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# Positive solutions of nonlinear third-order boundary value problem with integral boundary conditions

Habib Djourdem[1]* and Slimane Benaicha[2]

**Abstract**
In this paper, we study the existence of positive solutions for boundary value problems of third-order two-point differential equations with integral boundary conditions. we mainly use the Krasnoselskii's fixed point theorem Value problem, at least there is a positive solution, and give an example to verify the conclusion.

**Keywords**
Krasnoselskii's fixed point theorem, Third-order boundary value problem, Cone, Existence.

**AMS Subject Classification**
34B15, 34B18.

[1,2] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO)*
*University of Oran1, Ahmed Benbella. Algeria.*
***Corresponding author**: [1] djourdem.habib7@gmail.com; [2]slimanebenaicha@yahoo.fr
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## Contents



## 1. Introduction

In this work, we will study the existence of positive solutions of nonlinear two-point boundary value problem (BVP) for the following third-order differential equation:

$$u'''(t)+f(u(t))=0, \qquad ,0<t<1, \tag{1.1}$$

subject to the two-point boundary conditions

$$u'(0)=u'(1)=0, \;\; u(0)=\alpha\int_0^\eta u(s)\,ds, \tag{1.2}$$

$\alpha$, $\eta$ are constants with $0<\alpha<\frac{1}{n+1}$, $\eta\in(0,1)$ and $f\in C([0,\infty),[0,\infty))$.

Ordinary differential equation has been widely used in many fields of mathematics and physics, so the third-order boundary-value problems have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions [2, 8, 10]. Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Such kind of BVPs in Banach space has been studied by some researchers (see [6, 11, 13, 19]).

In 2011, Zhao and Wang [18] considered the third order two-point boundary value problem

$$u'''(t)+f(t,u(t))=\theta, \quad t\in[0,1], \tag{1.3}$$

subject to one of the following integral boundary conditions:

$$\begin{aligned} &u(0)=\theta, \quad u''(0)=\theta, \quad u(1)=\int_0^1 g(t)u(t)\,dt, \\ &u(0)=\int_0^1 g(t)u(t)\,dt, \quad u''(0)=\theta, \quad u(1)=\theta, \end{aligned} \tag{1.4}$$

In 2013, Yanping et Fei. [5] studied the third order boundary value problem

$$\begin{aligned} &u'''(t)+f\left(t,u(t),u'(t)\right)=0, \quad t\in[0,1], \\ &u(0)=0,\; u''(0)=0, \quad u(1)=\int_0^1 g(t)u(t)\,dt. \end{aligned} \tag{1.5}$$

In 2015, Galvis, Rojas and Sinitsyn. [3] considered the second order boundary value problem

$$\begin{aligned} &u''(t)+a(t)f(u)=0, \\ &u(0)=0, \quad \alpha\int_0^\eta u(s)\,ds=u(1), \quad \eta\in(0,1). \end{aligned} \tag{1.6}$$

In 2016, Benaicha and Haddouchi. [1] studied the existence of positive solutions of a nonlinear two-point boundary value problem (BVP) for the following fourth-order differential equation

$$\begin{aligned} & u''''(t)+f(u(t))=0, \quad t\in(0,1) \\ & u'(0)=u'(1)=u''(0)=0, \quad u(0)=\int_0^1 a(s)u(s)\,ds. \end{aligned} \tag{1.7}$$

For some other results on boundary value problem, we refer the reader to the papers [4, 7, 12, 14–17].

Motivated by the results obtained in the papers mentioned above the aim of this paper is to establish some sufficient conditions for the existence of at least one positive solutions of the BVP $(1.1)-(1.2)$.

We shall first construct the Green's function for the associated linear boundary value problem and then determine the properties of the Green's function for associated linear boundary value problems. Finally, existence results for at least one positive solution for the above problem are established when $f$ is superlinear or sublinear. As applications, two examples are presented to illustrate the main results.

