Malaya Journal of Matematik, Vol. 7, No. 2, 259-263, 2019

https://doi.org/10.26637/MJM0702/0017

Some results on fractional semilinear impulsive integro-differential equations

S. Suresh[1]* and G. Thamizhendhi[2]

Abstract

This paper is devoted to study the existence and uniqueness of solution for non-local impulsive fractional integro-differential equations involving the Caputo fractional derivative in a Banach Space. The arguments are based upon contraction mapping principle and Krasnoselskii's fixed point theorem.

Keywords

Fractional integro-differential equations, Fixed point methods, Impulsive condition.

AMS Subject Classification

26A33, 34A37, 34K05.

[1] *Department of Mathematics, Kongu Arts and Science College, Erode, Tamilnadu, India.*

[2] *Department of Mathematics, Vellalar College for Women, Erode, Tamilnadu, India.*

***Corresponding author**: [1] sureshkongucas@gmail.com; [2]gthamil@rediffmail.com

Article History: Received **2** February **2019**; Accepted **11** April **2019**

Contents

1. Introduction

The theory of fractional differential equations is a new branch of mathematics by valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [6, 11–13, 16, 19]). There has been a significant development in fractional differential and partial differential equations in recent years, see the monographs of Kilbas *et al* [14], Miller and Ross [17], Samko *et al* [21] and see [1–3, 8, 10, 18, 23] and the references therein.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, see for instance the monographs by Benchohra *et al* [9], Lakshmikantham *et al* [15], and Samoilenko and Perestyuk [22], K.Balachandran and J.Y.Park [5] and the references therein.

Archana chauhan et al [4] studied the existence of mild solutions for impulsive fractional- order semilinear evolution equations with nonlocal conditions and Bashir Ahmad & S. Sivasundara [7] investigated the some existence results for fractional integro - differetial equations with nonlinear conditions in a Banach space.

In [25] investigated fractional evolution equations with nonlocal conditions of the form

$$D^{\alpha}x(t) = Ax(t) + f(t,x(t)), \quad t \neq t_i, t \in J,\ 0 < \alpha < 1,$$
$$\Delta x|_{t=t_i} = I_i(x(t)),\ i = 1,2,...m,\ x(0) = g(x)$$

These results are obtained using Banach contraction fixed point theorem.

In this paper studies existence and uniqueness results in a Banach space for a impulsive fractional Integro-Differential equations

$$\begin{cases} {}^{c}D_t^q x(t) = A(t)x(t) + f(t,x(t)) + \int_0^t k(t,s,x(s))ds, \\ \qquad t \in J' = J/\{t_1,...,t_m\}, J := [0,T], \\ x(t_k^+) = x(t_k^-) + y_k,\ \ k = 1,2,...,m \quad y_k \in X \\ x(0) + g(x) = x_0 \end{cases} \tag{1.1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order q, $0 < q < 1$. $A(t)$ is a bounded linear operator on Banach space X. $f : J \times X \to X,\ k : J \times J \times X \to X$ are jointly continuous, $g : C \to X$ is continuous, t_k satisfy $0 = t_0 < t_1 < ... < t_m < t_{m+1} =$

T, $x(t_k^+) = \lim_{\varepsilon\to 0^+} x(t_k+\varepsilon)$ and $x(t_k^-) = \lim_{\varepsilon\to 0^-} x(t_k+\varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$.

2. Preliminaries

In this section, we study notations, definitions and preliminary facts. We introduce the Banach space $PC(J,X) = \{x : J \to X : x \in C(t_k,t_{k+1}],X\}, k = 0,1,2,...,m$ and their exist $x(t_k^-)$ and $x(t_k^+), k = 0,1,2,..,m$ with $x(t_k^-) = x(t_k)$ with the norm $||x||_{PC} := sup\{||x(t)|| : t \in J\}$.

Definition 2.1. *The fractionl integral of order γ with the lower limit zero for a funtion $f : [0,\infty) \to X$ is defined as*

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}ds,\ t>0,\ \gamma>0$$

provied the right side is point-wise defined on $[0,\infty)$, where $\Gamma(.)$ is the gamma funtion.

