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Fixed point theorem of a set valued map on Cone metric space

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Abstract
This paper presents some extensions of the result that has been proved in [5]. We obtain a result on common fixed point theorem in cone metric space for two set valued maps.

Keywords
Cone metric space, H-cone metric, Fixed point.

AMS Subject Classification
47H10, 54H25, 54C60.

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Contents

1. Introduction

In the year 1969, S.B.Nadler [1] gave a generalisation of Banach contraction principle in case of set-valued maps in metric space. In 1968 [2] Kannan established a fixed point theorem for mapping satisfying: $d(Tx,Ty) \leq \alpha \ \{d(x,Tx) + d(y,Ty)\}$, where $\alpha \in [0, \frac{1}{2})$. Kannan's paper was extended by many authors during the period of time, one of which came in the year 1971, by S. Riech in his paper [3]. Where a map satisfies the contraction condition: $d(Tx,Ty) \leq \{a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,y)\}$, where $a_i \geq 0$, $\forall i = 1,2,3$ and $\sum_{i=1}^{3} a_i < 1$. Then in 1972, Chatterjea [23] established a fixed point theorem for mapping satisfying: $d(Tx,Ty) \leq \alpha \ \{d(x,Ty) + d(y,Tx)\}$, where $\alpha \in [0, \frac{1}{2})$.

It is during the year 2007 when Huang and Zhang [4] introduced the concept of cone metric space by replacing the range set of non negative real numbers of the metric d by the ordered Banach space. Since then many other authors in [12]-[19] and the references therein, have obtained the fixed point theorems on single valued maps.In [5], the existence of a fixed point in cone metric space for set valued mappings has been obtained by the concept of H-Cone metric. For more recent fixed point theorems in cone metric spaces for multivalued mappings we refer [[5]-[11]] and references therein.

2. Preliminaries

Let E be a real Banach space P $\subset E$. Then P is said to a cone if it satisfies the following conditions:

1. P is a nonempty closed subset and P $\neq \emptyset$.
2. $x,y \in$ P and $a,b \in R$ where $a \geq 0$ and $b \geq 0$then $ax + by \in P$.
3. $If x \in P$ and $-x \in P$, then $x = \theta$.

Cone induces a Partial order relation We can define a partial order relation $\preceq$ on E with respect to the cone P in the following way: $x \preceq y$ if and only if $y - x \in P$. Also $x << y$ if and only if $y \preceq x \in IntP$ and $x < y$ implies $x \preceq y$ but $x \neq y$. If $IntP \neq \theta$ then the cone is a solid cone.

Definition 2.1. *[4] Let X be a non empty set and $d : X \times X \to E$ satisfying*

1. *$\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$.*
2. *$d(x,y) = d(y,x)$.*
3. *$d(x,y) \preceq d(x,z) + d(z,y)$, $\forall\, x,y \in X$.*

Then d is called the cone metric and the pair (X,d) is called the cone metric space.

Example 1 [5]: Let $E=R^2$ and P=$\{(x,y)\in R^2 : x\geq 0$ and $y\geq 0\}$, $X=R^2$ and $d(x,y) = (|x-y|,\alpha|x-y|)$, $\forall x,y\in X$ and $\alpha\geq 0$. Then (X,d) is a cone metric space and P is a normal cone with normal constant 1.

There are two different kinds of cones: Normal (with a normal constant) and Non-Normal cones. Let E be a real Banach space, $P\subset E$ be a cone and $\preceq$ be the partial ordering defined by P . Then P is said to be normal if there exist positive real number $\mathscr{K}>0$ such that, for all $x,y\in E$, $\theta\preceq x\preceq y \Rightarrow \|x\|\leq \mathscr{K}\ \|y\|$. Or, equivalently if $x_n\preceq y_n\preceq z_n$ and $\lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=x$, then $\lim_{n\to\infty}y_n=n$. The least of all such constant $\mathscr{K}$ is known as normal constant.

Definition 2.2. *[4]: Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x\in X$. If for every $\varepsilon\in E$ with $\varepsilon>>\theta$ there is N such that for all $n>N$, $d(x_n,x)<<\varepsilon$. Then $\{x_n\}$ is said to be convergent and x is the limit of $\{x_n\}$. We denote this by, $lim_{n\to\infty}x_n=x$ as $n\to\infty$.*

Definition 2.3. *: Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X.If for any $\varepsilon\in E$ with $\varepsilon>>\theta$ there is a positive integer N such that for all $n,m>N$, $d(x_n,x_m)<<\varepsilon$. Then $\{x_n\}$ is said to be a Cauchy sequence in X.*

Definition 2.4. *: If every Cauchy sequence $\{x_n\}\subset M$ is convergent in $x\in M$, then (M,d) is called a complete cone metric space.*

Lemma 2.5. *[21] Let E be a Banach space.*
(i) If $a,b,c\in E$ and $a\preceq b\ll c$, then $a\ll c$.
(ii) If $\theta\preceq a\ll c$ for each $c\gg\theta$, then $a=\theta$.
(iii) If E is a real Banach space with cone P and if $a\preceq\lambda a$ where $a\in P$ and $\lambda\in(0,1)$, then $a=\theta$.

Remark 2.6. *[21]: If $c\gg\theta$, $\theta\ll a_n$ and $a_n\to\theta$, then there exists N, such that for all $n>N$, we have $a_n\ll c$.*

A set $A\subset M$ is closed if for any sequence $\{x_n\}\subset A$ convergent to x, then $x\in A$.