## 2. Preliminaries

We shall consider the Banach space $C([0,1])$ equipped with the sup norm

$$\|u\|=\sup_{t\in[0,1]}|u(t)|$$

**Definition 2.1.** *Let $E$ be a real Banach space. A nonempty, closed, convex set $K\subset E$ is a cone if it satisfies the following two conditions:*

*$(i)$ $x\in K$, $\lambda\geq 0$ imply $\lambda x\in K$;*

*$(ii)$ $x\in K$, $-x\in K$ imply $x=0$.*

**Definition 2.2.** *An operator $T:E\to E$ is completely continuous if it is continuous and maps bounded sets into relatively compat sets.*

To prove some of our results, we will use the following fixed point theorem, which is due to Krasnoselskii's [9]

**Theorem 2.3.** *[9] Let be a Banach space, and let $K\subset E$, be a cone. Assume that $\Omega_1$ and $\Omega_2$ are open subsets of $E$ with $0\in\Omega_1$, $\Omega_1\subset\Omega_2$ and let*

$$A:K\cap\left(\overline{\Omega_2}\setminus\Omega_1\right)\to K$$

*be acompletely continuous operator such that*

*$(i)$ $\|Au\|\leq\|u\|$, $u\in K\cap\partial\Omega_1$, and $\|Au\|\geq\|u\|$, $u\in K\cap\partial\Omega_2$;*

*$(ii)$ $\|Au\|\geq\|u\|$, $u\in K\cap\partial\Omega_1$, and $\|Au\|\leq\|u\|$, $u\in K\cap\partial\Omega_2$.*

*Then $A$ has a fixed point in $K\cap\left(\overline{\Omega_2}\setminus\Omega_1\right)$.*

Consider the two-point boundary value problem

$$u'''+y(t)=0, \qquad 0<t<1, \tag{2.1}$$

subject to the two-point boundary conditions

$$u'(0)=u'(1)=0,\ u(0)=\alpha\int_0^\eta u(s)\,ds. \tag{2.2}$$

**Lemma 2.4.** *Let $0<\alpha<\frac{1}{n+1}$. Then for $y\in C([0,1],\mathbb{R})$, the problem $(2.1)-(2.2)$ has a unique solution*

$$u(t)=\int_0^1 G(t,s)y(s)\,ds,$$

*where $G(t,s):[0,1]\times[0,1]\to\mathbb{R}$ is the Green's function defined by*

$$G(t,s)=K(t,s)+\frac{\alpha}{1-\alpha\eta}\int_0^1 K(n,s)\,d\eta$$

*and*

$$K(t,s)=\frac{1}{2}\begin{cases} t^2(1-s)-(t-s)^2, & 0\leq s\leq t\leq 1 \\ t^2(1-s) & 0\leq t\leq s\leq 1 \end{cases} \tag{2.3}$$

*Proof.* integrating $(2.1)$ over the integral $[0,t]$ for $t\in[0,1]$, we obtain

$$u''(t)=-\int_0^t y(s)\,ds+C_1,$$

$$u'(t)=-\int_0^t (t-s)y(s)\,ds+C_1t+C_2,$$

$$u(t)=-\frac{1}{2}\int_0^t (t-s)^2y(s)\,ds+\frac{1}{2}C_1t^2+C_2t+C_3. \tag{2.4}$$

From the boundary conditions $(2.2)$, we get

$$C_1=\int_0^1(1-s)y(s)\,ds,\ C_2=0,$$

Integrating again from 0 to $\eta$ the expression $(2.4)$ where $\eta\in(0,1)$

$$\begin{aligned}\int_0^\eta u(s)\,ds=&-\frac{\alpha}{6}\int_0^\eta(\eta-s)^3y(s)\,ds+\frac{\eta^3\alpha}{6}\int_0^1(1-s)y(s)\,ds \\ &+\alpha C_3\eta\end{aligned}$$

From $u(0)=\alpha\int_0^\eta u(s)\,ds$, we have

$$C_3=-\frac{\alpha}{6}\int_0^\eta(\eta-s)^3y(s)\,ds+\frac{\eta^3\alpha}{6}\int_0^1(1-s)y(s)\,ds+\alpha C_3\eta$$

which implies

$$\begin{aligned}C_3=&-\frac{\alpha}{6(1-\alpha\eta)}\int_0^\eta(\eta-s)^3y(s)\,ds \\ &+\frac{\eta^3\alpha}{6(1-\alpha\eta)}\int_0^1(1-s)y(s)\,ds\end{aligned}$$

