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Analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queueing system with priority services, orbital search, compulsory short vacation, optional long vacation, working breakdown and repair under Bernoulli schedule controlled policy

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Abstract
In this paper, we consider $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue with priority services, orbital search, compulsory short vacation, optional long vacation, working breakdown and repair under Bernoulli schedule controlled policy. There are two types of customers, namely high priority and low priority arriving in batches. After completion of each high priority service, the server goes for short vacation and after completion of each low priority service, the server can option to take a long vacation. Further, the server subject to breakdown, the system either goes to repair immediately or continue slower rate service for current customer with some probability. The primary customers who find the server is long vacation are allowed to balk. After completion of low priority service (if the server not taking vacation), repair or both types of vacation the server can search for the customers in the orbit or remains idle, if there are no customers in the high priority queue. We use the established norm which is the corresponding steady state results for time dependent probability generating functions are obtained. Along with that, the expected waiting time for the expected number of customers for both high and low priority queues are computed. Numerical results along with the graphical representations are shown elaborately.

Keywords
Batch arrival; Priority queue; Orbital search; Working breakdown; Compulsory vacation; Optional vacation.

AMS Subject Classification
60K25, 68M30, 90B22.

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Contents

1. Introduction

The priority queue has received considerable attention in the literature. Two well-known priority disciplines in queueing literature are the non-preemptive and the preemptive disciplines. Under the non-preemptive rule which was introduced by Cobham [2], if arrival of high priority customers when low priority is being served, it will wait until the low priority completes its service. The second discipline interrupted the low priority service. Using matrix geometric method, Ning Zhao et al [13] computed sojourn time of an $MAP/PH/1$ queue with discretionary priority based on service stages. Ayyappan and Thamizhselvi [1] obtained time dependent results as well as steady state results for $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queueing system with priority services, working vacations, vacation interruption and delayed repair.

Madan [9] discussed an $M/G/1$ queueing system with compulsory server vacation. Saravanarajan and Chandrasekaran [17] discussed an $M^X/G/1$ feedback queue with two-phase service, compulsory server vacation and random breakdowns. A batch arrival retrial queue with Bernoulli vacation policy and server breakdown studied by Shweta Upadhyaya [19]. Krishnamoorthy et al. [7] presented orbital search in $M/G/1$ retrial queue with nonpersistent customers. Rajadurai et al. [15] extended to $M^X/G/1$ retrial queue with negative arrival, orbital search and feedback under Bernoulli vacation. Ke [5] studied optimal control of $M/G/1$ queueing system with server startup and two vacation types by using supplementary variable method. Madan and Hadjar [10] obtained time-dependent and steady state solution of an $M^X/G/1$ queueing system with server's long and short vacations. Recently, $M/G/1$ queue with compulsory short vacation and reneging during optional long vacation examined by Maragathasundari et al. [11], using supplementary variable method.

The idea of working breakdown is not the same as the idea of working vacations. This idea was introduce by Kalidass and Kasturi [6]. Using matrix-geometric method, Dong and Chen [3] studied $M/M/1$ queue with second optional service, working breakdowns and repairs and $MAP/M/1$ queue with working breakdowns analysed by Ye and Liu [14]. Li and Zhang [8] discussed an $M/G/1$ retrial G-queue with general retrial times and working breakdowns by using method of supplementary variable technique.

Shan and Zaiming [18] studied an $M/G/1$ queue with single working vacation and vacation interruption under Bernoulli schedule to obtain the stationary queue length at departure epochs, by using matrix analytic method and the stationary queue length at the arbitrary epoch, by using the supplementary variable technique. Rajadurai et al. [16] extended $M^X/G/1$ retrial G-queue with working vacations and vacation interruption under Bernoulli schedule to obtain explicit expressions of the performance measures, reliability measures and stochastic decomposition by using the supplementary variable technique. Tao Jiang and Baogui Xin [20] recently presented $M/M/1$ queue with working breakdowns and delaying repair under Bernoulli-schedule-controlled policy by using Matrix analytic method and Spectral expansion method to computed the performance measures and sojourn time of an arbitrary customer.

In this paper, we consider a single server batch arrival priority based retrial queueing system with orbital search, compulsory short vacation, optional long vacation which consists of working breakdown and repair under Bernoulli schedule controlled policy. We assume that customers arrive according to compound Poisson process in which high priority customers are assigned to type 1 and type 2 customers (retrial customers) are of low priority. At end of the each high priority service, the server compulsory go for a short vacation as well as completion of each low priority service server has an option to takes long vacation. At the completion of the short vacation the server search the orbit, if there are no customers in the high priority queue. Both type of customers balks during long vacation. Whenever, the system is subjected to breakdowns; the breakdowns occur according to Poisson process. Once the system breaks down, the server to decide stop the service immediately with probability $(1-p)$ or the service continues only to the current customer at a slower rate with probability p.

The paper is arranged as follows. The description of the mathematical model are mentioned in section 2, equations defining the model and the time dependent solutions are obtained in section 3. The steady state results are derived in section 4. The expected queue length and expected waiting time are derived in section 5 and 6, respectively. Some particular cases are mentioned in section 7 and in section 8, numerical results and their graphical representations are presented.

2. Mathematical Description

1. High priority and low priority customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda_h c_i dt$ $(i = 1,2,3,...)$ and $\lambda_l c_j dt$ $(j = 1,2,3,...)$ be the first order probability that a batch of i and j customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \le c_i \le 1, \sum_{i=1}^{\infty} c_i = 1$, $0 \le c_j \le 1, \sum_{j=1}^{\infty} c_j = 1$, and $\lambda_h > 0, \lambda_l > 0$ are the mean arrival rate for high and low priority customers entering into the system. Note that low priority customers will be served only when there are no high priority customers in the queue. Consequently, high priority customers have non-preemptive priority over low priority customers.
2. The retrial customers are the customers with low priority. A new batch of low priority customers who find the server idle begins to be one of the customer served immediately and remaining customer joins the orbit. A low priority customer in the orbit always returns to the orbit when he finds the server busy on his retrial attempt.
3. For each customer under high and low priority service provided by a single service channel on a 'First In - First Out' service basis.
4. The system may breakdown at any point of time during busy period and breakdowns are assumed to occur according to a Poisson process with breakdown rate $\alpha > 0$. However, the server either works slower than the regular service rate for only the current customer with probability p or stop service and repair immediately with probability $(1-p)$.
5. After completion of each high priority service, the server will take compulsory short vacation of random

length. If high priority queue is empty, the server goes in search of customers from the orbit after completion of short vacation with probability r or remains idle with probability $(1-r)$.

6. After completion of each low priority service, the server has an option to go for long vacation with probability θ and with probability $(1-\theta)$ it search next customer from orbit.

7. If the server is on long vacation then the arriving customers either join the queue with probability b or balks with probability $(1-b)$.

8. The stochastic processes involved in the system are assumed to be independent of each other.

2.1 Definitions

Let

- $N_1(t)$ be the high priority queue size at time t.
- $N_2(t)$ be the orbit size at time t.
- $B_i^0(t), i = 1,2,3,4$ be the elapsed service time of the high and low priority and working breakdown services respectively.
- $V_S^0(t)$ be the elapsed short vacation time.
- $V_L^0(t)$ be the elapsed long vacation time.
- $R^0(t)$ be the elapsed repair time for after working breakdown service.
- $R_1^0(t)$ be the elapsed repair time for high priority service.
- $R_2^0(t)$ be the elapsed repair time for low priority service.
- $Y(t)$ denote the server state at time t is given by,

$$
Y(t)=\begin{cases}
0, & \text{if the server is idle,}\\
1, & \text{if the server is busy with high priority service,}\\
2, & \text{if the server is busy with low priority service,}\\
3, & \text{if the server is busy with high priority service during working breakdown period,}\\
4, & \text{if the server is busy with low priority service during working breakdown period,}\\
5, & \text{if the server is under repair for after working breakdown service}\\
6, & \text{if the server is under repair for high priority,}\\
7, & \text{if the server is under repair for low priority,}\\
8, & \text{if the server is on short vacation,}\\
9, & \text{if the server is on long vacation.}
\end{cases}
$$

The high and low priority service time, working breakdown (WB) service time, vacation time, repair time follows general (arbitrary) distribution and the notions used for their

Table 1. Notations

Server state	CDF	PDF	LST	Hazard rate
Retrial	$A(t)$	$a(t)$	$\overline{M}(s)$	$\beta(\kappa)$
High priority service	$B_1(t)$	$b_1(t)$	$\overline{B}_1(s)$	$\mu_1(\kappa)$
Low priority service	$B_2(t)$	$b_2(t)$	$\overline{B}_2(s)$	$\mu_2(\kappa)$
High priority service for WB period	$B_1(t)$	$b_3(t)$	$\overline{B}_3(s)$	$\mu_3(\kappa)$
Low priority service for WB period	$B_4(t)$	$b_4(t)$	$\overline{B}_4(s)$	$\mu_4(\kappa)$
Repair for after WB service	$R(t)$	$r(t)$	$\overline{R}(s)$	$\eta(\kappa)$
Repair for high priority	$R_1(t)$	$r_1(t)$	$\overline{R}_1(s)$	$\eta_1(y)$
Repair for low priority	$R_2(t)$	$r_2(t)$	$\overline{R}_2(s)$	$\eta_2(y)$
Short vacation	$V_S(t)$	$v_S(t)$	$\overline{V}_S(s)$	$\gamma_1(\kappa)$
Long vacation	$V_L(t)$	$v_L(t)$	$\overline{V}_L(s)$	$\gamma_2(\kappa)$

cumulative distribution function (CDF), the probability density function (PDF) and the Laplace Stieltjes transform (LST) are given in table 1.

In the steady state, we assume that $B_i(0) = 0$, $B_i(\infty) = 1, V_S(0) = 0, V_S(\infty) = 1, V_L(0) = 0, V_L(\infty) = 1$, $R(0) = 0, R(\infty) = 1$ are continues at $\kappa = 0$ $(i = 1,2,3,4)$ and $R_1(0) = 0, R_1(\infty) = 1, R_2(0) = 0, R_2(\infty) = 1$, are continues at $y = 0$.

Next, we define the probability $I_0(t) = \Pr\{N_1(t) = 0, N_2(t) = 0, Y(t) = 0\}$ and probability densities

$$
\begin{aligned}
& I_{0,n}(t,\kappa)d\kappa = \Pr\{N_1(t) = 0, N_2(t) = n, Y(t) = 0; \kappa \le M^0(t) \\
& < \kappa + d\kappa\},\ n \ge 1,\\
& P^{(1)}_{m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 1; \kappa \le B_1^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& P^{(2)}_{m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 2; \kappa \le B_2^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& Q^{(1)}_{m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 3; \kappa \le B_3^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& Q^{(2)}_{m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 4; \kappa \le B_4^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& R_{m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 5; \kappa \le R^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& R^{(1)}_{m,n}(\kappa,y,t)dy = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 6; y \le R_1^0(t) \\
& < y + dy/B_1^0(t) = \kappa\}\ n \ge 0,\\
& R^{(2)}_{m,n}(\kappa,y,t)dy = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 7; y \le R_1^0(t) \\
& < y + dy/B_2^0(t) = \kappa\},\\
& V_{S,m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 8; \kappa \le V_S^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,\\
& V_{L,m,n}(\kappa,t)d\kappa = \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 9; \kappa \le V_L^0(t) \\
& < \kappa + d\kappa\}\ n \ge 0,
\end{aligned}
$$

for $\kappa \ge 0$, $y \ge 0$ $t \ge 0$, $m \ge 0$.