Definition 2.2. *The Riemann-Liouville derivative of order γ with the lower limit zero for a funtion $f : [0,\infty) \to X$ can be written as*

$$^L D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds,\quad t>0,\ n-1<\gamma<n.$$

Definition 2.3. *The Caputo derivative of order γ for function $f : [0,\infty) \to X$ can be written as*

$$^c D_t^\gamma f(t) = {}^L D^\gamma\left[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right],\quad t>0,\ n-1<\gamma<n.$$

Definition 2.4. *A function $x \in PC^1(J,X)$ is said to be a solution of the problem (1.1) if x satisfies the equation*

$$^c D_t^q x(t) = A(t)x(t) + f(t,x(t)) + \int_0^t k(t,s,x(s))ds,\ t\in J'$$

a.e on J', $g : C \to X$ is continuous and the conditions $x(t_k^+) = x(t_k^-) + y_k$, $k = 1,2,...,m$ and $x(0)+g(x) = x_0$.

Lemma 2.5. *Let $q \in (0,1)$ and $h : J \to X$ be continuous,$g : C \to X$ is continuous. A function $x \in C(J,X)$ is a solution of the fractional integral equation*

$$\begin{aligned} x(t) = x_0 - g(x) &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ &- \frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}h(s)ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds, \end{aligned} \tag{2.1}$$

if and only if u is a solution of the following fractional Cauchy problems

$$\begin{cases} ^c D_t^q x(t) = A(t)x(t) + h(t),\ t\in J, \\ x(0) + g(x) = x_0, a>0. \end{cases} \tag{2.2}$$

As a consequence of Lemma 2.6 we have the following result which is useful in what follows.

Lemma 2.6. *Let $q \in (0,1)$ and $h : J \to X$ be continuous,$g : C \to X$ is continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\ for\ t\in[0,t_1] \\ x_0 - g(x) + y_1 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\ for\ t\in(t_1,t_2] \\ x_0 - g(x) + y_1 + y_2 \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\ for\ t\in(t_2,t_3] \\ \vdots \\ x_0 - g(x) + \sum_{i=0}^m y_i \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds\ for\ t\in(t_m,T] \end{cases} \tag{2.3}$$

if and only if u is a solution of the following impulsive problem

$$\begin{cases} ^c D_t^q x(t) = A(t)x(t) + h(t),\ t\in J', \\ x(t_k^+) = x(t_k^-) + y_k,\ \ k = 1,2,...,m \\ x(0) + g(x) = x_0 \end{cases} \tag{2.4}$$

Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.7. *(Krasnoselskii Theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$*

(ii) *A is compact and continuous*

(iii) *B is a contraction mapping.*

Then there exists $Z \in M$ such that $Z = Az + Bz$.

3. Main Results

This section deals with the existence and uniqueness of solutions for the problem (1.1). Before stating and proving the main result, We introduce the following hypotheses.

(H_1) $A(t)$ is a bounded linear operator and

$$\max_{t\in J}||A(t)|| = C$$

and let us take

$$\gamma = \frac{T^q}{\Gamma(q+1)}, \gamma_1 = \frac{qT^{q+1}}{\Gamma(q+2)}$$

.

(H_2) There exists a constant $L_1 > 0$ such that

$$||f(t,x_1)-f(t,x_2)|| \leq L_1||x_1-x_2||,$$

$\forall\, t \in J^{'}\ \ x_1,x_2 \in X.$

(H_3) $k:\Delta\times X \to X$ is continuous

$$||k(t,s,x_1)-k(t,s,x_2)|| \leq L_2||x_1-x_2||,$$

$\forall\, t,s \in J^{'}\ \ x_1,x_2 \in X.$

(H_4) g is continuous, and there exists a constant $\lambda < 1$ such that

$$||g(x_1)-g(x_2)|| \leq \lambda||x_1-x_2|| \forall\, x_1,x_2 \in X,$$

and $M = ||g(0)||.$

(H_5) There exists functions $\mu,\sigma \in L^1_{Loc}(I,R^+)$ such that

$$||f(t,x)|| \leq \mu(t),\ (t,x)\in[0,T]\times X$$

$$||k(t,s,x)|| \leq \sigma(t),\ (t,s,x)\in[0,T]\times[0,T]\times X$$

Theorem 3.1. *Assume that* $(H_1)-(H_4)$ *hold. If*

$$\lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma + L_1\gamma + L_2 q\gamma_1 < 1 \qquad (3.1)$$