We denote $N(M)$ as the collection fo all nonempty subsets of M and $C(M)$ as collection of all nonempty closed subsets of M.

Definition 2.7. *An element $x\in M$ is said to be a fixed point of a set-valued mapping $T: M\to N(M)$ if $x\in Tx$. Denote Fix(T)=$\{x\in M : x\in Tx\}$.*

The following is the definition of H-cone metric as given by Wardowski in [6] came in the year 2011.

Definition 2.8. *Let (M,d) be a cone metric space and $\mathscr{A}$ be the collection of all nonempty subsets of M. A map $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ is called an H-cone metric with respect to d if for any $A_1,A_2\in\mathscr{A}$ the following conditions hold:*

1. *$\mathscr{H}$ $(A_1,A_2)=0\Rightarrow A_1=A_2$.*
2. *$\mathscr{H}$ (A_1,A_2)=$\mathscr{H}$ (A_2,A_1).*
3. *$\forall\ \varepsilon\in E$ with $\theta\ll\varepsilon$, $\forall x\in A_1$, $\exists$ at least one $y\in A_2$, such that $d(x,y)\preceq\mathscr{H}$ (A_1,A_2)+ε .*
4. *anyone of the following holds there exist*

   (a) $\forall\ \varepsilon\in E$ with $\theta\ll\varepsilon$, $\exists$ at least one $x\in A_1$, such that $\mathscr{H}(A_1,A_2)\preceq d(x,y)$+$\varepsilon$. $\forall y\in A_2$.

   (b) $\forall\ \varepsilon\in E$ and $\theta\ll\varepsilon$, $\exists$ at least one $x\in A_2$, such that $\mathscr{H}(A_1,A_2)\preceq d(x,y)$+$\varepsilon$. $\forall y\in A_1$.

For examples we refer [6] to the readers. The author in [6] have proved that if (M,d) is a cone metric space and $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ is H-cone metric with respect to d then the pair $(\mathscr{A},\mathscr{H})$ is a cone metric space.

In [6], the author have proved the following result.

Theorem 2.9. *Let (M,d) be a complete cone metric space with a normal cone P with a normal constant $\mathscr{H}$. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and let $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ be an H-cone metric with respect to d. If for a map $T: M\to\mathscr{A}$ $\exists\ \lambda\in(0,1)$ such that $\forall x,y\in M$, $\mathscr{H}$(Tx,Ty) $\preceq\lambda d(x,y)$, then FixT $\neq\phi$.*

In the year 2013, H-cone metric in the sense of Arshad and Ahmad [11] was defined in the following way to make it more comparable with a standard metric.

Definition 2.10. *[11]: Let (M,d) be a cone metric space and $\mathscr{A}$ be a collection of all nonempty subsets of M. A map $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ is called an H-cone metric in the sense of Arshad and Ahmad if the following conditions hold:*

1. *$\theta\preceq H(A,B)$ for all A,B $\mathscr{A}$ and $\mathscr{H}$ (A,B) = θ if and only if A = B;*
2. *$\mathscr{H}$(A,B) = $\mathscr{H}$(B,A), $\forall$ A,B $\in\mathscr{A}$;*
3. *$\mathscr{H}$(A,B) $\preceq\mathscr{H}$(A,C) + $\mathscr{H}$(C,B), $\forall$ A,B,C $\in\mathscr{A}$;*
4. *if $A,B\in\mathscr{A}$, $\theta<\varepsilon\in E$ with $\mathscr{H}$(A,B) $<\varepsilon$, then for each $a\in A$ there exists $b\in B$ such that $d(a,b)<\varepsilon$.*

Using this H-cone metric the following result [[11], Th.3] was proved

Theorem 2.11. *[11] Let (M,d) be a complete cone metric space. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and let $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ be an H-cone metric induced by d. If for a map $T: M\to\mathscr{A}$ $\exists\ \lambda\in(0,1)$ such that $\forall x,y\in M$ $\mathscr{H}$ (Tx,Ty) $\preceq\lambda d(x,y)$, then FixT $\neq\phi$.*

The following example has been shown in [[14], Eg 1.10] which indicates that Definition 2.10 is different from Definition 2.8.

Example 2.12. *Let $X=\{a,b,c\}$ and $d: X\times X\to[0,+\infty)$ be defined by*
$d(a,b)=d(b,a)=\frac{1}{2}$, $d(a,c)=d(c,a)=d(b,c)=d(c,b)=1$, $d(a,a)=d(b,b)=d(c,c)=0$. Let $A=\{\{a\},\{b\},\{c\}\}$, $\mathscr{H}$: $A\times A\to[0,+\infty)$ as
$\mathscr{H}$ $(\{a\},\{b\})$ = $\mathscr{H}$ $(\{b\},\{a\})$ = 1,

$\mathscr{H}$ $(\{a\},\{c\})$ =$\mathscr{H}$ $(\{c\},\{a\})$ =$\mathscr{H}$ $(\{b\},\{c\})$ = $\mathscr{H}$ $(\{c\},\{b\})$ = 2,
$\mathscr{H}$ $(\{a\},\{a\})$ =$\mathscr{H}$ $(\{b\},\{b\})$ = $\mathscr{H}$ $(\{c\},\{c\})$ = 0.
Then $\mathscr{H}$ is an H-cone metric which satisfies Definition 2.10 but not Definition 2.8. In fact, (iv) of Definition 2.8 does not hold.