3. Equation Governing the System

Here, we construct a set of Kolmogorov forward equations using supplementary variable technique as follows:

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_h+\lambda_l)I_{0,0}(t) + (1-\theta)\int_0^\infty P_{0,0}^{(2)}(\kappa,t)\mu_2(\kappa)d\kappa + \int_0^\infty V_{S,0,0}(\kappa,t)\gamma_1(\kappa)d\kappa + \int_0^\infty V_{L,0,0}(\kappa,t)\gamma_2(\kappa)d\kappa + \int_0^\infty R_{0,0}(\kappa,t)\eta_3(\kappa)d\kappa, \tag{3.1}$$

$$\frac{\partial}{\partial t}I_{0,n}(\kappa,t) + \frac{\partial}{\partial\kappa}I_{0,n}(\kappa,t) = -(\lambda_h+\lambda_l+\beta(\kappa))I_{0,n}(\kappa,t), \tag{3.2}$$

$$\frac{\partial}{\partial t}P_{m,n}^{(1)}(\kappa,t) + \frac{\partial}{\partial\kappa}P_{m,n}^{(1)}(\kappa,t) = \int_0^\infty R_{m,n}^{(1)}(\kappa,y,t)\eta_1(y)dy - (\lambda_h+\lambda_l+\alpha+\mu_1(\kappa))P_{m,n}^{(1)}(\kappa,t) + \lambda_h\sum_{i=1}^m c_iP_{m-i,n}^{(1)}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jP_{m,n-j}^{(1)}(\kappa,t), \tag{3.3}$$

$$\frac{\partial}{\partial t}P_{m,n}^{(2)}(\kappa,t) + \frac{\partial}{\partial\kappa}P_{m,n}^{(2)}(\kappa,t) = \int_0^\infty R_{m,n}^{(2)}(\kappa,y,t)\eta_2(y)dy - (\lambda_h+\lambda_l+\alpha+\mu_2(\kappa))P_{m,n}^{(2)}(\kappa,t) + \lambda_h\sum_{i=1}^m c_iP_{m-i,n}^{(2)}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jP_{m,n-j}^{(2)}(\kappa,t), \tag{3.4}$$

$$\frac{\partial}{\partial t}Q_{m,n}^{(1)}(\kappa,t) + \frac{\partial}{\partial\kappa}Q_{m,n}^{(1)}(\kappa,t) = \lambda_h\sum_{i=1}^m c_iQ_{m-i,n}^{(1)}(\kappa,t) - (\lambda_h+\lambda_l+\mu_3(\kappa))Q_{m,n}^{(1)}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jQ_{m,n-j}^{(1)}(\kappa,t), \tag{3.5}$$

$$\frac{\partial}{\partial t}Q_{m,n}^{(2)}(\kappa,t) + \frac{\partial}{\partial\kappa}Q_{m,n}^{(2)}(\kappa,t) = \lambda_h\sum_{i=1}^m c_iQ_{m-i,n}^{(1)}(\kappa,t) - (\lambda_h+\lambda_l+\mu_4(\kappa))Q_{m,n}^{(2)}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jQ_{m,n-j}^{(2)}(\kappa,t), \tag{3.6}$$

$$\frac{\partial}{\partial t}R_{m,n}^{(1)}(\kappa,y,t) + \frac{\partial}{\partial\kappa}R_{m,n}^{(1)}(\kappa,y,t) = \lambda_h\sum_{i=1}^m c_iR_{m-i,n}^{(1)}(\kappa,y,t) - (\lambda_h+\lambda_l+\eta_1(\kappa))R_{m,n}^{(1)}(\kappa,y,t) + \lambda_l\sum_{j=1}^n c_jR_{m,n-j}^{(2)}(\kappa,y,t), \tag{3.7}$$

$$\frac{\partial}{\partial t}R_{m,n}^{(2)}(\kappa,y,t) + \frac{\partial}{\partial\kappa}R_{m,n}^{(2)}(\kappa,y,t) = \lambda_h\sum_{i=1}^m c_iR_{m-i,n}^{(2)}(\kappa,y,t) - (\lambda_h+\lambda_l+\eta_2(\kappa))R_{m,n}^{(2)}(\kappa,y,t) + \lambda_l\sum_{j=1}^n c_jR_{m,n-j}^{(2)}(\kappa,y,t), \tag{3.8}$$

$$\frac{\partial}{\partial t}R_{m,n}(\kappa,t) + \frac{\partial}{\partial\kappa}R_{m,n}(\kappa,t) = -(\lambda_h+\lambda_l+\eta_3(\kappa))R_{m,n}(\kappa,t) + \lambda_h\sum_{i=1}^m c_iR_{m-i,n}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jR_{m,n-j}(\kappa,t), \tag{3.9}$$

$$\frac{\partial}{\partial t}V_{S,m,n}(\kappa,t) + \frac{\partial}{\partial\kappa}V_{S,m,n}(\kappa,t) = \lambda_h\sum_{i=1}^m c_iV_{S,m-i,n}(\kappa,t) - (\lambda_h+\lambda_l+\gamma_1(\kappa))V_{S,m,n}(\kappa,t) + \lambda_l\sum_{j=1}^n c_jV_{S,m,n-j}(\kappa,t), \tag{3.10}$$

$$\frac{\partial}{\partial t}V_{L,m,n}(\kappa,t) + \frac{\partial}{\partial\kappa}V_{L,m,n}(\kappa,t) = \lambda_h b\sum_{i=1}^m c_iV_{L,m-i,n}(\kappa,t) - (\lambda_h+\lambda_l+\gamma_2(\kappa))V_{L,m,n}(\kappa,t) + \lambda_l b\sum_{j=1}^n c_jV_{L,m,n-j}(\kappa,t) + (1-b)\lambda_hV_{L,m,n} + (1-b)\lambda_lV_{L,m,n}(\kappa,t). \tag{3.11}$$

The above set of equation are to be solved under the following boundary conditions at $\kappa = 0$ are

$$I_{0,n}(0,t) = (1-r)\Big\{(1-\theta)\int_0^\infty P_{0,n}^{(2)}(\kappa,t)\mu_2(\kappa)d\kappa + \int_0^\infty V_{S,0,n}(\kappa,t)\gamma_1(\kappa)d\kappa + \int_0^\infty V_{L,0,n}(\kappa,t)\gamma_2(\kappa)d\kappa + \int_0^\infty R_{0,n}(\kappa,t)\eta_3(\kappa)d\kappa\Big\}, \tag{3.12}$$

$$P_{m,n}^{(1)}(0,t) = \lambda_hc_{m+1}I_{0n}(t) + (1-\theta)\int_0^\infty P_{m+1,n}^{(2)}(\kappa,t)\mu_2(\kappa)d\kappa + \int_0^\infty V_{S,m+1,n}(\kappa,t)\gamma_1(\kappa)d\kappa + \int_0^\infty V_{L,m+1,n}(\kappa,t)\gamma_2(\kappa)d\kappa + \int_0^\infty R_{m+1,n}(\kappa,t)\eta_3(\kappa)d\kappa, \tag{3.13}$$

$$P_{0,n}^{(2)}(0,t) = \lambda_lc_{n+1}I_{0,0}(t) + \lambda_l\sum_{j=1}^n c_j\int_0^\infty I_{0,n+1-j}(\kappa,t)d\kappa + \int_0^\infty I_{0,n+1}(\kappa,t)\beta(\kappa)d\kappa + r\Big\{\int_0^\infty R_{0,n+1}(\kappa,t)\eta_3(\kappa)d\kappa + \int_0^\infty V_{S,0,n+1}(\kappa,t)\gamma_1(\kappa)d\kappa + \int_0^\infty V_{L,0,n+1}(\kappa,t)\gamma_2(\kappa)d\kappa + (1-\theta)\int_0^\infty P_{0,n+1}^{(2)}(\kappa,t)\mu_2(\kappa)d\kappa\Big\}, \tag{3.14}$$

$$Q_{m,n}^{(i)}(0,t) = \alpha p\int_0^\infty P_{m,n}^{(i)}(\kappa,t)d\kappa, \quad i=1,2, \tag{3.15}$$

$$R_{m,n}^{(i)}(\kappa,0,t) = \alpha(1-p)\int_0^\infty P_{m,n}^{(i)}(\kappa,t)d\kappa, \quad i=1,2, \tag{3.16}$$

$$R_{m,n}(0,t) = \int_0^\infty Q_{m,n}^{(1)}(\kappa,t)\mu_3(\kappa)d\kappa + \int_0^\infty Q_{m,n}^{(2)}(\kappa,t)\mu_4(\kappa)d\kappa, \tag{3.17}$$

$$V_{S,m,n}(0,t) = \int_0^\infty P_{m,n}^{(1)}(\kappa,t)\mu_1(\kappa)d\kappa, \tag{3.18}$$

$$V_{L,m,n}(0,t) = \theta\int_0^\infty P_{m,n}^{(2)}(\kappa,t)\mu_2(\kappa)d\kappa. \tag{3.19}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are,

$$P_{m,n}^{(i)}(0) = Q_{m,n}^{(i)}(0) = R_{m,n}^{(i)}(0) = R_{m,n}(0) = V_{S,m,n}(0) = V_{L,m,n}(0) = 0,\ m,n\ge 0,\ I_{0,n}(0)=0,\ n\ge 1,\ i=1,2 \text{ and } I_{0,0}(0)=1. \tag{3.20}$$

The Probability Generating Function (PGF) of this model:

$$\left.\begin{aligned} I(\kappa,t,z_l) &= \sum_{n=1}^{\infty} z_l^n I(\kappa,t) \\ A(\kappa,t,z_h,z_l) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_h^m z_l^n A_{m,n}(\kappa,t) \\ A(\kappa,t,z_h) &= \sum_{m=0}^{\infty} z_h^m A_m(\kappa,t) \\ A(\kappa,t,z_l) &= \sum_{n=0}^{\infty} z_l^n A_n(\kappa,t) \end{aligned}\right\} \tag{3.21}$$

where $A = P^{(i)},\ Q^{(i)},\ V_S,\ V_L,\ R,\ R^{(i)}$. Which are convergent inside the circle given by $\mid z_h \mid \leq 1, \mid z_l \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Now taking Laplace transforms for equation (3.1) to (3.19) and using (3.20) , we get

$$(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s) - 1 = (1-\theta)\int_0^{\infty}\overline{P}^{(2)}_{0,0}(\kappa,s)\mu_2(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{S,0,0}(\kappa,s)\gamma_1(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{L,0,0}(\kappa,s)\gamma_2(\kappa)d\kappa + \int_0^{\infty}\overline{R}_{0,0}(\kappa,s)\eta_3(\kappa)d\kappa, \tag{3.22}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\beta(\kappa))\bar{I}_{0,n}(\kappa,s) = 0, \tag{3.23}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\alpha+\mu_1(\kappa))\overline{P}^{(1)}_{m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{P}^{(1)}_{m-i,n}(\kappa,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{P}^{(1)}_{m,n-j}(\kappa,s) + \int_0^{\infty}\overline{R}^{(1)}_{m,n}(\kappa,y,s)\eta_1(y)dy, \tag{3.24}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\alpha+\mu_2(\kappa))\overline{P}^{(2)}_{m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{P}^{(2)}_{m-i,n}(\kappa,s) + \overline{q}\lambda_l\sum_{j=1}^{n}c_j\overline{P}^{(2)}_{m,n-j}(\kappa,s) + \int_0^{\infty}\overline{R}^{(2)}_{m,n}(\kappa,y,s)\eta_2(y)dy, \tag{3.25}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\mu_3(\kappa))\overline{Q}^{(1)}_{m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{Q}^{(1)}_{m-i,n}(\kappa,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{Q}^{(1)}_{m,n-j}(\kappa,s), \tag{3.26}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\mu_4(\kappa))\overline{Q}^{(2)}_{m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{Q}^{(2)}_{m-i,n}(\kappa,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{Q}^{(2)}_{m,n-j}(\kappa,s), \tag{3.27}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\eta_1(\kappa))\overline{R}^{(1)}_{m,n}(\kappa,y,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}^{(1)}_{m-i,n}(\kappa,y,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{R}^{(1)}_{m,n-j}(\kappa,y,s), \tag{3.28}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\eta_2(\kappa))\overline{R}^{(2)}_{m,n}(\kappa,y,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}^{(2)}_{m-i,n}(\kappa,y,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{R}^{(2)}_{m,n-j}(\kappa,y,s), \tag{3.29}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\eta_3(\kappa))\overline{R}_{m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}_{m-i,n}(\kappa,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{R}_{m,n-j}(\kappa,s), \tag{3.30}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\gamma_1(\kappa))\overline{V}_{S,m,n}(\kappa,s) = \lambda_h\sum_{i=1}^{m}c_i\overline{V}_{S,m-i,n}(\kappa,s) + \lambda_l\sum_{j=1}^{n}c_j\overline{V}_{S,m,n-j}(\kappa,s), \tag{3.31}$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h b+\lambda_l b+\gamma_2(\kappa))\overline{V}_{L,m,n}(\kappa,s) = \lambda_h b\sum_{i=1}^{m}c_i\overline{V}_{L,m-i,n}(\kappa,s) + \lambda_l b\sum_{j=1}^{n}c_j\overline{V}_{L,m,n-j}(\kappa,s), \tag{3.32}$$