Then the problem (1.1) *has a unique solution provided*

$$\lambda < \frac{1}{2},\ L_1 \leq \frac{\Gamma(q+1)}{4T^q},\ L_2 \leq \frac{\Gamma(q+2)}{4qT^{q+1}}.$$

Proof. We transform the problem (1.1) into a fixed point problem. Consider the operator $\theta: PC(J,X)\to PC(J,X)$ defined by

$$(\theta x)(t) = \begin{cases} x_0 - g(x) \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \text{ for } t\in[0,t_1) \\ x_0 - g(x) + y_1 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \text{ for } t\in(t_1,t_2] \\ x_0 - g(x) + y_1 + y_2 \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \text{ for } t\in(t_2,t_3] \\ \vdots \\ x_0 - g(x) + \sum_{i=0}^{m} y_i \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x(s)ds \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(t,x(t)) \\ \quad +\int_0^t k(\sigma,s,x(s))d\sigma]ds \text{ for } t\in(t_m,T] \end{cases}$$

Clearly, the fixed points of the operator θ are solution of the problem (1.1).

We shall use the Banach contraction principle to prove that F has a fixed point.

We shall show that θ is a contraction. Let us set

$$\sup_{t\in J'}||f(t,0)|| = M_1,$$

and

$$\sup_{t,s\in J'}||k(t,s,0)|| = M_2,$$

it can be shown that $\theta B_r \subset B_r$, where $B_r = \{x\in X; ||x||\leq r\}$. Choose

$$r \geq 2\Big(x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma + (L_1 r + M_1)\gamma + (L_2 r + M_2)\gamma_1\Big)$$

$$\begin{aligned} ||(\theta x)(t)|| &\leq x_0 + \lambda||x|| + M + ||\sum_{i=0}^{m} y_i|| \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_m}^{T}(t-s)^{q-1}||A(s)||||x(s)||ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_m}^{T}(t-s)^{q-1}\Big[||f(t,x(t))|| \\ &\quad + \int_{t_m}^{T}||k(\sigma,s,x(s))||d\sigma\Big]ds \\ &\leq x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\frac{T^q}{\Gamma(q+1)} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_m}^{T}(t-s)^{q-1}\Big[||f(t,x(t)) - f(s,0)|| \\ &\quad + ||f(s,0)|| + \int_{t_m}^{T}\Big(||k(\sigma,s,x(s)) \\ &\quad - k(\sigma,s,0)|| + ||k(\sigma,s,0)||\Big)d\sigma\Big]ds \\ &\leq x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma \\ &\quad + \frac{L_1 r + M_1}{\Gamma(q)}\int_{t_m}^{T}(t-s)^{q-1}ds \\ &\quad + \frac{L_2 r + M_2}{\Gamma(q)}\int_{t_m}^{T}(t-s)^{q}ds \\ &\leq x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma \\ &\quad + (L_1 r + M_1)\gamma + (L_2 r + M_2)\gamma_1 \\ &\leq r \end{aligned}$$

$$||(\theta x)(t)|| \leq r$$

Now, for $x_1, x_2 \in X$, We obtain

$$
\begin{aligned}
&||(\theta x_1)(t) - (\theta x_2)(t)|| \\
&\leq ||g(x_1) - g(x_2)|| + ||\sum_{i=0}^{m} y_i|| \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} ||A(s)(x_1(s) - x_2(s))|| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_m}^{T} (t-s)^{q-1} \Big[||f(t, x_1(t)) - f(t, x_2(t))|| \\
&\quad + \int_{t_m}^{T} ||k(\sigma, s, x_1(s)) - k(\sigma, s, x_2(s))|| d\sigma \Big] ds \\
&\leq \Lambda_{\lambda + C\gamma, L_1, L_2, T, q} ||x_1 - x_2||
\end{aligned}
$$

where $\Lambda_{\lambda+C\gamma,L_1,L_2,T,q} = \lambda + ||\sum_{i=0}^{m} y_i|| + C\gamma + L_1\gamma + L_2\gamma_1$.