Doric in the paper [[13], Th 2.3], have used the H-cone metric due to Arshad and Ahmad to prove the following result.

Theorem 2.13. *Let E be a Banach space, let P be a solid not necessarily normal cone of E and let (X,d) be a cone metric space over E. Let $\mathscr{A}$ be a family of nonempty closed, and bounded subsets of X and let there exist H-cone metric $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ induced by d. Suppose that $T,S\colon X\to\mathscr{A}$ be two cone multivalued mappings and suppose that there is $\lambda\in(0,1)$ such that $\forall x,y\in X$ at least one of the following is holds:*

1. *$\mathscr{H}$ $(Tx,Sy)\preceq d(x,y)$;*
2. *$\mathscr{H}$ $(Tx,Sy)\preceq d(x,u)$ for each fixed $u\in Tx$;*
3. *$\mathscr{H}$ $(Tx,Sy)\preceq d(y,v)$ for each fixed $v\in Sy$;*
4. *$\mathscr{H}$ $(Tx,Sy)\preceq\lambda\ \frac{d(x,v)+d(y,u)}{2}$ for each fixed $v\in Sy$ and $u\in Tx$.*

Then T and S have a common fixed point.

The following example is given by Wardowski [[6], Ex. 3.3] which satisfies Defn. 2.8.

Example 2.14. *Let $M=[0,1]$, $E=R^2$ be a Banach space with the standard norm, $P=\{(x,y)\in R^2 : x\geq 0 and y\geq 0\}$ be a normal cone and let $d\colon M\times M\to E$ be of the form $d(x,y)$=$(|x-y|,\frac{1}{2}|x-y|)$. Let $\mathscr{A}$ be a family of subsets of M of the form $\mathscr{A}=\{[0,x]:x\in M\}\cup\{\{x\}:x\in M\}$.*
We define an H-cone metric $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to\mathscr{E}$ with respect to d by the formulae

$$H(A,B)=\begin{cases}(|x-y|,\frac{1}{2}|x-y|), & for\ A=[0,x], B=[0,y],\\ (|x-y|,\frac{1}{2}|x-y|), & for\ A=\{x\},\ B=\{y\},\\ (max\{y,|x-y|\},\frac{1}{2}max\{y,|x-y|\}), & for\ A=[0,x], B=\{y\},\\ (max\{x,|x-y|\},\frac{1}{2}max\{x,|x-y|\}), & for\ A=\{x\}, B=[0,y],\end{cases}$$

In [5], the author have given the following example that satisfies defn 2.8.

Example 2.15. *Let $M=\{(1,0),(0,1),(0,0)\}$, $E=R^2$ be a Banach space with the standard norm, $P=\{(x,y)\in R^2 : x\geq 0$ and $y\geq 0\}$ be a normal cone and let $d\colon M\times M\to E$ be defined by*
$d((0,0),(0,1))$ = $d((0,1),(0,0))$ = $(1,\frac{2}{3})$.
$d((1,0),(0,0))$ = $d((0,0),(1,0))$ = $(\frac{4}{3},1)$.
$d((1,0),(0,1))$ = $d((0,1),(1,0))$ = $(\frac{7}{3},\frac{5}{3})$.
$d((1,0),(1,0))$ = $d((0,1),(0,1))$ = $(0,0)$ = $d((0,0),(0,0))$.

Then the pair (M,d) is a complete cone metric space.
Let $\mathscr{A}$ =$\{\{(0,0)\},\{(0,1)\},\{(1,0)\}\}$ be a family of subsets of M of the form
Define an H-cone metric $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ with respect to d by the formulae
$\mathscr{H}$ $(\{(0,0)\},\{(0,1)\})$=$\mathscr{H}$ $(\{(0,1)\},\{(0,0)\})$=$(1,\frac{2}{3})$.
$\mathscr{H}$ $(\{(1,0)\},\{(0,0)\})$=$\mathscr{H}$ $(\{(0,0)\},\{(1,0)\})$=$(\frac{4}{3},1)$.
$\mathscr{H}$ $(\{(1,0)\},\{(0,1)\})$=$\mathscr{H}$ $(\{(0,1)\},\{(1,0)\})$=$(\frac{7}{3},\frac{5}{3})$.
$\mathscr{H}$ $(\{(1,0)\},\{(1,0)\})$=$\mathscr{H}$ $(\{(0,1)\},\{(0,1)\})$=$\mathscr{H}$ $(\{(0,0)\},\{(0,0)\})$ =$(0,0)$.
Here we present the result by considering H-cone metric as defined by Wardowski.

