a non-increasing function of link weights [17].

3. Some results on τ, R and κ

First we present a known result [8] that gives a relation between network criticality τ and effective graph resistance R, but with a different proof based on laplacian spectrum. Then an interesting relation between τ and κ will be presented.

Theorem 3.1. *The network criticality τ is twice of the effective graph resistance R, i.e., $\tau = 2R$.*

Proof. Let $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the laplacian spectrum of a graph G with n vertices. We know that $L^+ = (L + \frac{J}{n})^{-1} - \frac{J}{n}$ [3], where J is an $n \times n$ matrix whose entries are all equal to 1. So, we can see that

$$\tau = 2n\,trace(L^+) = 2n\,trace((L + \frac{J}{n})^{-1} - \frac{J}{n})$$
$$= 2n\,(trace(B) - 1), \qquad (3.1)$$

where $B = (L + \frac{J}{n})^{-1}$.

Since $L\frac{J}{n} = 0 = \frac{J}{n}L$, the eigenvalues of $L + \frac{J}{n}$ are the sum of the corresponding eigenvalues of $\frac{J}{n}$ and L. The eigenvalues of $\frac{J}{n}$ are 1, 0, 0,..., 0, so eigenvalues of $L + \frac{J}{n}$ are 1, μ_1, μ_2, ..., μ_{n-1}. And hence the eigenvalues of $B = (L + \frac{J}{n})^{-1}$ are 1, $\frac{1}{\mu_1}$, $\frac{1}{\mu_2}$, ..., $\frac{1}{\mu_{n-1}}$, and $trace(B) = 1 + \sum_{i=1}^{n-1} \frac{1}{\mu_i}$. Now from expression (3.1), we have

$$\tau = 2n \sum_{i=1}^{n-1} \frac{1}{\mu_i}. \qquad (3.2)$$

From theorem 2.2 and equation (3.2), we have the result. $\square$

A relation between τ and κ can be obtained involving the characteristic polynomial of the laplacian of the graph, which is as follows:

Theorem 3.2. *Let G be a graph with tree-number $\kappa(G)$, then network criticality τ of G satisfies*

$$\tau = \frac{(-1)^n \sigma''(G,0)}{\kappa(G)},$$

where $\sigma''(G,0)$ is the value of the second derivative of the characteristic polynomial of the laplacian of G at 0.

Proof. Let $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the laplacian spectrum of graph G with n vertices. From expression (2.3), we have

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i} = \frac{\sum_{i=1}^{n-1} \mu_1 \mu_2 \ldots \mu_{i-1}\mu_{i+1} \ldots \mu_{n-1}}{\prod_{i=1}^{n-1} \mu_i} = \frac{S}{n\kappa(G)}, \quad (3.3)$$

where $S = \sum_{i=1}^{n-1} \mu_1 \mu_2 \ldots \mu_{i-1}\mu_{i+1} \ldots \mu_{n-1}$. Again from (3.2) and (3.3)

$$\tau = \frac{2S}{\kappa(G)}. \quad (3.4)$$

Again if $\sigma(G,x)$ is the characteristic polynomial of the laplacian i. e. $\sigma(G,x)=\det(xI - L)$, then $\sigma(G,x) = x(x - \mu_1)(x - \mu_2)\ldots(x - \mu_{n-1})$. Now,

$$\sigma'(G,x) = (x - \mu_1)(x - \mu_2)\ldots(x - \mu_{n-1}) + xf(x),$$

where $f(x) = (x - \mu_2)(x - \mu_3)\ldots(x - \mu_{n-1}) + (x - \mu_1)(x - \mu_3)\ldots(x - \mu_{n-1}) + \ldots + (x - \mu_1)(x - \mu_2)\ldots(x - \mu_{n-2})$. And $\sigma''(G,x) = 2f(x) + xf'(x)$. Clearly, $f(0) = (-1)^{n-2}S$. So, $\sigma''(G,0) = (-1)^n 2S$. Hence from equation (3.4), we have the required result. □

A lower bound of τ_e, which is the network criticality of the reduced graph $G \setminus e$ may be derived, as shown in our next theorem.

Theorem 3.3. *Let a link e (which is not a bridge) be removed from the graph G. Then a lower bound of the network criticality τ_e of the reduced graph $G \setminus e$ is*

$$\tau_e \geq \frac{n(n-1)^2}{m-1},$$

where m is the number of edges of the original graph G and n is the number of nodes of G.