$$\bar{I}_{0,n}(0,s) = (1-r)\Big\{(1-\theta)\int_0^{\infty}\overline{P}^{(2)}_{0,n}(\kappa,s)\mu_2(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{S,0,n}(\kappa,s)\gamma_1(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{L,0,n}(\kappa,s)\gamma_2(\kappa)d\kappa + \int_0^{\infty}\overline{R}_{0,n}(\kappa,s)\eta_3(\kappa)d\kappa\Big\}, \tag{3.33}$$

$$\overline{P}^{(1)}_{m,n}(0,s) = \lambda_h c_{m+1}\bar{I}_{0n}(s) + \int_0^{\infty}\overline{R}_{m+1,n}(\kappa,s)\eta_3(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{S,m+1,n}(\kappa,s)\gamma_1(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{L,m+1,n}(\kappa,s)\gamma_2(\kappa)d\kappa + (1-\theta)\int_0^{\infty}\overline{P}^{(2)}_{m+1,n}(\kappa,s)\mu_2(\kappa)d\kappa, \tag{3.34}$$

$$\overline{P}^{(2)}_{0,n}(0,s) = \lambda_l c_{n+1}\bar{I}_{0,0}(s) + \lambda_l\sum_{j=1}^{n}c_j\int_0^{\infty}\bar{I}_{0,n+1-j}(\kappa,s)d\kappa + \int_0^{\infty}\bar{I}_{0,n+1}(\kappa,s)\beta(\kappa)d\kappa + r\Big\{\int_0^{\infty}\overline{R}_{0,n+1}(\kappa,s)\eta_3(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{S,0,n+1}(\kappa,s)\gamma_1(\kappa)d\kappa + \int_0^{\infty}\overline{V}_{L,0,n+1}(\kappa,s)\gamma_2(\kappa)d\kappa + (1-\theta)\int_0^{\infty}\overline{P}^{(2)}_{0,n+1}(\kappa,s)\mu_2(\kappa)d\kappa\Big\}, \tag{3.35}$$

$$\overline{R}^{(i)}_{m,n}(\kappa,0,s) = \alpha(1-p)\int_0^{\infty}\overline{P}^{(i)}_{m,n}(\kappa,s)d\kappa,\ \ i=1,2, \tag{3.36}$$

$$\overline{Q}^{(i)}_{m,n}(0,s) = \alpha p\int_0^{\infty}\overline{P}^{(i)}_{m,n}(\kappa,s)d\kappa,\ \ i=1,2, \tag{3.37}$$

$$\overline{R}_{m,n}(0,s) = \int_0^{\infty}\overline{Q}^{(1)}_{m,n}(\kappa,s)\mu_3(\kappa)d\kappa + \int_0^{\infty}\overline{Q}^{(2)}_{m,n}(\kappa,s)\mu_4(\kappa)d\kappa, \tag{3.38}$$

$$\overline{V}_{S,m,n}(0,s) = \int_0^{\infty}\overline{P}^{(1)}_{m,n}(\kappa,s)\mu_1(\kappa)d\kappa, \tag{3.39}$$

$$\overline{V}_{L,m,n}(0,s) = \theta\int_0^{\infty}\overline{P}^{(2)}_{m,n}(\kappa,s)\mu_2(\kappa)d\kappa. \tag{3.40}$$

We multiply equations (3.23) to (3.32) by z_l^n, summing over n from 0 to ∞ and using (3.21), we have

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l+\beta(\kappa))\bar{I}_0(\kappa,s,z_l)=0, \quad (3.41)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\alpha+\mu_1(\kappa))\overline{P}_m^{(1)}(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{P}_{m-i}^{(1)}(\kappa,s,z_l)+\int_0^\infty\overline{R}_m^{(1)}(\kappa,y,s,z_l)\eta_1(y)dy, \quad (3.42)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\alpha+\mu_2(\kappa))\overline{P}_m^{(2)}(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{P}_{m-i}^{(2)}(\kappa,s,z_l)+\int_0^\infty\overline{R}_m^{(2)}(\kappa,y,s,z_l)\eta_2(y)dy, \quad (3.43)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\mu_3(\kappa))\overline{Q}_m^{(1)}(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{Q}_{m-i}^{(1)}(\kappa,s,z_l), \quad (3.44)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\mu_4(\kappa))\overline{Q}_m^{(2)}(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{Q}_{m-i}^{(2)}(\kappa,s,z_l), \quad (3.45)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\eta_3(\kappa))\overline{R}_m(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}_{m-i}(\kappa,s,z_l), \quad (3.46)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\eta_1(\kappa))\overline{R}_m^{(1)}(\kappa,y,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}_{m-i}^{(1)}(\kappa,y,s,z_l), \quad (3.47)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\eta_2(\kappa))\overline{R}_m^{(2)}(\kappa,y,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{R}_{m-i}^{(2)}(\kappa,y,s,z_l), \quad (3.48)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h+\lambda_l(1-C(z_l))+\gamma_1(\kappa))\overline{V}_{S,m}(\kappa,s,z_l) = \lambda_h\sum_{i=1}^{m}c_i\overline{V}_{S,m-i}(\kappa,s,z_l), \quad (3.49)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_hb+\lambda_lb(1-C(z_l))+\gamma_2(\kappa))\overline{V}_{L,m}(\kappa,s,z_l) = \lambda_hb\sum_{i=1}^{m}c_i\overline{V}_{L,m-i}(\kappa,s,z_l). \quad (3.50)$$

We multiply equations (3.42) to (3.50) by z_h^m summing over n from 0 to ∞ and use PGF, we get

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\alpha+\mu_1(\kappa)) \times\overline{P}^{(1)}(\kappa,s,z_h,z_l)=\int_0^\infty\overline{R}^{(1)}(\kappa,y,s,z_h,z_l)\eta_1(y)dy, \quad (3.51)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\alpha+\mu_2(\kappa)) \times\overline{P}^{(2)}(\kappa,s,z_h,z_l)=\int_0^\infty\overline{R}^{(2)}(\kappa,y,s,z_h,z_l)\eta_2(y)dy, \quad (3.52)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\mu_3(\kappa)) \times\overline{Q}^{(1)}(\kappa,s,z_h,z_l)=0, \quad (3.53)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\mu_3(\kappa)) \times\overline{Q}^{(2)}(\kappa,s,z_h,z_l)=0, \quad (3.54)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\eta_1(\kappa)) \times\overline{R}^{(1)}(\kappa,s,z_h,z_l)=0, \quad (3.55)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\eta_2(\kappa)) \times\overline{R}^{(2)}(\kappa,s,z_h,z_l)=0, \quad (3.56)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\eta_3(\kappa)) \times\overline{R}(\kappa,s,z_h,z_l)=0, \quad (3.57)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_h(1-C(z_h))+\lambda_l(1-C(z_l))+\gamma_1(\kappa)) \times\overline{V}_S(\kappa,s,z_h,z_l)=0, \quad (3.58)$$

$$(\frac{\partial}{\partial\kappa}+s+\lambda_hb(1-C(z_h))+\lambda_lb(1-C(z_l))+\gamma_2(\kappa)) \times\overline{V}_L(\kappa,s,z_h,z_l)=0, \quad (3.59)$$

Next, we multiply both sides of equation (3.33) by z_l^n summing over n from 1 to ∞, we obtain

$$\bar{I}_0(0,s,z_l)=(1-r)\Big\{(1-\theta)\int_0^\infty\overline{P}_0^{(2)}(\kappa,s,z_l)\mu_2(\kappa)d\kappa +\int_0^\infty\overline{V}_{S,0}(\kappa,s,z_l)\gamma_1(\kappa)d\kappa+\int_0^\infty\overline{V}_{L,0}(\kappa,s,z_l)\gamma_2(\kappa)d\kappa +\int_0^\infty\overline{R}_0(\kappa,s,z_l)\eta_3(\kappa)d\kappa\Big\}-(s+\lambda_h+\lambda_l)+1, \quad (3.60)$$

Next, we multiply both sides of equation (3.35) by z_l^{n+1} summing over n from 0 to ∞, we have

$$z_l\overline{P}_0^{(2)}(0,s,z_l)=\lambda_lC(z_l)\bar{I}_{0,0}(s)+\lambda_lC(z_l)\int_0^\infty\bar{I}_0(\kappa,s,z_l)d\kappa +\int_0^\infty\bar{I}_0(\kappa,s,z_l)\beta(\kappa)d\kappa+r\Big\{\int_0^\infty\overline{R}_0(\kappa,s,z_l)\eta_3(\kappa)d\kappa\Big\} +\int_0^\infty\overline{V}_{S,0}(\kappa,s,z_l)\gamma_1(\kappa)d\kappa+\int_0^\infty\overline{V}_{L,0}(\kappa,s,z_l)\gamma_2(\kappa)d\kappa +(1-\theta)\int_0^\infty\overline{P}_0^{(2)}(\kappa,s,z_l)\mu_2(\kappa)d\kappa, \quad (3.61)$$

We multiply both sides of equations (3.34) and (3.36) to (3.40) by z_l^n summing over n from 0 to ∞, we have

$$\overline{P}_m^{(1)}(0,s,z_l) = \lambda_h c_{m+1}\overline{I}_0(s,z_l) + \int_0^\infty \overline{R}_{m+1}(\kappa,s,z_l)\eta_3(\kappa)d\kappa + \int_0^\infty \overline{V}_{S,m+1}(\kappa,s,z_l)\gamma_1(\kappa)d\kappa + \int_0^\infty \overline{V}_{L,m+1}(\kappa,s,z_l)\gamma_2(\kappa)d\kappa + (1-\theta)\int_0^\infty \overline{P}_{m+1}^{(2)}(\kappa,s,z_l)\mu_2(\kappa)d\kappa, \quad (3.62)$$

$$\overline{R}_m^{(i)}(\kappa,0,s,z_l) = \alpha(1-p)\int_0^\infty \overline{P}_m^{(i)}(\kappa,s,z_l)d\kappa, \;\; i=1,2, \quad (3.63)$$

$$\overline{Q}_m^{(i)}(0,s,z_l) = \alpha p\int_0^\infty \overline{P}_m^{(i)}(\kappa,s,z_l)d\kappa, \;\; i=1,2, \quad (3.64)$$

$$\overline{R}_m(0,s,z_l) = \int_0^\infty \overline{Q}_m^{(1)}(\kappa,s,z_l)\mu_3(\kappa)d\kappa + \int_0^\infty \overline{Q}_m^{(2)}(\kappa,s,z_l)\mu_4(\kappa)d\kappa, \quad (3.65)$$

$$\overline{V}_{S,m}(0,s,z_l), = \int_0^\infty \overline{P}_m^{(1)}(\kappa,s,z_l)\mu_1(\kappa)d\kappa, \quad (3.66)$$

$$\overline{V}_{L,m}(0,s,z_l) = \theta\int_0^\infty \overline{P}_m^{(2)}(\kappa,s,z_l)\mu_2(\kappa)d\kappa. \quad (3.67)$$