3. Main Results

Theorem 3.1. *Let (M,d) be a complete cone metric space. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and $T_1,T_2\colon M\to\mathscr{A}$ be the set valued maps. Consider an H-cone metric with respect to d, $\mathscr{H}$: $\mathscr{A}\times\mathscr{A}\to E$ satisfying Defn. 2.8. Then if T_1 and T_2 satisfies the contraction condition $\mathscr{H}$ $(T_1x,T_2y)\preceq\lambda d(x,y)$, $\forall x,y\in M$, $\lambda\in(0,1)$. Then T_1 and T_2 has a common fixed point.*

Proof : Suppose that $\varepsilon_n\in$ E and $\varepsilon_n\gg\theta$, $\forall n$ and $\varepsilon_n\to\theta$. as $n\to\infty$. Let $x_0\in$ M be arbitrary and fixed.
Then $T_1(x_0)\in\mathscr{A}$. Let $x_1\in T_1(x_0)$ and $x_2\in T_2(x_1)$, then by the definition of $\mathscr{H}$ we have $d(x_1,x_2)\preceq\mathscr{H}$ $(T_1(x_0),T_2(x_1))$+ε_1.
Now for $x_1\in M$. We have $x_2\in T_2(x_1)$, there exist $x_3\in T_1(x_2)$ such that $d(x_2,x_3)\preceq\mathscr{H}$ $(T_2(x_1),T_1(x_2))$ + ε_2.
For $x_2\in M$ and $x_3\in T_1(x_2)$, $\exists$ $x_4\in T_2(x_3)$ such that $d(x_3,x_4)$ $\preceq\mathscr{H}$ $(T_1(x_2),T_2(x_3))$+ε_3.
So now for $x_3\in M$ and $x_4\in T_2(x_3)$, $\exists$ $x_5\in T_1(x_4)$ such that $d(x_4,x_5)\preceq\mathscr{H}$ $(T_2(x_3),T_1(x_4))$+ε_4.
So, for $n\geq 1$ we have, $x_{2n-1}\in T_1(x_{2n-2})$ and $x_{2n}\in T_2(x_{2n-1})$.
Now $d(x_{2n-1},x_{2n})\preceq\mathscr{H}$ $(T_1(x_{2n-2}),T_2(x_{2n-1}))$+$\varepsilon_{2n-1}$.
$\preceq\lambda$ $d(x_{2n-2},x_{2n-1})$+ε_{2n-1}.
$\preceq\lambda$ $d(x_{2n-1},x_{2n-2})$+ε_{2n-1}.
$\preceq\lambda$ $(\mathscr{H}$ $(T_1(x_{2n-2}),T_2(x_{2n-3}))$+$\varepsilon_{2n-2})$+$\varepsilon_{2n-1}$.
$\preceq\lambda^2$ $d(x_{2n-2},x_{2n-3})$+λ ε_{2n-2}+ε_{2n-1}.
$\preceq\lambda^2$ $d(x_{2n-3},x_{2n-2})$+λ ε_{2n-2}+ε_{2n-1}
$\preceq\lambda^2$ $(\mathscr{H}$ $(T_1(x_{2n-4}),T_2(x_{2n-3}))$+$\varepsilon_{2n-3})$+$\lambda$ ε_{2n-2}+ε_{2n-1}.
$\preceq\lambda^3($ $d(x_{2n-4},x_{2n-3}))$+λ^2 ε_{2n-3}+λ ε_{2n-2}+ε_{2n-1}.
$\preceq\lambda^4($ $d(x_{2n-5},x_{2n-4}))$+λ^3 ε_{2n-4}+λ^2 ε_{2n-3}+λ ε_{2n-2}+ε_{2n-1}.
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$\preceq\lambda^{2n-1}$ $d(x_0,x_1)$+λ^{2n-2} ε_1+λ^{2n-3} ε_2+λ^{2n-4} ε_3+....................+ λ^3 ε_{2n-4}+λ^2 ε_{2n-3}+$\lambda\varepsilon_{2n-2}$+$\varepsilon_{2n-1}$.
That is
$d(x_{2n},x_{2n-1})\preceq\lambda^{2n-1}$ $d(x_0,x_1)$+$\sum_{i=1}^{2n-1}\lambda^{2n-1-i}\varepsilon_i$.
suppose $m\geq n$,
$d(x_{2n},x_{2m})\preceq\sum_{j=n+1}^{m}$ $d(x_{2j},x_{2j+1})$.
$\preceq\sum_{j=n+1}^{m}$ $[\lambda^{2j-1}$ $d(x_0,x_1)$ + $\sum_{i=1}^{2j-1}$ λ^{2j-1-i} $\varepsilon_i]$.
$\preceq d(x_0,x_1)$ $\sum_{j=n+1}^{m}\lambda^{2j-1}$+$\sum_{j=n+1}^{m}$ $\sum_{i=1}^{2j-1}$ $\lambda^{2j-1-i}\varepsilon_i$.
$d(x_{2n},x_{2m})\preceq d(x_0,x_1)$ $\sum_{j=n+1}^{m}\lambda^{2j-1}$+$\sum_{j=n+1}^{m}$ $\sum_{i=1}^{2j-1}$ $\lambda^{2j-1-i}\varepsilon_i$.
Taking limit n $\to\infty$ we get,
$d(x_0,x_1)$ $\sum_{j=n+1}^{m}\lambda^{2j-1}\to\theta$ and $\sum_{j=n+1}^{m}$ $\sum_{i=1}^{2j-1}$ $\lambda^{2j-1-i}\varepsilon_i\to\theta$.
Let $c\in IntP$. Then there exist a natural number N such that we have,