Proof. Let the laplacian spectrum of the reduced graph $G \setminus e$ be $0 \leq \mu_1' \leq \mu_2' \leq \ldots \leq \mu_{n-1}'$ and then $\tau_e = 2n\sum_{i=1}^{n-1} \frac{1}{\mu_i'}$. For positive real numbers $a_1, a_2, \ldots, a_n$, the HM-GM-AM inequality is

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq (\prod_{i=1}^{n-1} a_i)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n-1} a_i. \quad (3.5)$$

Since e is not a bridge so all μ_i' are positive (as $\mu_1' \neq 0$). And applying inequality (3.5) for $\mu_1', \mu_2', \ldots, \mu_{n-1}'$, we have

$$\frac{n-1}{\sum_{i=1}^{n} \frac{1}{\mu_i'}} \leq (\prod_{i=1}^{n-1} \mu_i')^{1/(n-1)} \leq \frac{1}{n-1}\sum_{i=1}^{n-1} \mu_i'. \quad (3.6)$$

From (3.6), we can see that $\sum_{i=1}^{n-1} \frac{1}{\mu_i'} \geq \frac{(n-1)^2}{\sum_{i=1}^{n-1} \mu_i'}$. Since the sum of the laplacian eigenvalues is equal to sum of the degree of the nodes and is again equal to twice the number of edges. So, $\sum_{i=1}^{n-1} \mu_i' = 2(m-1)$. And hence the result. □

Another lower bound of τ_e may be derived using tree-number of the graph, which is as follows.

Theorem 3.4. *Let a link e (which is not a bridge) be removed from the graph G. Then a lower bound of the network criticality τ_e of the reduced graph $G \setminus e$ is*

$$\tau_e \geq \frac{2n^{\frac{n-2}{n-1}}(n-1)}{\kappa^{\frac{1}{n-1}}},$$

where κ is the tree-number of $G \setminus e$.

Proof. This result can be proved by considering the first part of the inequality (3.6), and from the result of the theorem 2.2. □

An upper bound of the ratio $\frac{\tau}{\tau_e}$ is the same as that of the ratio $\frac{R}{R_e}$, which can be derived similarly as shown in [20].

Theorem 3.5. *Let a link e be removed from the graph G, and the network criticality of the reduced graph $G \setminus e$ be τ_e. Then*

$$\frac{\tau_e}{\tau} \leq \max_i \frac{\mu_i}{\mu_{i-1}},$$

where $2 \leq i \leq n-1$.

Proof. For positive real numbers $a_1, a_2, \ldots, a_n$ and real numbers $b_1, b_2, \ldots, b_n$, it holds

$$\min_i \frac{a_i}{b_i} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \max_i \frac{a_i}{b_i}. \quad (3.7)$$

Let $a_i = \frac{1}{\mu_i'}$ and $b_i = \frac{1}{\mu_i}$ in the inequality (3.7), we get

$$\frac{1}{1 - \min_i \frac{\mu_i - \mu_i'}{\mu_i}} \leq \frac{\sum_{i=1}^{n-1} \frac{1}{\mu_i'}}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \leq \frac{1}{1 - \max_i \frac{\mu_i - \mu_i'}{\mu_i}}. \quad (3.8)$$

After a link removal interlacing property gives us $\mu_{i-1} \leq \mu_i' \leq \mu_i$, where $2 \leq i \leq n-1$. Using this fact, the right side of the inequality (3.8) can be written as

$$\frac{1}{1 - \min_i \frac{\mu_i - \mu_i'}{\mu_i}} \leq \frac{1}{1 - \max_i \frac{\mu_i - \mu_{i-1}}{\mu_i}} = \frac{1}{1 - (1 - \min_i \frac{\mu_{i-1}}{\mu_i})}$$

$$= \frac{1}{\min_i \frac{\mu_{i-1}}{\mu_i}} = \max_i \frac{\mu_i}{\mu_{i-1}}.$$

And hence the result. □

4. A measure of network functionality

First let us consider a simple example which will motivate us for the following discussion.

4.1 A motivational example

In figure 1, two graphs G and G' each of 3 nodes, and weights on the edges are considered in such a way that G and G' will have almost same network criticality (i.e. the values of $\tau(\approx 2)$ are almost the same).
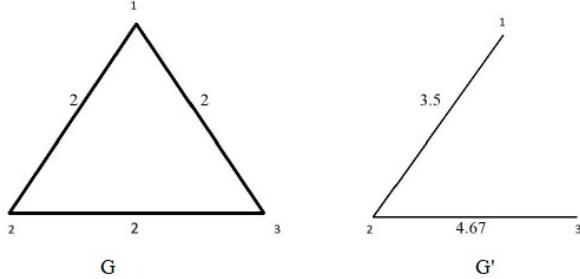

Figure 1. Two graphs G and G' with almost same network criticality.

Now if we closely look at the graphs G and G' we find that removal of an edge will disconnect G' but not G. So from the perspective of random failures or hostile attacks G' is more vulnerable than G. In other words we can say that the robustness of G and G' are not same, which is not captured by network criticality.