We multiply both sides of equation (3.62), by z_h^{m+1} summing over n from 0 to ∞, we have

$$z_h\overline{P}^{(1)}(0,s,z_h,z_l) = (1-\theta)\int_0^\infty \overline{P}^{(2)}(\kappa,s,z_h,z_l)\mu_2(\kappa)d\kappa + \int_0^\infty \overline{V}_S(\kappa,s,z_h,z_l)\gamma_1(\kappa)d\kappa + \int_0^\infty \overline{V}_L(\kappa,s,z_hz_l)\gamma_2(\kappa)d\kappa + \int_0^\infty \overline{R}(\kappa,s,z_h,z_l)\eta_3(\kappa)d\kappa + \lambda_h C(z_h)\overline{I}_0(s,z_l) - \Big\{(1-\theta)\int_0^\infty \overline{P}_0^{(2)}(\kappa,s,z_l)\mu_2(\kappa)d\kappa + \int_0^\infty \overline{V}_{S,0}(\kappa,s,z_l)\gamma_1(\kappa)d\kappa + \int_0^\infty \overline{V}_{L,0}(\kappa,s,z_l)\gamma_2(\kappa)d\kappa + \int_0^\infty \overline{R}_0(\kappa,s,z_l)\eta_3(\kappa)d\kappa\Big\}, \quad (3.68)$$

We multiply both sides of equations (3.63) to (3.67), by z_h^m summing over n from 0 to ∞, we have

$$\overline{R}^{(i)}(\kappa,0,s,z_h,z_l) = \alpha(1-p)\int_0^\infty \overline{P}^{(i)}(\kappa,s,z_h,z_l)d\kappa, i=1,2, \quad (3.69)$$

$$\overline{Q}^{(i)}(0,s,z_h,z_l) = \alpha p\int_0^\infty \overline{P}^{(i)}(\kappa,s,z_h,z_l)d\kappa, \;\; i=1,2, \quad (3.70)$$

$$\overline{R}(0,s,z_h,z_l) = \int_0^\infty \overline{Q}^{(1)}(\kappa,s,z_h,z_l)\mu_3(\kappa)d\kappa + \int_0^\infty \overline{Q}^{(2)}(\kappa,s,z_h,z_l)\mu_4(\kappa)d\kappa, \quad (3.71)$$

$$\overline{V}_S(0,s,z_h,z_l) = \int_0^\infty \overline{P}^{(1)}(\kappa,s,z_h,z_l)\mu_1(\kappa)d\kappa, \quad (3.72)$$

$$\overline{V}_L(0,s,z_h,z_l) = \theta\int_0^\infty \overline{P}^{(2)}(\kappa,s,z_h,z_l)\mu_2(\kappa)d\kappa. \quad (3.73)$$

Integration of equations (3.41) and (3.53) to (3.59) between 0 and κ we get

$$\overline{I}_0(\kappa,s,z_l) = \overline{I}_0(0,s,z_l)e^{-(s+\lambda_h+\lambda_l)\kappa - \int_0^\kappa \beta(t)dt}, \quad (3.74)$$

$$\overline{Q}^{(1)}(\kappa,s,z_h,z_l) = \overline{Q}^{(1)}(0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \mu_3(t)dt}, \quad (3.75)$$

$$\overline{Q}^{(2)}(\kappa,s,z_h,z_l) = \overline{Q}^{(2)}(0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \mu_4(t)dt}, \quad (3.76)$$

$$\overline{R}^{(1)}(\kappa,y,s,z_h,z_l) = \overline{R}^{(1)}(\kappa,0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \eta_1(t)dt}, \quad (3.77)$$

$$\overline{R}^{(2)}(\kappa,y,s,z_h,z_l) = \overline{R}^{(2)}(\kappa,0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \eta_2(t)dt}, \quad (3.78)$$

$$\overline{R}(\kappa,s,z_h,z_l) = \overline{R}(0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \eta_3(t)dt}, \quad (3.79)$$

$$\overline{V}_S(\kappa,s,z_h,z_l) = \overline{V}_S(0,s,z_h,z_l)e^{-A(s,z)\kappa - \int_0^\kappa \gamma_1(t)dt}, \quad (3.80)$$

$$\overline{V}_L(\kappa,s,z_h,z_l) = \overline{V}_L(0,s,z_h,z_l)e^{-B(s,z)\kappa - \int_0^\kappa \gamma_2(t)dt}. \quad (3.81)$$

Multiplying equations (3.74) to (3.81) by $\beta(\kappa)$, $\mu_3(\kappa)$, $\mu_4(\kappa)$, $\eta_1(\kappa)$, $\eta_2(\kappa)$, $\eta_3(\kappa)$, $\gamma_1(\kappa)$ and $\gamma_2(\kappa)$ respectively,

$$\int_0^\infty \overline{I}_0(\kappa,s,z_l)\beta(\kappa)d\kappa = \overline{I}_0(0,s,z_l)\overline{M}(s+\lambda_h+\lambda_l), \quad (3.82)$$

$$\int_0^\infty \overline{Q}^{(1)}(\kappa,s,z_h,z_l)\mu_3(\kappa)d\kappa = \overline{Q}^{(1)}(0,s,z_h,z_l)\overline{B}_3(A(s,z)), \quad (3.83)$$

$$\int_0^\infty \overline{Q}^{(2)}(\kappa,s,z_h,z_l)\mu_4(\kappa)d\kappa = \overline{Q}^{(2)}(0,s,z_h,z_l)\overline{B}_4(A(s,z)), \quad (3.84)$$

$$\int_0^\infty \overline{R}^{(1)}(\kappa,y,s,z_h,z_l)\eta_1(\kappa)d\kappa = \overline{R}^{(1)}(\kappa,0,s,z_h,z_l)\overline{R}_1(A(s,z)), \quad (3.85)$$

$$\int_0^\infty \overline{R}^{(2)}(\kappa,y,s,z_h,z_l)\eta_2(\kappa)d\kappa = \overline{R}^{(2)}(\kappa,0,s,z_h,z_l)\overline{R}_2(A(s,z)), \quad (3.86)$$

$$\int_0^\infty \overline{R}(\kappa,s,z_h,z_l)\eta_3(\kappa)d\kappa = \overline{R}(0,s,z_h,z_l)\overline{R}(A(s,z)), \quad (3.87)$$

$$\int_0^\infty \overline{V}_S(\kappa,s,z_h,z_l)\gamma_1(\kappa)d\kappa = \overline{V}_S(0,s,z_h,z_l)\overline{V}_S(A(s,z)), \quad (3.88)$$

$$\int_0^\infty \overline{V}_L(\kappa,s,z_h,z_l)\gamma_2(\kappa)d\kappa = \overline{V}_L(0,s,z_h,z_l)\overline{V}_L(B(s,z)). \quad (3.89)$$

where,

$$A(s,z) = s + \lambda_h(1-C(z_h)) + \lambda_l(1-C(z_l))$$
$$B(s,z) = s + \lambda_h b(1-C(z_h)) + \lambda_l b(1-C(z_l))$$

at $z_h = 0$, equations (3.85) to (3.89), becomes

$$\int_0^\infty \overline{R}_0^{(1)}(\kappa,y,s,z_l)\eta_1(\kappa)d\kappa = \overline{R}_0^{(1)}(\kappa,0,s,z_l)\overline{R}_1(A_1(s,z)), \quad (3.90)$$

$$\int_0^\infty \overline{R}_0^{(2)}(\kappa,y,s,z_l)\eta_2(\kappa)d\kappa = \overline{R}_0^{(2)}(\kappa,0,s,z_l)\overline{R}_2(A_1(s,z)), \quad (3.91)$$

$$\int_0^\infty \overline{R}_0(\kappa,s,z_l)\eta_3(\kappa)d\kappa = \overline{R}_0(0,s,z_l)\overline{R}(A_1(s,z)), \quad (3.92)$$

$$\int_0^\infty \overline{V}_{S,0}(\kappa,s,z_l)\gamma_1(\kappa)d\kappa = \overline{V}_{S,0}(0,s,z_l)\overline{V}_S(A_1(s,z)), \quad (3.93)$$

$$\int_0^\infty \overline{V}_{L,0}(\kappa,s,z_l)\gamma_2(\kappa)d\kappa = \overline{V}_{L,0}(0,s,z_l)\overline{V}_L(B_1(s,z)), \quad (3.94)$$

where,

$$A_1(s,z) = s+\lambda_h+\lambda_l(1-C(z_l))$$
$$B_1(s,z) = s+\lambda_h b+\lambda_l b(1-C(z_l))$$

substitute equation (3.69) into (3.85) and (3.86), we get

$$\int_0^\infty \overline{R}^{(i)}(\kappa,y,s,z_h,z_l)\eta_1(\kappa)d\kappa = \alpha(1-p)\overline{P}^{(i)}(\kappa,s,z_h,z_l)\overline{R}_i(A(s,z)) \quad i=1,2 \quad (3.95)$$

then (3.51) and (3.52) becomes,

$$\{\frac{\partial}{\partial\kappa}+A(s,z)+\alpha[1-(1-p)\overline{R}^{(i)}(A(s,z))]+\mu_i(\kappa)\} \times\overline{P}^{(i)}(\kappa,s,z_h,z_l)=0, \ \ i=1,2. \quad (3.96)$$

solving (3.96) we get

$$\overline{P}^{(i)}(\kappa,s,z_h,z_l) = \overline{P}^{(i)}(0,s,z_h,z_l)\ e^{-(\phi_i(s,z))\kappa-\int_0^\kappa\mu_i(t)dt}, i=1,2, \quad (3.97)$$

at $z_h=0$

$$\int_0^\infty \overline{P}_0^{(i)}(\kappa,s,z_l)\mu_i(\kappa)d\kappa = \overline{P}_0^{(i)}(0,s,z_l)\overline{B}_i(\psi_1(s,z)), \ i=1,2. \quad (3.98)$$

where,

$$\phi_i(s,z) = A(s,z)+\alpha[1-(1-p)\overline{R}^{(i)}(A(s,z))]$$
$$\psi_i(s,z) = A_1(s,z)+\alpha[1-(1-p)\overline{R}^{(i)}(A_1(s,z))]$$