Replacing these expressions in (2.4), we get

$$\begin{aligned}
u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2}t^2\int_0^1 (1-s)\,y(s)\,ds \\
&\quad - \frac{\alpha}{6(1-\alpha\eta)}\int_0^\eta (\eta-s)^3 y(s)\,ds \\
&\quad + \frac{\eta^3\alpha}{6(1-\alpha\eta)}\int_0^1 (1-s)\,y(s)\,ds \\
&= \frac{1}{2}\int_0^t \left[t^2(1-s)-(t-s)^2\right] y(s)\,ds \\
&\quad + \frac{1}{2}t^2\int_t^1 (1-s)\,y(s)\,ds \\
&\quad + \frac{1}{2}\times\frac{\alpha}{(1-\alpha\eta)}\int_0^\eta \frac{1}{3}\left[\eta^3(1-s)-(\eta-s)^3\right] y(s)\,ds \\
&\quad + \frac{\alpha}{6(1-\alpha\eta)}\int_\eta^1 \eta^3(1-s)\,y(s)\,ds \\
&= \int_0^1 \left[K(t,s)+\frac{\alpha}{(1-\alpha\eta)}\int_0^1 K(\eta,s)\,d\eta\right] y(s)\,ds.
\end{aligned}$$

□

**Lemma 2.5.** *Let $\theta \in \left]0,\frac{1}{2}\right[$ be fixed. Then*
*$(i)$ $K(t,s)\geq 0$, for all $t,s\in[0,1]$*
*$(ii)$ $\frac{\theta^2}{2}s(1-s)\leq K(t,s)\leq \frac{1}{2}s(1-s)$*
*for all $(t,s)\in[\theta,1-\theta]\times[0,1]$.*

*Proof.* $(i)$ We will show that $K(t,s)\geq 0$ for all $t,s\in[0,1]$. Since it is obvious for $t\leq s$, we only need to prove the case $s\leq t$. Now we suppose that $s\leq t$. Then

$$\begin{aligned}
k(t,s) &= \frac{1}{2}\left[t^2(1-s)-(t-s)^2\right] = \frac{1}{2}\left[t(t-ts)-(t-s)^2\right] \\
&\geq \frac{1}{2}\left[t(t-s)-(t-s)^2\right] \\
&= \frac{1}{2}(t-s)\,s \geq 0.
\end{aligned} \tag{2.5}$$

$(i)$ If $s\leq t$, from (2.3) , we have

$$\begin{aligned}
k(t,s) &= \frac{1}{2}\left[t^2(1-s)-(t-s)^2\right] \geq \frac{1}{2}\left[t^2(1-s)^2-(t-s)^2\right] \\
&= \frac{1}{2}\left[t(1-s)-(t-s)\right]\left[t(1-s)+(t-s)\right] \\
&= \frac{1}{2}(s-ts)(t-ts+t-s) \geq \frac{1}{2}s(1-t)(t-ts) \\
&= \frac{1}{2}s(1-t)\,t(1-s) = \frac{1}{2}t(1-t)\,s(1-s).
\end{aligned} \tag{2.6}$$

On other hand

$$\begin{aligned}
K(t,s)-\frac{1}{2}s(1-s) &= \frac{1}{2}\left[t^2(1-s)-(t-s)^2\right]-\frac{1}{2}s(1-s) \\
&= \frac{1}{2}t^2-\frac{1}{2}t^2s-\frac{1}{2}t^2-\frac{1}{2}s^2+ts-\frac{1}{2}s+\frac{1}{2}s^2 \\
&= -\frac{1}{2}t^2s+ts-\frac{1}{2}s = -\frac{1}{2}s(t-1)^2 \leq 0,
\end{aligned} \tag{2.7}$$

if $t\leq s$, from (2.3) we have

$$K(t,s) = \frac{1}{2}t^2(1-s) \geq \frac{1}{2}t^2 s(1-s) \tag{2.8}$$

and

$$K(t,s) = \frac{1}{2}t^2(1-s) \leq \frac{1}{2}s(1-s). \tag{2.9}$$

Let

$$\begin{cases} t^2, & t\leq\frac{1}{2} \\ (1-t)\,t, & t\geq\frac{1}{2}. \end{cases} \tag{2.10}$$