4.2 The Measure

To capture the effect of this kind of random failures or hostile attacks we define a measure based on the same network criticality.

Since network criticality is a non-increasing function of link weights, so it is expected that the removal of an edge (reduction of link weight) will result in an increment in the value of network criticality. But if the removed edge happens to be a bridge, then that does not hold. Because of the fact that L^+ of a disconnected network will have lesser number of non-zero laplacian eigenvalues and hence the value τ will drop down dramatically.

Let $G \setminus e$ be the graph obtained by removing the edge e, and the laplacian spectrum of $G \setminus e$ is $0 \le \mu'_1 \le \mu'_2 \le \ldots \le \mu'_{n-1}$. Then by interlacing property of laplacian eigenvalues, we have $0 \le \mu'_1 \le \mu_1 \le \mu'_2 \le \mu_2 \le \ldots \le \mu'_{n-1} \le \mu_{n-1}$. And hence we have the following theorem.

Theorem 4.1. *Let τ and τ_e be the network criticality of G and $G \setminus e$. Then $\tau_e \ge \tau$, if e is not a bridge. And $\tau_e \ge \tau - \frac{2n}{\mu}$, if e is a bridge, where μ is the smallest non-zero laplacian eigenvalue of G.*

Proof. Since $\tau = 2n\sum_{i=1}^{n-1}\frac{1}{\mu_i}$ and $\tau_e = 2n\sum_{i=1}^{n-1}\frac{1}{\mu'_i}$, so from the interlacing property $\tau_e \ge \tau$, if e is not a bridge. Now if e is a bridge, then the components of the graph $G \setminus e$ will increase and also μ', the corresponding eigenvalue μ will be zero . And hence we have the other inequality. $\square$

	Watts-Strogatz Network				Barabási-Albert Network Network			
	$\bar{\eta}$	$\bar{B}_c$	C_c	$\bar{E}_c$	$\bar{\eta}$	$\bar{B}_c$	C_c	$\bar{E}_c$
$\bar{\eta}$	1	0.3158	0.6040	-0.7968	1	-0.6693	0.8637	0.6190
$\bar{B}_c$		1	0.8011	-0.2102		1	-0.7967	-0.0032
C_c			1	-0.5394			1	0.4323
$\bar{E}_c$				1				1

Table 1. Correlation between average network functionality and different graph measures.

Definition 4.2. *The average network functionality, $\bar{\eta}$ of a graph $G = (V,E)$, with $|E| = m$ is defined as*

$$\bar{\eta} = 1 - \frac{1}{m}\sum_{e \in E}\left[H^+(\tau_e - \tau)\frac{\tau}{\tau_e} + H^-(\tau_e - \tau)\frac{\tau}{\frac{2n}{\mu} + \tau_e}\right],$$

where μ is the smallest non-zero laplacian eigenvalue of G, $H^+(x) = 1$ if $x \ge 0$ and 0 otherwise, and $H^-(x) = 1$ if $x < 0$ and 0 otherwise.

For a connected network G, $\mu = \mu_1$ which is known as the algebraic connectivity of G, and from theorem 4.1, it is clear that $0 \le \bar{\eta} \le 1$. This $\bar{\eta}$ can be considered as a measure of functionality of the network. A higher value of $\bar{\eta}$ means a higher degree of functionality of the network.

4.3 A statistical study of $\bar{\eta}$

In this subsection we study the measure, $\bar{\eta}$ in two well studied networks namely Watts-Strogatz network and Barabási-Albert network for different rewiring probabilities and different sizes of seed. The results obtained for different number of nodes (50, 100, 150, 200, 250, 300) are presented in figure 2 and figure 3. In case of Barabási-Albert network, the average path length increases approximately logarithmically as the size of the network increases and in case of Watts-Strogatz network as the rewiring probability increases, the connections with neighbours break down with rewiring edges. So in case of both the networks one can expect that the functionality of the network drops down as the size of the network or rewiring probability increases. We see that the aforesaid property is also reflected in figure 2 and figure 3.

In figure 4 and figure 5, the results of the comparison between average network functionality and other popular graph measures like average betweenness ($\bar{B}_c$), clustering co-efficient (C_c) and average eigenvector centrality ($\bar{E}_c$) are presented. In case of both the networks, we see that the average network functionality is bounded by average betweenness and clustering co-efficient or average eigenvector centrality, which is because of the fact that with increase of size of the network average betweenness increases very rapidly whereas the rewiring edges leads to breakdown of cliques with neighbours.