Now, using equations (3.82) to (3.98) into (3.60), (3.61) and (3.68), we get

$$\begin{aligned}\bar{I}_0(0,s,z_l) &= 1-(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)+\overline{P}_0^{(1)}(0,s,z_l)(1-r)\\ &\times\Big\{\overline{B}_1(\psi_1(s,z))\overline{V}_S(A_1(s,z))+\alpha p[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\\ &\times\overline{B}_3(A_1(s,z))\overline{R}(A_1(s,z))\Big\}+\overline{P}_0^{(2)}(0,s,z_l)(1-r)\\ &\times\Big\{(1-\theta)\overline{B}_2(\psi_2(s,z))+\theta\overline{B}_2(\psi_2(s,z))\overline{V}_L(B_1(s,z))\\ &+\alpha p[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\overline{B}_4(A_1(s,z))\overline{R}(A_1(s,z))\Big\}, \quad (3.99)\end{aligned}$$

$$\begin{aligned}&\overline{P}^{(1)}(0,s,z_h,z_l)\{z_h-\overline{B}_1(\phi_1(s,z))\overline{V}_S(A(s,z))\\ &-\alpha p[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}]\overline{B}_3(A(s,z))\overline{R}(A(s,z))\}\\ &=\lambda_h C(z_h)\bar{I}_0(0,s,z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l))}{s+\lambda_h+\lambda_l}]-\overline{P}_0^{(1)}(0,s,z_l)\\ &\times\Big\{\overline{B}_1(\psi_1(s,z))\overline{V}_S(A_1(s,z))+\alpha p[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\\ &\times\overline{B}_3(A_1(s,z))\overline{R}(A_1(s,z))\Big\}+\overline{P}_0^{(2)}(0,s,z_l)\Big\{(1-\theta)\\ &\times\{\overline{B}_2(\phi_2(s,z))-\overline{B}_2(\psi_2(s,z))\}+\theta\{\overline{B}_2(\phi_2(s,z))\\ &\times\overline{V}_L(B(s,z))-\overline{B}_2(\phi_2(s,z))\overline{V}_L(B_1(s,z))\}+\alpha p\\ &\times\{[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}]\overline{B}_4(A(s,z))\overline{R}(A(s,z))\\ &-[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\overline{B}_4(A_1(s,z))\overline{R}(A_1(s,z))\}\Big\}, \quad (3.100)\end{aligned}$$

$$\begin{aligned}&\overline{P}_0^{(2)}(0,s,z_l)\Big\{z_l-r\{(1-\theta)\overline{B}_2(\psi_2(s,z))+\theta\overline{B}_2(\psi_2(s,z))\\ &\times\overline{V}_L(B_1(s,z))+\alpha p[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\overline{B}_4(A_1(s,z))\\ &\times\overline{R}(A_1(s,z))\}\Big\}=\lambda_l C(z_l)\bar{I}_{0,0}(s)+\bar{I}_0(0,s,z_l)\\ &\times\Big\{\overline{M}(s+\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\Big\}\\ &+r\overline{P}_0^{(1)}(0,s,z_l)\Big\{\overline{B}_1(\psi_1(s,z))\overline{V}_S(A_1(s,z))\\ &+\alpha p[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\overline{B}_3(A_1(s,z))\overline{R}(A_1(s,z))\Big\} \quad (3.101)\end{aligned}$$

We have to solve (3.99), (3.100) and (3.101). Letting $z_h = g(z_l)$ in (3.100), we get

$$\begin{aligned}&\overline{P}_0^{(1)}(0,s,z_l)\Big\{\overline{B}_1(\psi_1(s,z))\overline{V}_S(A_1(s,z))+\alpha p[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\\ &\times\overline{B}_3(A_1(s,z))\overline{R}(A_1(s,z))\Big\}=\lambda_h C(g(z_l))\bar{I}_0(0,s,z_l)\\ &\times[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]+\overline{P}_0^{(2)}(0,s,z_l)\Big\{(1-\theta)\{\overline{B}_2(\sigma_2(s,z))\\ &-\overline{B}_2(\psi_2(s,z))\}+\theta\{\overline{B}_2(\sigma_2(s,z))\overline{V}_L(B_2(s,z))\\ &-\overline{B}_2(\psi_2(s,z))\overline{V}_L(B_1(s,z))\}+\alpha p\{[\frac{1-\overline{B}_2(\sigma_2(s,z))}{\sigma_2(s,z)}]\\ &\times\overline{B}_4(A_2(s,z))\overline{R}(A_2(s,z))-[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\\ &\times\overline{B}_4(A_1(s,z))\overline{R}(A_1(s,z))\}\Big\}, \quad (3.102)\end{aligned}$$

substituting the above in (3.99), we get

$$\begin{aligned}&\bar{I}_0(0,s,z_l)\Big\{1-(1-r)\lambda_h C(g(z_l))[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\Big\}\\ &=1-(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)+(1-r)\overline{P}_0^{(2)}(0,s,z_l)\end{aligned}$$

$$\times\Big\{(1-\theta)\overline{B}_2(\sigma_2(s,z))+\theta\overline{B}_2(\sigma_2(s,z))\overline{V}_L(B_2(s,z)) + \alpha p\big[\frac{1-\overline{B}_2(\sigma_2(s,z))}{\sigma_2(s,z)}\big]\overline{B}_4(A_2(s,z))\overline{R}(A_2(s,z))\Big\}, \quad (3.103)$$

substituting (3.102) and (3.103) in (3.101), we get

$$\overline{P}_0^{(2)}(0,s,z_l)=\frac{N_2(s,z)}{D(s,z)} \quad (3.104)$$

substituting (3.104) in (3.103), we get

$$\bar{I}_0(0,s,z_l)=\frac{N_1(s,z)}{D(s,z)} \quad (3.105)$$

Finally substituting (3.102), (3.104) and (3.105) in (3.100), we get

$$\overline{P}^{(1)}(0,s,z_h,z_l)=\frac{N_0(s,z)}{D_0(s,z)} \quad (3.106)$$

where,

$$N_0(s,z)=\lambda_h\{C(z_h)-C(g(z_l))\}\bar{I}_0(0,s,z_l)\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big] + \overline{P}_0^{(2)}(0,s,z_l)\Big\{(1-\theta)\big\{\overline{B}_2(\phi_2(s,z))-\overline{B}_2(\sigma_2(s,z))\big\} + \theta\big\{\overline{B}_2(\phi_2(s,z))\overline{V}_L(B(s,z))-\overline{B}_2(\sigma_2(s,z))\overline{V}_L(B_2(s,z))\big\} + \alpha p\big\{\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big]\overline{B}_4(A(s,z))\overline{R}(A(s,z)) - \big[\frac{1-\overline{B}_2(\sigma_2(s,z))}{\sigma_2(s,z)}\big]\overline{B}_4(A_2(s,z))\overline{R}(A_2(s,z))\big\}\Big\}$$

$$N_1(s,z)=\{1-(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)\}\{z_l-rT(s,z_l)\} + (1-r)T(s,z_l)\{\lambda_l C(z_l)\bar{I}_{0,0}(s)\}$$

$$N_2(s,z)=\{1-(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)\}\{\overline{M}(s+\lambda_h+\lambda_l) + \lambda_l C(z_l)\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big]\}+\{\lambda_l C(z_l)\bar{I}_{0,0}(s)\} \times\{1-(1-r)\lambda_h C(g(z_l))\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big]\}$$

$$D_0(s,z)=z_h-\alpha p\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big]\overline{B}_3(A(s,z))\overline{R}(A(s,z)) - \overline{B}_1(\phi_1(s,z))\overline{V}_S(A(s,z))$$

$$D(s,z)=\{1-(1-r)\lambda_h C(g(z_l))\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big]\} \times\{z_l-rT(s,z_l)\}-(1-r)T(s,z_l)\{\overline{M}(s+\lambda_h+\lambda_l) + \lambda_l C(z_l)\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big]\}$$

$$T(s,z_l)=(1-\theta)\overline{B}_2(\sigma_2(s,z))+\theta\overline{B}_2(\sigma_2(s,z))\overline{V}_L(B_2(s,z)) + \alpha p\big[\frac{1-\overline{B}_2(\sigma_2(s,z))}{\sigma_2(s,z)}\big]\overline{B}_4(A_2(s,z))\overline{R}(A_2(s,z)).$$

the other boundary conditions are

$$\overline{Q}^{(i)}(0,s,z_h,z_l)=\alpha p\,\overline{P}^{(i)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_i(\phi_i(s,z))}{\phi_i(s,z)}\big], \quad (3.107)$$

$$\overline{R}^{(i)}(0,s,z_h,z_l)=\alpha(1-p)\overline{P}^{(i)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_i(\phi_i(s,z))}{\phi_i(s,z)}\big], \quad (3.108)$$

$$\overline{R}(0,s,z_h,z_l)=\alpha p\,\sum_{i=1}^{2}\overline{P}^{(i)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_i(\phi_i(s,z))}{\phi_i(s,z)}\big], \quad (3.109)$$

$$\overline{V}_L(0,s,z_h,z_l)=\overline{P}^{(1)}(0,s,z_h,z_l)\overline{B}_1(\phi_1(s,z)), \quad (3.110)$$

$$\overline{V}_S(0,s,z_l)=\theta\overline{P}_0^{(2)}(0,s,z_l)\overline{B}_2(\psi_1(s,z)). \quad (3.111)$$

Theorem 3.1. *The probability generating function of the Laplace transforms of the number of customers in the high and low priority queue while the system was in regular service, working breakdown service, repair and vacation are given by*

$$\bar{I}_0(s,z_l)=\bar{I}_0(0,s,z_l)\big[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}\big], \quad (3.112)$$

$$\overline{P}^{(1)}(s,z_h,z_l)=\overline{P}^{(1)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big], \quad (3.113)$$

$$\overline{P}^{(2)}(s,z_h,z_l)=\overline{P}_0^{(2)}(0,s,z_l)\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big], \quad (3.114)$$

$$\overline{Q}^{(1)}(s,z_h,z_l)=\alpha\overline{P}^{(1)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big] \times\big[\frac{1-\overline{B}_3(A(s,z))}{A(s,z)}\big], \quad (3.115)$$

$$\overline{Q}^{(2)}(s,z_h,z_l)=\alpha\overline{P}_0^{(2)}(0,s,z_l)\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big] \times\big[\frac{1-\overline{B}_4(A(s,z))}{A(s,z)}\big], \quad (3.116)$$

$$\overline{R}(s,z_h,z_l)=\alpha p\,\overline{P}^{(1)}(0,s,z_h,z_l)\overline{B}_3(A(s,z))\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big] \times\big[\frac{1-\overline{R}(A(s,z))}{A(s,z)}\big]+\alpha p\,\overline{P}_0^{(2)}(0,s,z_l)\overline{B}_4(A(s,z)) \times\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big]\big[\frac{1-\overline{R}(A(s,z))}{A(s,z)}\big], \quad (3.117)$$

$$\overline{R}^{(1)}(s,z_h,z_l)=\alpha(1-p)\,\overline{P}^{(1)}(0,s,z_h,z_l)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big] \times\big[\frac{1-\overline{R}^{(1)}(A(s,z))}{A(s,z)}\big], \quad (3.118)$$

$$\overline{R}^{(2)}(s,z_h,z_l)=\alpha(1-p)\,\overline{P}_0^{(2)}(0,s,z_l)\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big] \times\big[\frac{1-\overline{R}^{(2)}(A(s,z))}{A(s,z)}\big], \quad (3.119)$$

$$\overline{V}_S(s,z_h,z_l) = \overline{P}^{(1)}(0,s,z_h,z_l)\overline{B}_1(\phi_1(s,z)) \times [\frac{1-\overline{V}_S(A(s,z))}{A(s,z)}], \quad (3.120)$$

$$\overline{V}_L(s,z_h,z_l) = \theta \overline{P}_0^{(2)}(0,s,z_l)\overline{B}_2(\phi_2(s,z)) \times [\frac{1-\overline{V}_L(B(s,z))}{B(s,z)}]. \quad (3.121)$$

4. Steady State Analysis: Limiting Behaviour

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $I_{0,0}$, we use the normalizing condition

$$I_{0,0} + I_0(1) + \sum_{i=1}^{2}\{P^{(i)}(1,1) + Q^{(i)}(1,1) + R^{(i)}(1,1)\} + R(1,1) + V_S(1,1) + V_L(1,1) = 1.$$

Let $W_q(z_h,z_l)$ be the probability generating function of the queue size irrespective of the state of the system.