From (2.6), (2.7), (2.8) , (2.9) and (2.10) we have

$$\rho(t)\,s(1-s) \leq K(t,s) \leq \frac{1}{2}s(1-s)$$

For $\theta\in\left]0,\frac{1}{2}\right[$ we have

$$\frac{\theta^2}{2}s(1-s) \leq K(t,s) \leq \frac{1}{2}s(1-s),\ forall\ (t,s)\in[\theta,1-\theta]\times[0,1]$$

□

**Lemma 2.6.** *Let $y(t)\in C([0,1],[0,\infty))$. The unique solution of (2.1)-(2.2) is nonnegative and satisfies*

$$\min_{\theta\leq t\leq 1-\theta} u(t) \geq \theta^2\|u\|$$

*Proof.* From Lemma 2.4 and Lemma 2.5, $u(t)$ is nonnegative. For $t\in[0,1]$, from Lemma 2.4 and Lemma 2.5, we have that

$$\begin{aligned}
u(t) &= \int_0^1 \left(K(t,s)+\frac{\alpha}{1-\alpha\eta}\int_0^1 K(n,s)\,d\eta\right) y(s)\,ds, \\
&\leq \int_0^1 \left(\frac{1}{2}s(1-s)+\frac{\alpha}{1-\alpha\eta}\int_0^1 \frac{1}{2}s(1-s)\,d\eta\right) y(s)\,ds \\
&= \frac{1}{2}\times\frac{1-\alpha\eta+\alpha}{1-\alpha\eta}\int_0^1 s(1-s)\,y(s)\,ds.
\end{aligned}$$

Then

$$\|u\| \leq \frac{1}{2}\times\frac{1-\alpha\eta+\alpha}{1-\alpha\eta}\int_0^1 s(1-s)\,y(s) \tag{2.11}$$

and,

$$\begin{aligned}
u(t) &= \int_0^1 \left(K(t,s)+\frac{\alpha}{1-\alpha\eta}\int_0^1 K(n,s)\,d\eta\right) y(s)\,ds \\
&= \frac{\theta^2}{2}\int_0^1 \left(s(1-s)+\frac{\alpha}{1-\alpha\eta}(1-s)\int_\theta^{1-\theta} d\eta\right) y(s)\,ds \\
&\geq \frac{\theta^2}{2}\int_0^1 \left(s(1-s)+\frac{\alpha}{1-\alpha\eta}s(1-s)\right) y(s)\,ds \\
&= \frac{\theta^2}{2}\times\frac{1-\alpha\eta+\alpha}{1-\alpha\eta}\int_0^1 s(1-s)\,y(s)\,ds.
\end{aligned}$$

(2.12)

From (2.11), (2.12), we obtain

$$\min_{t\in[\theta,1-\theta]} u(t) \geq \theta^2 \|u\|$$

□

Define the cone

$$\Gamma = \left\{ u \in C([0,1],\mathbb{R}), u \geq 0 : \min_{t\in[\theta,1-\theta]} u(t) \geq \theta^2 \|u\| \right\}$$

and the operator $A : \Gamma \to [0,1]$ by

$$A(u(t)) = \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\,d\eta \right) f(u(s))\,ds \tag{2.13}$$

**Remark 2.7.** *By Lemma 2.4 problem (1.1), (1.2) has a positive solution $u(t)$ if and only if $u$ is a fixed point of $A$.*