Consequently by (3.1), θ is a contraction. As a consequence of Banach fixed point theorem, we deduce that θ has a fixed point which is a solution of the problem (1.1). This completes the proof of the theorem. □

Theorem 3.2. *Assume that $(H_1), (H_4), (H_5)$ hold. Then the problem (1.1) has atleast one solution on $[0, T]$.*

Proof. Choose

$$
\begin{aligned}
r \geq 2\Big(& x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma \\
& + \frac{||\mu||_{L'} T^q}{\Gamma(q+1)} + \frac{q||\sigma||_{L'} T^{q+1}}{\Gamma(q+2)} \Big),
\end{aligned}
$$

Consider $B_r = \{x \in X : ||x|| < r\}$. We define the operators Φ and Ψ on B_r as

$$
\begin{aligned}
(\Phi x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s) x(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^t k(\sigma, s, x(s)) d\sigma \Big] ds \\
(\Psi x)(t) &= x_0 - g(x) + \sum_{i=1}^{m} y_i
\end{aligned}
$$

For $x, y \in B_r$. We find that

$$
\begin{aligned}
||\Phi x + \Psi y|| &= ||x_0 - g(x) \\
&\quad + \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s) x(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^t k(\sigma, s, x(s)) d\sigma \Big] ds||
\end{aligned}
$$

$$
\begin{aligned}
||\Phi x + \Psi y|| &\leq x_0 + ||g(x) - g(0)|| + ||g(0)|| + ||\sum_{i=1}^{m} y_i|| \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ||A(s)|| ||x(s)|| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[||f(s, x(s))|| \\
&\quad + \int_0^t ||k(\sigma, s, x(s))|| d\sigma \Big] ds \\
&\leq x_0 + \lambda r + M + ||\sum_{i=0}^{m} y_i|| + Cr\gamma \\
&\quad + \frac{||\mu||_{L'} T^q}{\Gamma(q+1)} + \frac{q||\sigma||_{L'} T^{q+1}}{\Gamma(q+2)} \\
&\leq r.
\end{aligned}
$$

Thus, $\Phi x + \Psi y \in B_r$.

If follows that the assumption (H_4) that Ψ is a contraction mapping. continuity of f and k demanded in (1.1) implies that the operator Φ is continuous.

Also Φ is uniformly bounded on B_r as

$$
||\Phi|| \leq \left(\frac{||\mu||_{L'} T^q}{\Gamma(q+1)} + \frac{q||\sigma||_{L'} T^{q+1}}{\Gamma(q+2)} \right)
$$

Now, we prove the compactness of the operator Φ. Since f and k are respectively bounded on the compact sets $\Omega_1 = [0, T] \times X$ and $\Omega_2 = [0, T] \times [0, T] \times X$, therefore, we define $\sup_{(t,x)\in\Omega_1} ||f(t, x)|| = C_1$, $\sup_{(t,s,x)\in\Omega_2} ||k(t, s, x)|| = C_2$.

For $t_2, t_1 \in [0, T]$, $x \in B_r$, Now we see $(\Phi x)(t_2)$ and $(\Phi x)(t_1)$ equation are

$$
\begin{aligned}
(\Phi x)(t_2) &= \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} A(s) x(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^{t_2} k(\sigma, s, x(s)) d\sigma \Big] ds \\
(\Phi x)(t_1) &= \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} A(s) x(s) dsv \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^{t_1} k(\sigma, s, x(s)) d\sigma \Big] ds
\end{aligned}
$$

Taking norm on both side, we get

$$
\begin{aligned}
&||(\Phi x)(t_2) - (\Phi x)(t_1)|| \\
&= \frac{1}{\Gamma(q)} || \int_0^{t_2} (t_2 - s)^{q-1} A(s) x(s) ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{q-1} A(s) x(s) ds + \int_0^{t_2} (t_2 - s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^{t_2} k(\sigma, s, x(s)) d\sigma \Big] ds - \int_0^{t_1} (t_1 - s)^{q-1} \Big[f(s, x(s)) \\
&\quad + \int_0^{t_1} k(\sigma, s, x(s)) d\sigma \Big] ds||
\end{aligned}
$$