$$W_q(z_h,z_l) = I_{0,0}\frac{Nr(z_h,z_l)}{Dr(z_h,z_l)}, \quad (4.1)$$

where,

$$Nr(z_h,z_l) = \phi_2(z)\sigma_2(z)N_1(z)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\{\lambda_h\big(C(z_h) - C(g(z_l))\big)B(z)S_1(z_h,z_l) + A(z)B(z)F_2(z_h,z_l)\} + N_2(z) \times \{B(z)S_1(z_h,z_l)F_1(z_h,z_l) + \sigma_2(z)S_2(z_h,z_l)F_2(z_h,z_l)\},$$

$$Dr(z_h,z_l) = A(z)B(z)\phi_2(z)\sigma_2(z)F_2(z_h,z_l),$$

$$S_1(z_h,z_l) = [1-\overline{B}_1(\phi_1(z))]\{A(z) + \alpha p[1-\overline{B}_3(A(z)) \times \overline{R}(A(z))] + \alpha(1-p)[1-\overline{R}^{(1)}(A(z))]\} + \phi_1(z)\overline{B}_1(\phi_1(z)) \times [1-\overline{V}_S(A(z))],$$

$$S_2(z_h,z_l) = B(z)[1-\overline{B}_2(\phi_2(z))]\{A(z) + \alpha p[1-\overline{B}_4(A(z)) \times \overline{R}(A(z))] + \alpha(1-p)[1-\overline{R}^{(2)}(A(z))]\} + \theta\phi_2(z)A(z)\overline{B}_2(\phi_2(z))[1-\overline{V}_L(B(z))],$$

$$F_1(z_h,z_l) = (1-\theta)\phi_2(z)\sigma_2(z)\{\overline{B}_2(\phi_2(z)) - \overline{B}_2(\sigma_2(z))\} + \theta\phi_2(z)\sigma_2(z)\{\overline{B}_2(\phi_2(z))\overline{V}_L(B(z)) - \overline{B}_2(\sigma_2(z)) \times \overline{V}_L(B_2(z))\} + \alpha p\{\sigma_2(z)[1-\overline{B}_2(\phi_2(z))]\overline{B}_4(A(z))\overline{R}(A(z)) - \phi_2(z)[1-\overline{B}_2(\sigma_2(z))]\overline{B}_4(A_2(z))\overline{R}(A_2(z))\},$$

$$F_2(z_h,z_l) = \phi_1(z)[z_h - \overline{B}_1(\phi_1(z))\overline{V}_S(A(z))] - \alpha p \times [1-\overline{B}_1(\phi_1(z))]\overline{B}_3(A(z))\overline{R}(A(z)).$$

Now using the normalizing condition, we get

$$I_{0,0} = \frac{D(1)\alpha p\left\{\begin{array}{l}[1-\overline{B}_1(\alpha p)]\{[\phi_1'(1)] + \alpha p[A'(1)] \\ \times\big(E(B_3)+E(R)\big)\} + \alpha p\{1 \\ +[A'(1)]\overline{B}_1(\alpha p)E(V_S)\}\end{array}\right\}}{\Omega}, \quad (4.2)$$

$$\begin{aligned}\Omega = & N_1(1)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\alpha p\Big\{\lambda_h\big(1-E(X_1)E(X)\big) \\ & \times\big\{[1-\overline{B}_1(\alpha p)]\{1+\alpha p\big(E(B_3)+E(R)\big) \\ & +\alpha(1-p)E(R_1)\} + \alpha p\overline{B}_1(\alpha p)E(V_S)\big\} + \big\{[1-\overline{B}_1(\alpha p)] \\ & \times\{[\phi_1'(1)] + \alpha p[A'(1)]\big(E(B_3)+E(R)\big)\} + \alpha p \\ & \times\{1+[A'(1)]\overline{B}_1(\alpha p)E(V_S)\}\big\}\Big\} + N_2(1)\Big\{\big\{[1-\overline{B}_1(\alpha p)] \\ & \times\{1+\alpha p\big(E(B_3)+E(R)\big) + \alpha(1-p)E(R_1)\} \\ & +\alpha p\overline{B}_1(\alpha p)E(V_S)\big\}\big\{\theta\alpha p\big([B_2'(1)] - [B'(1)]\big)\overline{B}_2(\alpha p) \\ & \times E(V_L) + [1-\overline{B}_2(\alpha p)]\{\big([\sigma_2'(1)] - [\phi_2'(1)]\big) \\ & +(\alpha p)\big([A_2'(1)] - [A'(1)]\big)\big(E(B_4)+E(R)\big)\}\big\} \\ & +\big\{[1-\overline{B}_2(\alpha p)]\{1+\alpha p\big(E(B_4)+E(R)\big) \\ & +\alpha(1-p)E(R_2)\} + \theta(\alpha p)\overline{B}_2(\alpha p)E(V_L)\big\} \\ & \times\big\{[1-\overline{B}_1(\alpha p)]\{[\phi_1'(1)] + \alpha p[A'(1)]\big(E(B_3)+E(R)\big)\} \\ & +\alpha p\{1+[A'(1)]\overline{B}_1(\alpha p)E(V_S)\}\big\}\Big\} + D(1)\alpha p \\ & \times\Big\{[1-\overline{B}_1(\alpha p)]\{[\phi_1'(1)] + \alpha p[A'(1)]\big(E(B_3)+E(R)\big)\} \\ & +\alpha p\{1+[A'(1)]\overline{B}_1(\alpha p)E(V_S)\}\Big\}.\end{aligned}$$

5. The Average Queue Length

The mean number of customers in the high priority queue under the steady state is

$$L_{q_1} = \frac{d}{dz_h}W_{q_1}(z_h,1)|_{z_h=1} \quad (5.1)$$

and the mean number of customers in the low priority queue under the steady state is

$$L_{q_2} = \frac{d}{dz_l}W_{q_2}(1,z_l)|_{z_l=1} \quad (5.2)$$

then

$$L_{q_1} = \frac{DR'''(1)NR^{(iv)}(1) - DR^{(iv)}(1)NR'''(1)}{4(DR'''(1))^2}, \quad (5.3)$$

$$L_{q_2} = \frac{dr'''(1)nr^{(iv)}(1) - dr^{(iv)}(1)nr'''(1)}{4(dr'''(1))^2}, \quad (5.4)$$

$$NR'''(1) = 6(\alpha p)N_1(1)\{(\alpha p)[\tilde{B}'(1)]\lambda_h[1-E(X_1)]E(X)S_1'(1) + [\tilde{A}'(1)][\tilde{B}'(1)]F_2'(1)\} + N_2(1)\{6[\tilde{B}'(1)]S_1'(1)F_1'(1) + 3(\alpha p)S_2''(1)F_2'(1)\},$$

$$NR^{(iv)}(1)=12(\alpha p)N_1(1)\Big\{2\big\{[\tilde{B}^{'}(1)][\tilde{\phi}_2^{'}(1)]\times\lambda_h[1-E(X_1)]E(X)S_1^{'}(1)\big\}+(\alpha p)\big\{\lambda_h[1-E(X_1)]E(X)\times[\tilde{B}^{''}(1)]S_1^{'}(1)+[\tilde{B}^{'}(1)]\lambda_h\{[1-E(X_1)]E(X^2)-E(X_1^2)E(X)\}S_1^{'}(1)+[\tilde{B}^{'}(1)][1-E(X_1)]E(X)S_1^{''}(1)+[\tilde{A}^{''}(1)][\tilde{B}^{'}(1)]F_2^{'}(1)+[\tilde{A}^{'}(1)][\tilde{B}^{''}(1)]F_2^{'}(1)+[\tilde{A}^{'}(1)][\tilde{B}^{''}(1)]F_2^{''}(1)\big\}\Big\}+N_2(1)\Big\{12\big\{[\tilde{B}^{''}(1)]S_1^{'}(1)F_1^{'}(1)+[\tilde{B}^{'}(1)]S_1^{''}(1)F_1^{'}(1)+[\tilde{B}^{'}(1)]S_1^{'}F_1^{''}\big\}+(\alpha p)\big\{4S_2^{'''}F_2^{'}(1)+6S_2^{''}(1)F_2^{''}(1)\big\}\Big\},$$

$$DR^{'''}(1)=6(\alpha p)[\tilde{A}^{'}(1)][\tilde{B}^{'}(1)]F_2^{'}(1),$$

$$DR^{(iv)}(1)=24(\alpha p)[\tilde{A}^{'}(1)][\tilde{B}^{'}(1)][\tilde{\phi}_2^{'}(1)]F_2^{'}(1)+12(\alpha p)^2\big\{[\tilde{A}^{''}(1)][\tilde{B}^{'}(1)]F_2^{'}(1)+[\tilde{A}^{'}(1)][\tilde{B}^{''}(1)]F_2^{'}(1)+[\tilde{A}^{'}(1)][\tilde{B}^{'}(1)]F_2^{''}(1)\big\},$$

$$S_1^{'}(1)=[1-\overline{B}_1(\alpha p)][\tilde{A}^{'}(1)]\big\{1+(\alpha p)\big(E(B_3)+E(R)\big)+\alpha(1-p)E(R_1)\big\}+(\alpha p)\overline{B}_1(\alpha p)[\tilde{A}^{'}(1)]E(V_S),$$

$$S_1^{''}(1)=[\tilde{A}^{''}(1)][1-\overline{B}_1(\alpha p)]\big\{1+(\alpha p)\big(E(B_3)+E(R)\big)+\alpha(1-p)E(R_1)\big\}-2[\tilde{A}^{'}(1)][\tilde{\phi}_1^{'}(1)]\overline{B}_1^{'}(\alpha p)\big\{1+(\alpha p)\times\big(E(B_3)+E(R)\big)+\alpha(1-p)E(R_1)\big\}-[\tilde{A}^{'}(1)]^2\times[1-\overline{B}_1(\alpha p)]\big\{(\alpha p)\big(E(B_3^2)+E(R^2)+2E(B_3)E(R)\big)+\alpha(1-p)E(R_1^2)\big\}+2[\tilde{A}^{'}(1)][\tilde{\phi}_1^{'}(1)]\overline{B}_1(\alpha p)E(V_S)+2(\alpha p)[\tilde{A}^{'}(1)][\tilde{\phi}_1^{'}(1)]\overline{B}_1^{'}(\alpha p)E(V_S)+(\alpha p)[\tilde{A}^{''}(1)]\times\overline{B}_1(\alpha p)E(V_S)-(\alpha p)[\tilde{A}^{'}(1)]^2\overline{B}_1(\alpha p)E(V_S^2),$$

$$S_2^{''}(1)=2[\tilde{A}^{'}(1)][\tilde{B}^{'}(1)]\big\{[1-\overline{B}_2(\alpha p)]\big\{1+(\alpha p)\big(E(B_4)+E(R)\big)+\alpha(1-p)E(R_2)\big\}+\theta(\alpha p)\overline{B}_2(\alpha p)E(V_L)\big\},$$

$$S_2^{'''}(1)=3[1-\overline{B}_2(\alpha p)]\big\{1+(\alpha p)\big(E(B_4)+E(R)\big)+\alpha(1-p)E(R_2)\big\}\big\{[\tilde{A}^{'}(1)][\tilde{B}^{''}(1)]+[\tilde{A}^{''}(1)][\tilde{B}^{'}(1)]\big\}-6[\tilde{A}^{'}(1)][\tilde{B}^{'}(1)][\tilde{\phi}_2^{'}(1)]\overline{B}_2^{'}(\alpha p)\big\{1+(\alpha p)\big(E(B_4)+E(R)\big)+\alpha(1-p)E(R_2)\big\}-3[\tilde{A}^{'}(1)]^2[\tilde{B}^{'}(1)][1-\overline{B}_2(\alpha p)]\big\{(\alpha p)\times\big(E(B_4^2)+E(R^2)+2E(B_4)E(R)\big)+\alpha(1-p)E(R_2^2)\big\},$$