$d(x_0,x_1)\sum_{j=n+1}^{m}\lambda^{2j-1} \ll \frac{c}{2}$ and $\sum_{j=n+1}^{m}\sum_{i=1}^{2j-1}\lambda^{2j-1-i}\varepsilon_i \ll \frac{c}{2}$, for all $n > N$.
Therefore, $d(x_{2n},x_{2m}) \ll c$, for all $n > N$.
which gives that $\{x_{2n}\}$ is a cauchy sequence. Since (M,d) is complete $\{x_{2n}\}$ is convergent in M. Let $x_{2n} \to x_0$. Now $x_{2n} \in T_2(x_{2n-1})\ \exists\ x^* \in T_1(x_0)$ such that
$d(x_{2n},x^*) \preceq \mathscr{H}\ (T_2(x_{2n-1}),T_1(x_0))+\varepsilon_n$.
$d(x_{2n},x^*) \preceq \lambda\ d(x_{2n-1},x_0)+\varepsilon_n$.
Taking limit n $\to \infty$ we get, $d(x_0,x^*) \preceq \varepsilon_n$.
Therefore, $x_0 = x^*$. But $x^* \in T_1(x_0)$. So, we have $x_0 \in T_1(x_0)$. That is x_0 is a fixed point of T_1. Similarly, $x_{2n-1} \in T_1(x_{2n-2})$ then $\exists\ x^{**} \in T_2(x_0)$ such that,
$d(x_{2n-1},x^{**}) \preceq \mathscr{H}\ (T_1(x_{2n-2}),T_2(x_0))+\varepsilon_n$.
$d(x_{2n-1},x^{**}) \preceq \lambda\ d(x_{2n-2},x_0)+\varepsilon_n$.
Taking limit n $\to \infty$ we get,
$d(x_0,x^{**}) \preceq \varepsilon_n$. Therefore, $x_0 = x^{**}$. But $x^{**} \in T_2(x_0)$.
So we have $x_0 \in T_2(x_0)$. That is x_0 is a fixed point of T_2.

If we take $T_1 = T_2$, in the above theorem we get the following result due to Wardowski [[6], Th 3.1].

Corollary 3.2. *Let (M,d) be a complete cone metric space with a normal coen P with a normal constant $\mathscr{K}$. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and $\mathscr{H} : \mathscr{A}\times\mathscr{A} \to E$ an be H-cone metric with respect to d. If for a map $T : M \to \mathscr{A}$ there exists $\lambda \in [0,1)$ such that $\mathscr{H}$ $(Tx,Ty) \preceq \lambda d(x,y)$, $\forall\ x,y \in M$. then $FixT \neq \phi$.*

Definition [5]: Suppose $D(x,Tx)$=$\{d(x,z) : z \in Tx\}$ and $S(x,Tx)$=$\{u \in D(x,Tx) : \|u\| = inf\{\|v\| : v \in D(x,Tx)\}$.

Theorem 3.3. *Let (M,d) be a complete cone metric space. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and $T : M \to \mathscr{A}$ be the set valued map. Consider an H-cone metric with respect to d $\mathscr{H} : \mathscr{A}\times\mathscr{A} \to E$ satisfying Defn. 2.8*
Then if T satisfies the contraction condition $\mathscr{H}\ (Tx,Ty) \preceq \lambda(S(x,Tx)+S(y,Ty))$, $\forall\ x,y \in M$. $\lambda \in [0,\frac{1}{2})$ then T has a fixed point.