Table 1 shows the correlations among average network functionality and various graph measures in Watts-Strogatz network and Barabási-Albert network respectively. In both networks we see that clustering co-efficient and average network functionality are highly correlated, which is because of
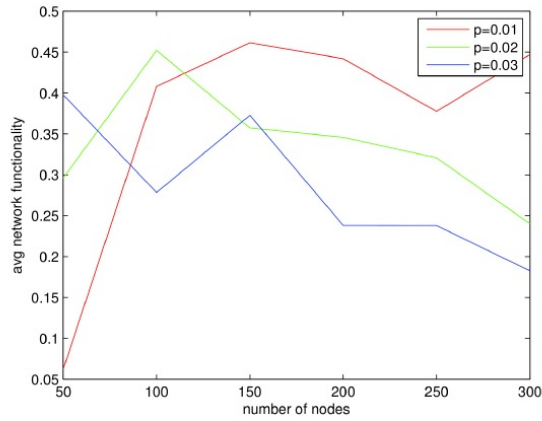
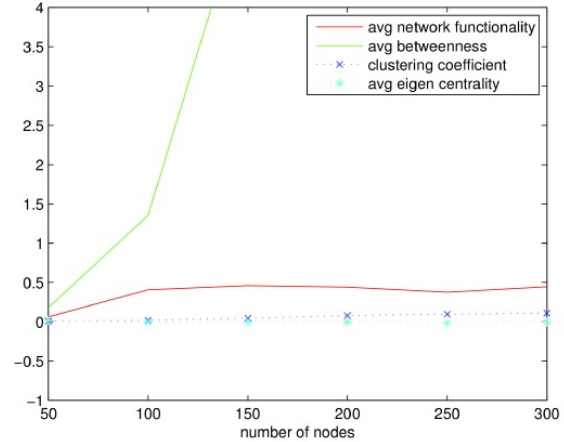**Figure 2.** Average network functionality in Watts-Strogatz network for different values rewiring of probability p.
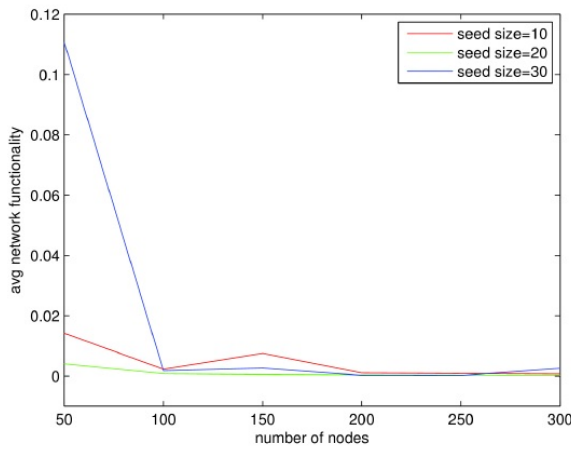

Figure 3. Average network functionality in Barabási-Albert network for different sizes of seed.

Figure 4. Comparison of average network functionality with other graph measures in Watts-Strogatz network.
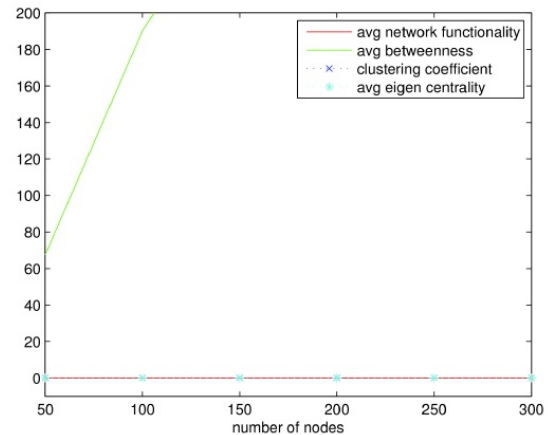

Figure 5. Comparison of average network functionality with other graph measures in Barabási-Albert network.

the fact that presence of more cliques reduces the chance of an edge to become a bridge. Average network functionality exhibits negative correlation with eigenvector centrality in Watts-Strogatz network, and with betweeness centrality in Barabási-Albert network. In Watts-Strogatz network it is also visible that the degree distribution among vertices is closely homogenous, which explains the reason of the negative correlation between average network functionality and eigenvector centrality. Again as Barabási-Albert model is based on a preferential attachment or a "rich get richer" effect, an edge is most likely to attach to nodes with higher degrees. So it reduces the chance of an edge to become a bridge, and hence we can expect a negative correlation between average network functionality and betweeness centrality.

5. Conclusion

In this paper we have studied the measure called network criticality and established its relation with some well known graph measures. Due to random or intentional attack if an edge is removed, we study the change in the value of network criticality and some interesting results have been presented. Then in the later part of this paper we propose a measure called average network functionality based on network criticality, which quantifies the effect of removal of an edge.

A statistical study of the proposed measure is also presented in case of two popular networks namely Watts-Strogatz network and Barabási-Albert network. In this study we have found some very interesting results which establish this measure as a measure of functionality of a network.

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