$$F_1^{'}(1)=-(\alpha p)[1-\overline{B}_2(\alpha p)]\big\{[\tilde{\phi}_2^{'}(1)]+(\alpha p)[\tilde{A}^{'}(1)]\big(E(B_4)+E(R)\big)\big\}-\theta(\alpha p)^2[\tilde{B}^{'}(1)]\overline{B}_2(\alpha p)E(V_L),$$

$$F_1^{''}(1)=-(\alpha p)[1-\overline{B}_2(\alpha p)]\big\{[\tilde{\phi}_2^{''}(1)]+(\alpha p)[\tilde{A}^{''}(1)]\times\big(E(B_4)+E(R)\big)-(\alpha p)[\tilde{A}^{'}(1)]^2\big(E(B_4^2)+E(R^2)+2E(B_4)E(R)\big)\big\}+2(\alpha p)[\tilde{\phi}_2^{'}(1)]\overline{B}_2^{'}(\alpha p)\big\{[\tilde{\phi}_2^{'}(1)]+(\alpha p)\{[\tilde{A}^{'}(1)]\big(E(B_4)+E(R)\big)-\theta[\tilde{B}^{'}(1)]E(V_L)\}\big\},$$

$$F_2^{'}(1)=[1-\overline{B}_1(\alpha p)]\big\{[\tilde{\phi}_1^{'}(1)]+(\alpha p)[\tilde{A}^{'}(1)]\big(E(B_3)+E(R)\big)\big\}+(\alpha p)\{1+[\tilde{A}^{'}(1)]\overline{B}_1(\alpha p)E(V_S)\},$$

$$F_2^{''}(1)=2[\tilde{\phi}_1^{'}(1)]+[1-\overline{B}_1(\alpha p)]\big\{[\tilde{\phi}_1^{''}(1)]+(\alpha p)\times\{[\tilde{A}^{''}(1)]\big(E(B_3)+E(R)\big)+[\tilde{A}^{'}(1)]^2\big(E(B_3^2)+E(R^2)+2E(B_3)E(R)\big)\}\big\},$$

$$nr^{'''}(1)=6N_1(1)(\alpha p)^2[\underline{B}^{'}(1)][\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\times\big\{[\underline{A}^{'}(1)]f_2^{'}(1)-\lambda_hE(X_1)E(X)s_1^{'}(1)\big\}+6N_2(1)\times\big\{[\underline{B}^{'}(1)]s_1^{'}(1)f_1^{'}(1)+(\alpha p)s_2^{''}(1)f_2^{'}(1)\big\},$$

$$nr^{iv}(1)=24(\alpha p)\big\{N_1(1)\big([\underline{\phi}_2^{'}(1)]+[\sigma_2^{'}(1)]\big)+(\alpha p)N_1^{'}(1)\big\}\times[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}][\underline{B}^{'}(1)]\big\{[\underline{A}^{'}(1)]f_2^{'}(1)-\lambda_hE(X_1)E(X)\times s_1^{'}(1)\big\}+12(\alpha p)^2N_1(1)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\Big\{-\lambda_hE(X_1)\times E(X)\{[\underline{B}^{''}(1)]s_1^{'}(1)+[\underline{B}^{'}(1)]s_1^{''}(1)\}-\lambda_h[E(X_1^2)E(X)+(E(X_1))^2E(X)][\underline{B}^{'}(1)]s_1^{'}(1)+[\underline{A}^{''}(1)][\underline{B}^{'}(1)]F_2^{'}(1)+[\underline{A}^{'}(1)][\underline{B}^{''}(1)]F_2^{'}(1)+[\underline{A}^{'}(1)][\underline{B}^{'}(1)]F_2^{''}(1)\Big\}+24N_2^{'}(1)\times\big\{[\underline{B}^{'}(1)]s_1^{'}(1)f_1^{'}(1)+(\alpha p)s_2^{''}(1)f_2^{'}(1)\big\}+N_2(1)\times\big\{12\{[\underline{B}^{''}(1)]s_1^{'}(1)f_1^{'}(1)+[\underline{B}^{'}(1)]s_1^{''}(1)f_1^{'}(1)+[\underline{B}^{'}(1)]s_1^{'}(1)f_1^{''}(1)+\sigma_2^{'}(1)s_2^{''}(1)f_2^{'}(1)\}+4(\alpha p)s_2^{'''}(1)f_2^{'}(1)+6(\alpha p)s_2^{''}(1)f_2^{''}(1)\big\},$$

$$dr^{'''}(1)=6(\alpha p)^2D(1)[\underline{A}^{'}(1)][\underline{B}^{'}(1)]f_2^{'}(1),$$

$$dr^{iv}(1)=24(\alpha p)\big\{D(1)\big([\underline{\phi}_2^{'}(1)]+[\sigma_2^{'}(1)]\big)+(\alpha p)D^{'}(1)\big\}\times\big\{[\underline{A}^{'}(1)][\underline{B}^{'}(1)]f_2^{'}(1)\big\}+6(\alpha p)^2D(1)\big\{[\underline{A}^{''}(1)][\underline{B}^{'}(1)]\times f_2^{'}(1)+[\underline{A}^{'}(1)][\underline{B}^{''}(1)]f_2^{'}(1)+[\underline{A}^{'}(1)][\underline{B}^{'}(1)]f_2^{''}(1)q\big\},$$

$$s_1^{'}(1)=[\underline{A}^{'}(1)][1-\overline{B}_1(\alpha p)]\big\{1+(\alpha p)\big(E(B_3)+E(R)\big)+\alpha(1-p)E(R_1)\big\}+(\alpha p)[\underline{A}^{'}(1)]\overline{B}_1(\alpha p)E(V_S),$$

$$s_1^{''}(1)=\big\{[1-\overline{B}_1(\alpha p)][\underline{A}^{''}(1)]-2[\underline{A}^{'}(1)][\phi_1^{'}]\overline{B}_1^{'}(\alpha p)\big\}\times\big\{1+(\alpha p)\big(E(B_3)+E(R)\big)+\alpha(1-p)E(R_1)\big\}-[\underline{A}^{'}(1)]^2[1-\overline{B}_1(\alpha p)]\big\{(\alpha p)\big(E(B_3^2)+E(R^2)+2E(B_3)E(R)\big)+\alpha(1-p)E(R_1^2)\big\}+2[\underline{A}^{'}(1)][\underline{\phi}_1^{'}(1)]\times\overline{B}_1(\alpha p)E(V_S)+2(\alpha p)[\underline{A}^{'}(1)][\underline{\phi}_1^{'}(1)]\overline{B}_1^{'}(\alpha p)E(V_S)+(\alpha p)[\underline{A}^{''}(1)]\overline{B}_1(\alpha p)E(V_S)-(\alpha p)[\underline{A}^{'}(1)]^2\overline{B}_1(\alpha p)E(V_S^2),$$

$$s_2^{''}(1)=2[\underline{A}^{'}(1)][\underline{B}^{'}(1)][1-\overline{B}_2(\alpha p)]\big\{1+(\alpha p)\big(E(B_4)+E(R)\big)+\alpha(1-p)E(R_2)\big\}+2(\alpha p)[\underline{A}^{'}(1)][\underline{B}^{'}(1)]\times\overline{B}_2(\alpha p)E(V_L),$$

$$s_2^{'''}1=\big\{3[\underline{A}^{'}(1)][\underline{B}^{''}(1)][1-\overline{B}_2(\alpha p)]-6[\underline{A}^{'}(1)][\underline{B}^{'}(1)]\times[\underline{\phi}_2^{'}(1)]\overline{B}_2^{'}(\alpha p)+3[\underline{A}^{''}(1)][\underline{B}^{'}(1)][1-\overline{B}_2(\alpha p)]\big\}\times\big\{1+(\alpha p)\big(E(B_4)+E(R)\big)+\alpha(1-p)E(R_2)\big\}$$

$$
\begin{aligned}
&-3[\underline{A}'(1)]^2[\underline{B}'(1)][1-\overline{B}_2(\alpha p)]\{(\alpha p)\big(E(B_4^2)+E(R^2)\\
&+2E(B_4)E(R)\big)+\alpha(1-p)E(R_2^2)\}+6\theta[\underline{A}'(1)][\underline{B}'(1)]\\
&\times E(V_L)\{[\underline{\phi}_2'(1)]\overline{B}_2(\alpha p)+(\alpha p)[\underline{\phi}_2'(1)]\overline{B}_2'(\alpha p)\}\\
&+3(\alpha p)\overline{B}_2(\alpha p)E(V_L)\{[\underline{A}''(1)][\underline{B}'(1)]+[\underline{A}'(1)][\underline{B}''(1)]\}\\
&-3(\alpha p)[\underline{A}'(1)][\underline{B}'(1)]^2\overline{B}_2(\alpha p)E(V_L^2),
\end{aligned}
$$

$$
\begin{aligned}
f_1'(1)=&(\alpha p)^2\Big\{\theta\overline{B}(\alpha p)E(V_L)\big([B_2'(1)]-[\underline{B}'(1)]\big)\\
&+[1-\overline{B}_2(\alpha p)]\big([A_2'(1)]-[\underline{A}'(1)]\big)\big(E(B_4)+E(R)\big)\Big\}\\
&+\alpha p\big([\sigma_2'(1)]-[\underline{\phi}_2'(1)]\big)[1-\overline{B}_2(\alpha p)],
\end{aligned}
$$

$$
\begin{aligned}
f_1''(1)=&(\alpha p)[1-\overline{B}_2(\alpha p)]\Big\{\big([\sigma_2''(1)]-[\underline{\phi}_2''(1)]\big)\\
&+2\big([\underline{\phi}_2'(1)][A_2'(1)]-[\sigma_2'(1)][\underline{A}'(1)]\big)\big(E(B_4)+E(R)\big)\\
&+(\alpha p)\big([A_2''(1)]-[\underline{A}''(1)]\big)\big(E(B_4)+E(R)\big)+(\alpha p)\\
&\big([A_2'(1)]^2-[\underline{A}'(1)]^2\big)\big(E(B_4^2)+E(R^2)+2E(B_4)E(R)\big)\Big\}\\
&+2(\alpha p)\overline{B}_2'(\alpha p)\Big\{(\alpha p)+[\underline{\phi}_2'(1)]^2-[\sigma_2'(1)]^2\Big\}-\theta(\alpha p)\\
&\times\Big\{2\big(\underline{\phi}_2'(1)+\sigma_2'(1)\big)\big([\underline{B}'(1)]-[B_2'(1)]\big)\overline{B}_2(\alpha p)E(V_L)\\
&+(\alpha p)\overline{B}_2(\alpha p)\big([\underline{B}''(1)]-[B_2''(1)]\big)E(V_L)+2(\alpha p)\\
&\times\big([\underline{\phi}_2'(1)][\underline{B}'(1)]-[\sigma_2'(1)][B_2'(1)]\big)\overline{B}_2'(\alpha p)E(V_L)\\
&+(\alpha p)\overline{B}_2(\alpha p)\big([\underline{B}'(1)]^2-[B_2'(1)]^2\big)E(V_L^2)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
f_2'(1)=&(\alpha p)[\underline{A}'(1)]\overline{B}_1(\alpha p)E(V_S)+[1-\overline{B}_1(\alpha p)]\\
&\times\Big\{[\underline{\phi}_1'(1)]+(\alpha p)[\underline{A}'(1)]\big(E(B_3)+E(R)\big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
f_2''(1)=&[\underline{\phi}_1''(1)][1-\overline{B}_1(\alpha p)]-2[\underline{\phi}_1'(1)]^2\overline{B}_1'(\alpha p)\\
&+2[\underline{A}'(1)][\underline{\phi}_1'(1)]\overline{B}_1(\alpha p)E(V_S)+2(\alpha p)[\underline{A}'(1)][\underline{\phi}_1'(1)]\\
&\times\overline{B}_1'(\alpha p)\big(E(V_S)-E(B_3)-E(R)\big)+(\alpha p)[\underline{A}''(1)]\\
&\times\{\overline{B}_1(\alpha p)E(V_S)+[1-\overline{B}_1(\alpha p)]\big(E(B_3)+E(R)\big)\}\\
&-(\alpha p)[\underline{A}'(1)]^2\{\overline{B}_1(\alpha p)E(V_S^2)+[1-\overline{B}_1(\alpha p)]\\
&\times\big(E(B_3^2)+E(R^2)+2E(B_3)E(R)\big)\}.
\end{aligned}
$$

6. The Expected Waiting Time in the Queue

Expected waiting time of high priority customers is

$$W_{q_1}=\frac{L_{q_1}}{\lambda_h} \tag{6.1}$$

Expected waiting time of low priority customers is

$$W_{q_2}=\frac{L_{q_2}}{\lambda_l} \tag{6.2}$$

7. Particular Cases

Case 1: If there is no low priority, no breakdown, no compulsory short vacation, no balking. i.e $\lambda_l=0$, $\alpha=0$, $b=0$. Then, our model can be reduced to $M^X/G/1$ queue.

$$W(z_h)=\frac{-I_{0,0}[1-\overline{B}_1(\lambda_h(1-C(z_h)))]}{z_h-\overline{B}_1(\lambda_h(1-C(z_h)))}.$$

The above result coincides with the result of Medhi. J [12].