**Remark 2.8.** *We can prouve that for $0 < \alpha < \frac{1}{1+\eta}$, and $\eta \in (0,1)$, we have*

$$0 < \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} < 1$$

**Lemma 2.9.** *The operator $A$ defined in (2.13) is completely continuous and satisfies $A\Gamma \subset \Gamma$.*

*Proof.* We shall prove that $A\Gamma \subset \Gamma$. Obviously, for $u \in \Gamma$, $A(u) \in C^+[0,1]$. For all $t \in [0,1]$, we have

$$\begin{aligned}
&\|Au(t)\| \\
&= \max_{0\leq t\leq 1} \left( \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\,d\eta \right) f(u(s))\,ds \right) \\
&\leq \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \int_0^1 s(1-s) f(u(s)),
\end{aligned}$$

and

$$\begin{aligned}
&Au(t) \\
&= \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\,d\eta \right) f(u(s))\,ds \\
&\geq \frac{\theta^2}{2} \int_\theta^{1-\theta} \Big(s(1-s) \\
&\quad + \frac{\alpha}{1-\alpha\eta} s(1-s) \int_\theta^{1-\theta} d\eta \Big) f(u(s))\,ds \\
&= \frac{\theta^2}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} (1-2\theta) \int_\theta^{1-\theta} s(1-s) f(u(s))\,ds \\
&\geq \theta^2 \|Au(t)\|.
\end{aligned}$$

Which give that $Au \in \Gamma$. Therefore $A : \Gamma \to \Gamma$.

Now, we shall prove that the operator $A$ is completely continuous. Let $D \subset \Gamma$ is a bounded subset. Then there exists a positive constant $M_1$ such that

$$\|u\| \leq M_1, \quad \forall u \in D$$

Let $M_2 = \sup_{0\leq t\leq 1} |f(u(t))|$ for all $(t,u) \in [0,1] \times [0,M_1]$. For any $k \in \mathbb{N}^*$, by (2.13), we have

$$\begin{aligned}
|y_k(t)| &= |A_n x_k(t)| \\
&= \left| \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\,d\eta \right) f(x_k(s))\,ds \right| \\
&\leq M_2 \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\,d\eta \right) \\
&\leq M_2 \times \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \int_0^1 s(1-s)\,ds \\
&= M_2 \times \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \times \frac{1}{6} \\
&\leq \frac{1}{6} M_2,
\end{aligned}$$

which implies that $(y_k(t))_{k\in\mathbb{N}^*}$ is uniformly bounded. Now, we show that $A$ is equicontinuous. For any $u \in \Gamma$, $n \geq 2$, and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
|y_k(t_1) - y_k(t_2)| &= |Au(t_1) - Au(t_2)| \\
&\leq \left| \int_0^1 (K(t_1,s) - K(t_2,s)) f(x_k(s))\,ds \right| \\
&\leq M_2 \int_0^1 |K(t_1,s) - K(t_2,s)|\,ds.
\end{aligned}$$

It follows from the uniform continuity of the function $K$ on $[0,1] \times [0,1]$, that for any $\varepsilon > 0$, we have

$$|K(t_1,s) - K(t_2,s)| \leq \frac{\varepsilon}{M_2}, \ for t_1, t_2 \in [0,1], |t_1 - t_2| < \delta$$

Then

$$\begin{aligned}
|y_k(t_1) - y_k(t_2)| &= |Au(t_1) - Au(t_2)| \\
&\leq M_2 \int_0^1 |K(t_1,s) - K(t_2,s)|\,ds \\
&\leq M_2 \times \frac{\varepsilon}{M_2} = \varepsilon.
\end{aligned}$$

Therefore, $A$ is equicontinuous. By the Ascoli-Arzela Theorem, we know that $A$ is completely continuous. □

In what follows, we will the following notations

$$f_0 = \lim_{u\to 0+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

We note that the case $f_0 = 0$ and $f_\infty = \infty$ corresponds to the superlinear case and $f_0 = \infty$ and $f_\infty = 0$ corresponds to the sublinear case.

## 3. Existence of positive solutions

In this section, we will state and prove our main results.