$$
\begin{aligned}
&||(\Phi x)(t_2)-(\Phi x)(t_1)|| \\
&\leq \frac{2C}{\Gamma(q+1)}|t_2-t_1|^q \\
&\quad+\frac{C_1}{\Gamma(q+1)}\left|2(t_2-t_1)^q+t_1^q-t_2^q\right| \\
&\quad+\frac{qC_2}{\Gamma(q+2)}\left|2(t_2-t_1)^{q+1}+t_1^{q+2}-t_2^{q+1}\right| \\
&\leq \frac{2}{\Gamma(q+1)}|t_2-t_1|^q(C+C_1) \\
&\quad+\frac{2qC_2}{\Gamma(q+2)}|t_2-t_1|^{q+1}
\end{aligned}
$$

which is independent of x. so Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumption of Theorem 3.2 are satisfied. Consequently, the conclusion of Theorem 3.2 applied and the problem (1.1) has atleast one solution. □

References

[1] A.Anguraj, M.Kasthuri, P. Karthikeyan, Existence of solutions for impulsive fractional differential equations with anti-periodic and integral jump conditions, *Nonlinear Studies,* 24(3)(2017), 501-510.

[2] R. P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Advanced Studies in Contemporary Mathematics*, 16(2)(2008), 181-196.

[3] G. Akram and F. Anjum, Existence and uniqueness of solution for differential equation of fractional order $2<\alpha<3$ with nonlocal multipoint integral boundary conditions, *Turkish Journal of Mathematics*, 42(2018), 2304-2324.

[4] A. Chauhan and J. Dabas, Studied the existence of mild solutions for impulsive fractional- order semilinear evoluation equations with nonlocal conditions, *Electronic journal of Differential Equations*, 2011(2011), 1-10.

[5] K.Balachandran and J.Y.Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equation, *Nonlinear Analysis*, 71(2009), 4471-4475.

[6] D. D. Bainov, P. S. Simeonov, *Systems with Impulsive effect,* Horwood, Chichister, 1989.

[7] B. Ahmad And S. Sivasundaram, Some existence results for fractional integro - differetial equations with nonlinear conditions, *Communications in Applied Analysis*, 12(2)(2008), 107-112.

[8] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Journal of Applied Analysis*, 87(7)(2008), 851-863.

[9] M. Benchohra, J. Henderson and S. K. Ntouyas, Impulsive Differential Equations and Inclusions, *Hindawi Publishing Corporation*, 2(2006).

[10] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *Journal of Mathematical Analysis and Applications*, 204(1996), 609-625.

[11] V. Gupta, J. Dabas, and M. Feckan, Existence results of solutions for impulsive fractional differential equations, *Nonautonomous Dynamical Systems*, 5(2018), 35-51.

[12] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophysical Journal*, 68(1995), 46-53.

[13] R. Hilfer, Applications of Fractional Calculus in Physics, *World Scientific*, Singapore, 2000.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, 204. *Elsevier*, Amsterdam, 2006.

[15] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, *Worlds Scientific*, Singapore, 1989.

[16] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *Journal of Chemical Physics*, 103(1995), 7180-7186.

[17] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, *John Wiley*, New York, 1993.

[18] P. Karthikeyan, R.Arul, Existence of solutions for Hadamard fractional hybrid differential equations with impulsive and nonlocal conditions, *Journal of Fractional Calculus and Applications*, 9(1)(2018), 232-240.

[19] K. B. Oldham and J. Spanier, The Fractional Calculus, *Academic Press*, New York, London, 1974.

[20] I. Podlubny, Fractional Differential Equation, *Academic Press*, San Diego, 1999.

[21] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, *Gordon and Breach*, Yverdon, 1993.

[22] A. M. Samoilenko, N. A. Perestyuk, Impulsive Differential Equations, *World Scientific*, Singapore, 1995.

[23] C. Yu and G. Gao, Existence of fractional differential equations, *Journal of Mathematical Analysis and Applications*, 310(2005), 26-29.

[24] Z.W. Lv, J. Liang and T.J Xiao, Solutions to Fractional Differential Equations with nonlocal initial condition in Banach Space, *Advances in Difference Equations*, 2010(2010), 1-10.

[25] Z. Dahmani and S. Belarbi, New results for fractional evolution equations using Banach fixed point theorem, *International Journal of Nonlinear Analysis and Applications*, 5(2)(2014), 22-30.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