Case 2: If there is no high priority, no breakdown, no balking, no orbit search. i.e $\lambda_h=0$, $\alpha=0$, $b=0$, $r=0$. Then, our model can be reduced to $M^X/G/1$ retrial queue under Bernoulli vacation schedule.

$$I(z_l)=\frac{\left\{\begin{array}{r}-I_{0,0}\{z_l-C(z_l)\overline{B}_2(\phi(z_l))[1-\theta+\theta\overline{V}_L(\phi(z_l))]\}\\ \times[1-\overline{M}(\lambda_l)]\end{array}\right\}}{\left\{\begin{array}{l}z_l-\overline{B}_2(\phi(z_l))[1-\theta+\theta\overline{V}_L(\phi(z_l))]\\ \times\{C(z_l)[1-\overline{M}(\lambda_l)]+\overline{M}(\lambda_l)\}\end{array}\right\}}$$

$$P(z_l)=\frac{-I_{0,0}\overline{M}(\lambda_l)[1-\overline{B}_2(\phi(z_l))]}{\left\{\begin{array}{l}z_l-\overline{B}_2(\phi(z_l))[1-\theta+\theta\overline{V}_L(\phi(z_l))]\\ \times\{C(z_l)[1-\overline{M}(\lambda_l)]+\overline{M}(\lambda_l)\}\end{array}\right\}}$$

$$V_L(z)=\frac{-\theta I_{0,0}\overline{M}(\lambda_l)\overline{B}_2(\phi(z_l))[1-\overline{V}_L(\phi(z_l))]}{\left\{\begin{array}{l}z_l-\overline{B}_2(\phi(z_l))[1-\theta+\theta\overline{V}_L(\phi(z_l))]\\ \times\{C(z_l)[1-\overline{M}(\lambda_l)]+\overline{M}(\lambda_l)\}\end{array}\right\}}$$

where, $\phi(z_l)=\lambda_l[1-C(z_l)]$. The above result coincides with the result of Choudhury and Ke [4].

8. Numerical Result

The above queueing model is analysed numerically with the following assumption. We consider that the service time in regular busy period, service time in working breakdown period, vacation time and repair time are to be exponential and Erlangian of order two.

Table 2 and 3 shows that an increase the high priority arrival rate, decreases the idle time and increases the expected queue length and waiting time of high priority and low priority queues for the arbitrary values, we choose $\lambda_l=1,\mu=18,\mu_w=8,\alpha=11,\eta=3.5,\eta_1=3,\eta_2=3,\theta=0.1,\gamma_1=15,\gamma_2=20,p=0.9,b=0.8,r=0.3,\beta=24,E(X)=1,E(X(X-1))=0$. While λ_h, varies from 0.1 to 0.4 such that the stability condition is satisfied. In figure 1 and 2, we compare the result for exponential and Erlang-2 distributions for idle time, expected orbit size under the value of table 2 and 3.

In Figures 3-4 Three dimensional graphs are illustrated. In Fig. 4, the surface displays an downward trend as expected for increasing the value of slow service rate (μ_w) and repair rate (η) against the expected orbit size (Lq_2). In Fig. 3, the surface displays upward trend as expected for increasing the value of slow service rate (μ_w) and repair rate (η) against the server idle probability under the value of the table 4.

Table 2. Impact of λ_h on various queue characteristics

Exponential						
λ_h	I_0	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.9756	0.0244	0.0138	1.3488	0.1380	1.3488
0.2	0.9612	0.0388	0.0287	1.8143	0.1434	1.8143
0.3	0.9524	0.0476	0.0445	2.0960	0.1482	2.0960
0.4	0.9472	0.0528	0.0609	2.2613	0.1522	2.2613

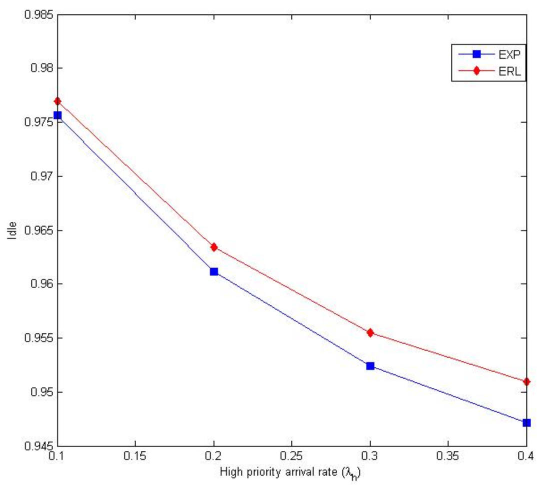

Figure 1. Idle versus high priority arrival rate (λ_h)

Table 3. Impact of λ_h on various queue characteristics

Erlang-2						
λ_h	I_0	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.9769	0.0225	0.0143	1.1565	0.1427	1.1565
0.2	0.9635	0.0365	0.0300	1.4223	0.1502	1.4223
0.3	0.9555	0.0445	0.0472	1.4895	0.1572	1.4895
0.4	0.9510	0.0490	0.0655	1.5280	0.1637	1.5280

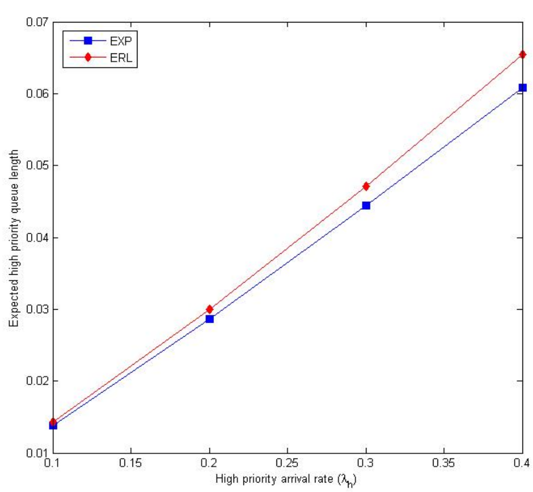

Figure 2. L_{q_1} versus high priority arrival rate (λ_h)

Table 4. Impact of η with varying value of μ_w on various queue characteristics

η	μ_w	I_0	Lq_1	Lq_2	Wq_1	Wq_2
7.0	10	0.8380	3.2293	7.9118	2.1528	3.1647
	11	0.8387	3.2270	7.8669	2.1514	3.1468
	12	0.8394	3.2251	7.8281	2.1500	3.1312
	13	0.8399	3.2233	7.7942	2.1489	3.1177
	14	0.8403	3.2218	7.7644	2.1479	3.1058
7.5	10	0.8388	3.2271	7.8651	2.1514	3.1460
	11	0.8395	3.2247	7.8183	2.1498	3.1273
	12	0.8401	3.2227	7.7779	2.1485	3.1112
	13	0.8406	3.2209	7.7426	2.1472	3.0971
	14	0.8411	3.2192	7.7117	2.1462	3.0847
8.0	10	0.8395	3.2251	7.8225	2.1500	3.1290
	11	0.8402	3.2226	7.7741	2.1484	3.1096
	12	0.8408	3.2205	7.7323	2.1470	3.0929
	13	0.8413	3.2186	7.6958	2.1457	3.0783
	14	0.8417	3.2169	7.6638	2.1446	3.0655
8.5	10	0.8401	3.2232	7.7837	2.1488	3.1135
	11	0.8408	3.2206	7.7337	2.1471	3.0935
	12	0.8414	3.2184	7.6907	2.1456	3.0763
	13	0.8419	3.2164	7.6532	2.1443	3.0613
	14	0.8423	3.2147	7.6203	2.1431	3.0481
9.0	10	0.8406	3.2214	7.7481	2.1476	3.0992
	11	0.8413	3.2188	7.6968	2.1459	3.0787
	12	0.8419	3.2165	7.6526	2.1443	3.0611
	13	0.8423	3.2145	7.6142	2.1430	3.0457
	14	0.8427	3.2127	7.5805	2.1418	3.0322

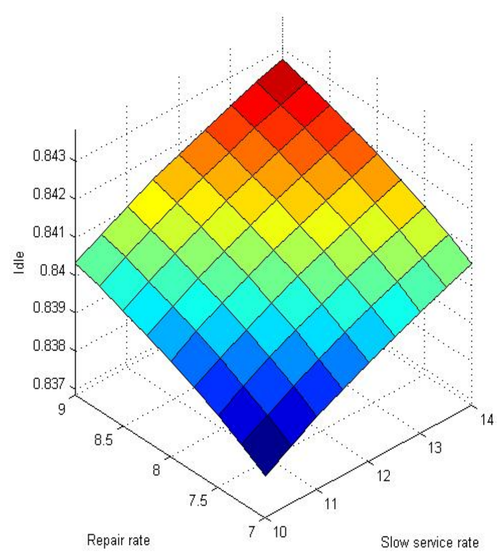

Figure 3. Idle versus slow service rate and repair rate

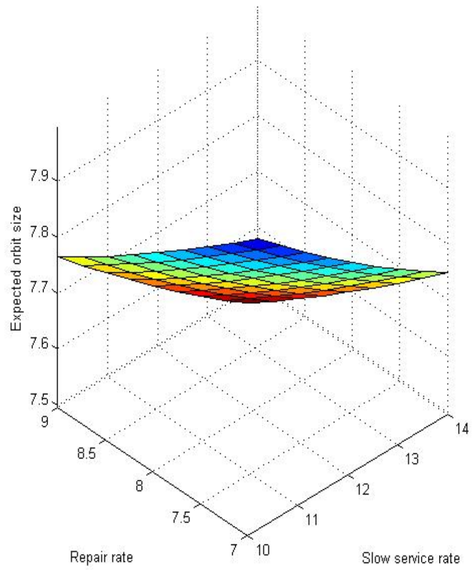

Figure 4. Expected orbit size versus slow service rate and repair rate

9. Conclusions

In this paper, a single server batch arrival priority based retrial queueing system with orbital search, compulsory short vacation, optional long vacation which consists of working breakdown and repair under Bernoulli schedule controlled policy is analyzed. The probability generating function of the queue size distribution at an arbitrary time is obtained and some performance measures are calculated. Finally, we present some numerical examples to study the effect of various parameters. For future research, the discretionary priority based on service consider the similar model.

ber:MHR-02-23-200-44".

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