**Theorem 3.1.** *Assume that $f_0 = 0$ and $f_\infty = \infty$. Then BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* Since $f_0 = 0$, there exists $M > 0$ such that $f(u) < \varepsilon u$, for $0 < u < M$, where $\varepsilon$ satisfies

$$\frac{\varepsilon}{6} \leq 1$$

$$\begin{aligned} Au(t) &= \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\, d\eta \right) f(u(s))\, ds \\ &\leq \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(\eta,s)\, d\eta \right) \varepsilon u(s)\, ds \\ &\leq \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \varepsilon \|u\| \int_0^1 s(1-s)\, ds \\ &\leq \frac{\varepsilon}{6} \|u\| \\ &\leq \|u\|. \end{aligned} \tag{3.1}$$

Therefore

$$\|Au\| \leq \|u\|, \forall u \in \Gamma \cap \partial\Omega_1$$

Now, since $f_\infty = \infty$, there exists $N > 0$ such that $f(u) > \delta u$, for $u > N$, where $\delta$ satisfies

$$\delta \geq \frac{12(1-\alpha\eta)}{\theta^4 (4\theta^3 - 6\theta^2 + 1)(1-2\theta)(1-\alpha\eta+\alpha)}$$

Let $N_1 = \max\left\{2M, \frac{N}{\theta^2}\right\}$ and $\Omega_2 = \{u \in E : \|u\| < N_1\}$, then $u \in K \cap \partial\Omega_2$ implies that

$$\min_{t\in[\theta,1-\theta]} u(t) \geq \theta^2 \|u\| = \theta^2 N_1 \geq N$$

by (2.13), we obtain

$$\begin{aligned} &Au(t) \\ &= \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(n,s)\, d\eta \right) f(u(s))\, ds \\ &\geq \frac{\theta^2}{2} \int_\theta^{1-\theta} (s(1-s) \\ &\quad + \frac{\alpha}{1-\alpha\eta} \frac{1}{2} s(1-s) \int_\theta^{1-\theta} d\eta \Bigg) f(u(s))\, ds \\ &= \frac{\theta^2}{2} \delta \int_\theta^{1-\theta} \left( s(1-s) \left( \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \right) (1-2\theta) \right) u(s)\, ds \\ &\geq \frac{\theta^2}{2} \delta \left( \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \right) (1-2\theta) \\ &\quad \times \min_{t\in[\theta,1-\theta]} u(t) \int_\theta^{1-\theta} s(1-s)\, ds \\ &\geq \theta^4 \delta \frac{1}{2} \left( \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \right) (1-2\theta) \frac{1}{6} \left(4\theta^3 - 6\theta^2 + 1\right) \|u\| \\ &\geq \|u\|. \end{aligned} \tag{3.2}$$

Hence $\|Au\| \geq u$, $\forall u \in \Gamma \cap \partial\Omega_2$. Therefore, from (3.1) and (3.2) and Theorem 2.3 the operator $A$ has at least one fixed point in $\Gamma \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ such that $M \leq \|u\| \leq N_1$,which is a positive solution of (1.1) and (1.2). ☐

**Theorem 3.2.** *Assume that $f_0 = \infty$ and $f_\infty = 0$. Then BVP (1.1) and (1.2) has at least one positive solution.*

*Proof.* Since $f_0 = \infty$ and $f_\infty = 0$, there exists $M > 0$ such that $f(u) > \gamma u$, for $u > M$, where $\gamma$ satisfies

$$\gamma \geq \frac{12(1-\alpha\eta)}{\theta^4 (4\theta^3 - 6\theta^2 + 1)(1-2\theta)(1-\alpha\eta+\alpha)}.$$

Thus, for $u \in \Gamma \cap \partial\Omega_1$with $\Omega_1 = \{u \in E : \|u\| < M\}$, we have from (3.2)

$$\begin{aligned} &Au(t) \\ &= \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(n,s)\, d\eta \right) f(u(s))\, ds \\ &\geq \frac{\theta^2}{2} \int_\theta^{1-\theta} \left( s(1-s) + \frac{\alpha}{1-\alpha\eta} \int_\theta^{1-\theta} s(1-s)\, d\eta \right) f(u(s))\, ds \\ &\geq \theta^4 \gamma \frac{1}{2} \left( \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} \right) \frac{1}{6} \left(4\theta^3 - 6\theta^2 + 1\right) (1-2\theta) \|u\| \\ &\geq \|u\|. \end{aligned}$$