Proof : Suppose that $\varepsilon_n \in$ E and $\varepsilon_n \gg \theta$, such that $\varepsilon_n \to \theta$, as $n \to \infty$. Let $x_0 \in$ M be arbitrary and fixed. Then $T(x_0) \in \mathscr{A}$. Let $x_1 \in T(x_0)$, be such that $\|d(x_0,x_1)\| = inf\{\|d(x_0,z)\|$, $\forall$ $z \in Tx_0\}$. Then $S(x_0,Tx_0) = d(x_0,x_1)$.
Let $x_2 \in T(x_1)$, such that $\|d(x_1,x_2)\|$=inf $\{\|d(x_1,z)\|,\forall z \in Tx_1\}$ .Then we have $S(x_1,Tx_1)$=$d(x_1,x_2)$.
Inductively we have for $x_{n+1} \in Tx_n$, $S(x_n,Tx_n)$=$d(x_n,x_{n+1})$.
Therefore, $d(x_n,x_{n+1}) \preceq \mathscr{H}\ (Tx_{n-1},Tx_n)$+$\varepsilon_n$.
$\preceq \lambda\ \{S(x_{n-1},Tx_{n-1})+S(x_n,Tx_n)\}$+$\varepsilon_n$.
$\preceq \lambda\ \{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}$+$\varepsilon_n$.
$d(x_n,x_{n+1}) \preceq (\frac{\lambda}{1-\lambda})d(x_{n-1},x_n)$+$\frac{\varepsilon_n}{(1-\lambda)}$.
So we have,
$\preceq (\frac{\lambda}{1-\lambda})[(\frac{\lambda}{1-\lambda})\ d(x_{n-2},x_{n-1})+\frac{\varepsilon_{n-1}}{(1-\lambda)}]+\frac{\varepsilon_n}{(1-\lambda)}$.
$\preceq (\frac{\lambda}{1-\lambda})^2\ d(x_{n-2},x_{n-1})+\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}+\frac{1}{(1-\lambda)}\ \varepsilon_n$.
$\preceq (\frac{\lambda}{1-\lambda})^3\ d(x_{n-3},x_{n-2})+\frac{\lambda^2}{(1-\lambda)^3}\ \varepsilon_{n-2}+\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}+\frac{1}{(1-\lambda)}\ \varepsilon_n$.
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$\preceq (\frac{\lambda}{1-\lambda})^n\ d(x_0,x_1)+\frac{\lambda^{n-1}}{(1-\lambda)^n}\ \varepsilon_1+\frac{\lambda^{n-2}}{(1-\lambda)^{n-1}}\ \varepsilon_2+\frac{\lambda^{n-3}}{(1-\lambda)^{n-2}}\ \varepsilon_3$
+..................+$\frac{\lambda^2}{(1-\lambda)^3}\ \varepsilon_{n-2}+\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}+\frac{1}{(1-\lambda)}\ \varepsilon_n$.
Therefore, we have
$d(x_n,x_{n+1}) \preceq \alpha^n\ d(x_0,x_1)+\sum_{r=1}^{n}\ \frac{\lambda^{n-r}}{(1-\lambda)^{n+1-r}}\ \varepsilon_r$. Where $\alpha = \frac{\lambda}{1-\lambda} < 1$.
For $m \geq n$, we have,
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1}\ d(x_j,x_{j+1})$.
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1}\ [\alpha^j\ d(x_0,x_1)+\sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r]$.
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1) + \sum_{j=n}^{m-1}\sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r$.
Taking limit n $\to \infty$ we get,
$\sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1) \to \theta$ and $\sum_{j=n}^{m-1}\sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r \to \theta$
Let $c \in IntP$, then there exist a natural number N such that,
$\sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1) \ll \frac{c}{2}$ and $\sum_{j=n}^{m-1}\sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r \ll \frac{c}{2}$, for all $n > N$.
Therefore, $d(x_n,x_m) \ll c$ That is, $\{x_n\}$ is a Cauchy sequence. Since, (M,d) is complete, $\{x_n\}$ is convergent. Let us suppose that $x_n \to x^*$ in M. We claim that x^* is the fixed point of T i.e., $x^* \in Tx^*$.
suppose that, $x_1 \in Tx^*$, such that, $\|d(x^*,x_1)\|$=inf $\{\|d(x^*,z)\|$: $z \in Tx^*\}$.
Now, for $x_n \in Tx_{n-1}$, $\exists\ x_1 \in Tx^*$, such that,
$d(x_n,x_1) \preceq \mathscr{H}\ (Tx_{n-1},Tx^*)$+$\varepsilon_n$.
$\preceq \lambda\ [S(x_{n-1},Tx_{n-1})+S(x^*,Tx^*)]$+$\varepsilon_n$.
$\preceq \lambda\ [d(x_{n-1},x_n)+d(x^*,x_1)]$+$\varepsilon_n$.
Taking $n \to \infty$, we get,
$d(x^*,x_1) \leq \lambda\ d(x^*,x_1)$. Since, $\lambda < 1\ d(x^*,x_1) = \theta$.
$x^* = x_1 \in Tx^*$. Hence, $x^* \in Tx^*$.

Example 3.4. *Consider the H-cone metric defined in Ex. 2.14. Let us define a mapping $T :\ M \to \mathscr{A}$ as follows:*

$$Tx = \begin{cases} \{0\}, & for\ x \in [0,\frac{1}{2}] \\ [0,\frac{x}{2}(x-\frac{1}{2})^2], & for\ x \in (\frac{1}{2},1] \end{cases}$$

Then T satisfies the contraction condition:
$\mathscr{H}\ (Tx,Ty) \preceq \lambda(S(x,Tx)+S(y,Ty)), \forall\ x,y \in M\ \lambda \in [0,\frac{1}{2})$.

Soln: **Case 1**: Let $x \in [0,\frac{1}{2}]$ and $y \in [0,\frac{1}{2}]$, then $Tx = \{0\}$ and $Ty = \{0\}$ so $D(x,Tx) = d(x,0) = (x,\frac{x}{2})$.
$S(x,Tx) = (x,\frac{x}{2})$ and $D(y,Ty) = d(y,0) = (y,\frac{y}{2})$, $S(y,Ty) = (y,\frac{y}{2})$.
$\mathscr{H}\ (A,B) = (\{0\},\{0\}) = (0,0)$, hence we have,
$\frac{1}{3}\ [S(x,Tx)+S(y,Ty)]$-$\mathscr{H}\ (Tx,Ty)$ = $(\alpha(x+y),\frac{\alpha}{2}(x+y)) \in P$, that is
$\mathscr{H}\ (Tx,Ty) \preceq \frac{1}{3}\ [S(x,Tx)+S(y,Ty)]$, $\forall\ x \in [0,\frac{1}{2}]$ and $y \in [0,\frac{1}{2}]$.

Case 2: Let $x \in [0,\frac{1}{2}]$ and $y \in (\frac{1}{2},1]$, then $Tx = \{0\}$ and $Ty = [0,\frac{1}{2}(y-\frac{1}{2})^2]$ so $D(x,Tx) = d(x,0) = (x,\frac{x}{2})$.
$S(x,Tx) = (x,\frac{x}{2})$ and $D(y,Ty) = \{d(y,z) : z \in Ty\} = \{d(y,z) : z \in [0,\frac{1}{2}(y-\frac{1}{2})^2]\}$ Now, $y \in (\frac{1}{2},1]$ and $z \in [0,\frac{1}{2}(y-\frac{1}{2})^2]$. Now $y \in (\frac{1}{2},1]$ implies $z \in (0,\frac{1}{8}]$.
Then it is clear that $inf\{\|d(y,z)\| : z \in Ty\}$ = $d(\frac{1}{2},\frac{1}{8})$.

So, $S(y,Ty) = d(\frac{1}{2},\frac{1}{8}) = (\frac{3}{8},\frac{3}{16})$.
Also, $\mathscr{H}\ (Tx,Ty) = (max\{0,|0-\frac{1}{2}(y-\frac{1}{2})^2|\},\frac{1}{2}max\{0,|0-\frac{1}{2}(y-\frac{1}{2})^2|\})$ = $(\frac{1}{2}(y-\frac{1}{2})^2,\frac{1}{4}(y-\frac{1}{2})^2)$.
Hence we have,
$\alpha\ [S(x,Tx)+S(y,Ty)]$ - $\mathscr{H}\ (Tx,Ty)$ = $(\alpha(x+\frac{3}{8}),\frac{\alpha}{2}(x+\frac{3}{8}))$ - $(\frac{1}{2}(y-\frac{1}{2})^2,\frac{1}{4}(y-\frac{1}{2})^2)$, that is
$\alpha\ [S(x,Tx)+S(y,Ty)]$ - $\mathscr{H}\ (Tx,Ty)$ = $(\alpha(x+\frac{3}{8})-\frac{1}{2}(y-\frac{1}{2})^2,\frac{\alpha}{2}(x+\frac{3}{8})-\frac{1}{4}(y-\frac{1}{2})^2))$.
We now find what could be the minimum value of $\alpha(x+\frac{3}{8})-\frac{1}{2}(y-\frac{1}{2})^2$ for $x\in[0,\frac{1}{2}]$ and $y\in(\frac{1}{2},1]$.
Observe that $\alpha(x+\frac{3}{8})-\frac{1}{2}(y-\frac{1}{2})^2$ is minimum. If $\frac{1}{2}(y-\frac{1}{2})^2$ is maximum and $\alpha(x+\frac{3}{8})$ is minimum.
But, $\frac{1}{2}(y-\frac{1}{2})^2$ is maximum if y is maximum i.e., $y=1$, so $\frac{1}{2}(y-\frac{1}{2})^2$ = $\frac{1}{8}$.
and $\alpha(x+\frac{3}{8})$ is minimum if x is minimum i.e., $x=0$, so $\alpha(x+\frac{3}{8})$ = $\alpha\frac{3}{8}$.
So, we have,
$\alpha\frac{3}{8}$ - $\frac{1}{8}\geq 0$.
$\alpha\frac{3}{8}\geq\frac{1}{8}$.
$\alpha\geq\frac{1}{3}$. Taking α = $\frac{1}{3}$, we get.
$\mathscr{H}\ (Tx,Ty)\preceq\frac{1}{3}\ [S(x,Tx)+S(y,Ty)]$, $\forall\ x\in[0,\frac{1}{2}]$ and $y\in(\frac{1}{2},1]$.
Hence, $\mathscr{H}\ (Tx,Ty)\preceq\alpha\ [S(x,Tx)+S(y,Ty)]$, $\forall x,y\in M$ where α = $\frac{1}{3}$.

Theorem 3.5. *Let (M,d) be a complete cone metric space. Let $\mathscr{A}$ be a nonempty collection of all nonempty closed subsets of M and $T: M\to\mathscr{A}$ be the set valued map. Consider an H-cone metric with respect to d $\mathscr{H}:\mathscr{A}\times\mathscr{A}\to E$ satisfying Defn. 2.8.*
Then if T satisfies the contraction condition
$\mathscr{H}\ (Tx,Ty)\preceq\lambda(S(x,Ty)+S(y,Tx))$, $\forall\ x,y\in M$. $\lambda\in[0,\frac{1}{2})$.
then T has a fixed point.