Then

$$\|Au\| \geq u, \forall u \in \Gamma \cap \partial\Omega_1.$$

On the other hand, since $f_\infty = 0$ there exists $N > 0$ such that $f(u) \leq \beta u$, for $u > N$, where $\beta$ satisfies

$$\frac{\beta}{6} \leq 1$$

We consider two cases:

**Case 1.** Suppose $f$ is bounded. Let $L$ be such that $f(u(t)) < L$ and $\Omega_2 = \{u \in E : \|u\| \leq N_1\}$ with $N_1 = \max\left\{2M, \frac{L}{6}\right\}$. Then for $u \in \Gamma \cap \partial\Omega_2$, we have

$$\begin{aligned} &Au(t) \\ &= \int_0^1 \left( K(t,s) + \frac{\alpha}{1-\alpha\eta} \int_0^1 K(n,s)\, d\eta \right) f(u(s))\, ds \\ &\leq \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta} L \int_0^1 s(1-s)\, ds \\ &\leq \frac{L}{6} \\ &\leq N_1 = \|u\| \end{aligned}$$

and we obtain, $\|Au\| \leq \|u\|$ for $u \in \Gamma \cap \partial\Omega_2$.

**Case 2.** Suppose that $f$ is unbounded. Since $f \in ([0,\infty), [0,\infty))$, there exists $N_1 > \max\{2M, N\}$ such that $f(u) < f(N_1)$ with $0 < u < N_1$

Now, we set $\Omega_2 = \{u \in E : \|u\| < N_1\}$. Then for $u \in \Gamma \cap \partial\Omega_2$, we have

$$\begin{aligned}
&Au(t)\\
&= \int_0^1 \left(K(t,s) + \frac{\alpha}{1-\alpha\eta}\int_0^1 K(n,s)\,d\eta\right) f(u(s))\,ds\\
&\leq \frac{1}{2} \times \frac{1-\alpha\eta+\alpha}{1-\alpha\eta}\beta N_1 \int_0^1 s(1-s)\,ds\\
&\leq \frac{\beta N_1}{6}\\
&\leq N_1 = \|u\|.
\end{aligned}$$

Thus, $\|Au\| \leq \|u\|$, for $u \in \Gamma \cap \partial\Omega_2$. By Theorem 2.3 $A$ has at least one fixed point, which is a positive solution of (1.1) and (1.2). □

## 4. Examples

**Example 4.1.** *Consider the boundary value problem*

$$u'''(t) + u^2\left(1 - e^{-u}\right) = 0, \tag{4.1}$$

*subject to the two-point boundary conditions*

$$u'(0) = u'(1) = 0,\ \ u(0) = \frac{1}{3}\int_0^{0,5} u(s)\,ds, \tag{4.2}$$

*where* $f(u) = u^2(1-e^{-u}) \in C([0,\infty), [0,\infty))$

*On the other hand*

$$\lim_{u\to 0+} \frac{f(u)}{u} = \lim_{u\to 0+} u\left(1-e^{-u}\right) = 0,$$

$$\lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} u\left(1-e^{-u}\right) = \infty.$$

*From Theorem 3.1, the problem (4.1) and (4.2) has at least one positive solution.*

**Example 4.2.** *Consider the boundary value problem*

$$u'''(t) + \sqrt{1+u} - \sqrt{u} = 0, \tag{4.3}$$

*subject to the two-point boundary conditions*

$$u'(0) = u'(1) = 0,\ \ u(0) = \frac{3}{7}\int_0^{0,75} u(s)\,ds. \tag{4.4}$$

*where* $f(u) = \sqrt{1+u} - \sqrt{u} \in C([0,\infty), [0,\infty))$

*On the other hand*

$$\lim_{u\to 0+} \frac{f(u)}{u} = \lim_{u\to 0+} \frac{\sqrt{1+u}-\sqrt{u}}{u} = +\infty,$$

$$\lim_{u\to +\infty} \frac{f(u)}{u} = \lim_{u\to +\infty} \frac{\sqrt{1+u}-\sqrt{u}}{u} = 0.$$

*From Theorem 3.2, the problem (4.3) and (4.4) has at least one positive solution.*

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