Proof : Suppose that $\varepsilon_n\in$ E and $\varepsilon_n\gg\theta$, such that $\varepsilon_n\to\theta$ as $n\to\infty$. Let $x_0\in$ M be arbitrary and fixed.
Then $T(x_0)\in\mathscr{A}$. Let $x_1\in T(x_0)$, then $S(x_1,Tx_0)=\theta$.
Let $x_2\in T(x_1)$, such that $\|d(x_0,x_2)\|$=inf $\{\|d(x_0,z)\|,\forall z\in Tx_1\}$. Then we have $S(x_0,Tx_1)$=$d(x_0,x_2)$.
Hence, we have, $d(x_1,x_2)$=$\mathscr{H}\ (Tx_0,Tx_1)$+ε_1.
$\preceq\lambda(S(x_0,Tx_1)+S(x_1,Tx_0))$+$\varepsilon_1$.
$\preceq\lambda d(x_0,x_2)$+$\varepsilon_1$.
$\preceq\lambda(d(x_0,x_1)+d(x_1,x_2))$+$\varepsilon_1$.
$d(x_1,x_2)\preceq\frac{\lambda}{1-\lambda}(d(x_0,x_1)$+$\frac{1}{1-\lambda}\varepsilon_1$.
Again, since $x_2\in T(x_1)$, which implies that $S(x_2,Tx_1)=\theta$.
Let $x_3\in T(x_2)$, such that $\|d(x_1,x_3)\|$=inf $\{\|d(x_1,z)\|,\forall z\in Tx_2\}$.
Then we have $S(x_1,Tx_2)$=$d(x_1,x_3)$.
Hence, we have, $d(x_2,x_3)$=$\mathscr{H}\ (Tx_1,Tx_2)$+ε_1.
$\preceq\lambda(S(x_1,Tx_2)+S(x_2,Tx_1))$+$\varepsilon_2$.
$\preceq\lambda d(x_1,x_3)$+$\varepsilon_2$.
$\preceq\lambda(d(x_1,x_2)+d(x_2,x_3))$+$\varepsilon_2$.
$d(x_2,x_3)\preceq\frac{\lambda}{1-\lambda}(d(x_1,x_2)$+$\frac{1}{1-\lambda}\varepsilon_2$.
Inductively we have for $x_{n+1}\in Tx_n$,
$d(x_n,x_{n+1})\preceq(\frac{\lambda}{1-\lambda})d(x_{n-1},x_n)$+$\frac{\varepsilon_n}{(1-\lambda)}$.
So we have,
$\preceq(\frac{\lambda}{1-\lambda})[(\frac{\lambda}{1-\lambda})\ d(x_{n-2},x_{n-1})$+$\frac{\varepsilon_{n-1}}{(1-\lambda)}]$+$\frac{\varepsilon_n}{(1-\lambda)}$.
$\preceq(\frac{\lambda}{1-\lambda})^2\ d(x_{n-2},x_{n-1})$+$\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}$+$\frac{1}{(1-\lambda)}\ \varepsilon_n$.
$\preceq(\frac{\lambda}{1-\lambda})^3\ d(x_{n-3},x_{n-2})$+$\frac{\lambda^2}{(1-\lambda)^3}\ \varepsilon_{n-2}$+$\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}$+$\frac{1}{(1-\lambda)}\ \varepsilon_n$.
. . . .
. . . .
$\preceq(\frac{\lambda}{1-\lambda})^n\ d(x_0,x_1)$+$\frac{\lambda^{n-1}}{(1-\lambda)^n}\ \varepsilon_1$+$\frac{\lambda^{n-2}}{(1-\lambda)^{n-1}}\ \varepsilon_2$+$\frac{\lambda^{n-3}}{(1-\lambda)^{n-2}}\ \varepsilon_3$
+..................+$\frac{\lambda^2}{(1-\lambda)^3}\ \varepsilon_{n-2}$+$\frac{\lambda}{(1-\lambda)^2}\ \varepsilon_{n-1}$+$\frac{1}{(1-\lambda)}\ \varepsilon_n$.
Therefore, we have,
$d(x_n,x_{n+1})\preceq\alpha^n\ d(x_0,x_1)$+$\sum_{r=1}^{n}\ \frac{\lambda^{n-r}}{(1-\lambda)^{n+1-r}}\ \varepsilon_r$. Where $\alpha=\frac{\lambda}{1-\lambda}<1$.
For $m\geq n$, we have,
$d(x_n,x_m)\preceq\sum_{j=n}^{m-1}\ d(x_j,x_{j+1})$.
$d(x_n,x_m)\preceq\sum_{j=n}^{m-1}\ [\alpha^j\ d(x_0,x_1)$+$\sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r]$.
$d(x_n,x_m)\preceq\sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1)$ + $\sum_{j=n}^{m-1}\ \sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r$.
Taking limit n $\to\infty$, we get, $\sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1)\to\theta$ and $\sum_{j=n}^{m-1}\ \sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r\to\theta$.
Let $c\in IntP$, then there exist a natural number N such that, $\sum_{j=n}^{m-1}\ \alpha^j\ d(x_0,x_1)\ll\frac{c}{2}$ and $\sum_{j=n}^{m-1}\ \sum_{r=1}^{j}\ \frac{\lambda^{j-r}}{(1-\lambda)^{j+1-r}}\ \varepsilon_r\ll\frac{c}{2}$, for all $n>N$.
Therefore, we have $d(x_n,x_m)\ll c$ for all $n>N$.
Hence, $\{x_n\}$ is a Cauchy sequence.
Since, (M,d) is complete, $\{x_n\}$ is convergent. Let us suppose that $x_n\to x^*$ in M. We claim that x^* is the fixed point of T i.e., $x^*\in Tx^*$.
Now since $x_n\to x^*$ as $n\to\infty$, we get,
$\|d(x_n,x^*)\|\to 0$ as$n\to\infty$, again since $x_n\in Tx_{n-1}$, therefore $S(x^*,Tx_{n-1})=\theta$.
Suppose that, $x_1\in Tx^*$, such that, $\|d(x_{n-1},x_1)\|$=inf $\{\|d(x_{n-1},z)\|: z\in Tx^*\}$.
So, $S(x_{n-1},Tx^*)$=$d(x_{n-1},x_1)$. Now, for $x_n\in Tx_{n-1}$, $\exists\ x_1\in Tx^*$, such that,
$d(x_n,x_1)\preceq\mathscr{H}\ (Tx_{n-1},Tx^*)$+$\varepsilon_n$.
$\preceq\lambda\ [S(x_{n-1},Tx^*)+S(x^*,Tx_{n-1})]$+$\varepsilon_n$.
$\preceq\lambda\ d(x_{n-1},x_1)$+$\varepsilon_n$.
Taking $n\to\infty$, we get,
$d(x^*,x_1)\leq\lambda\ d(x^*,x_1)$. Since, $\lambda<1$.
$d(x^*,x_1)=\theta$. $x^*=x_1\in Tx^*$. Hence, $x^*\in Tx^*$.

Example 3.6. *Consider the Example 2.15. There we take the following mapping $T((1,0))=\{(0,0)\}$, $T((0,0))=\{(0,1)\}$, $T((0,1))=\{(0,1)\}$.*

Case 1*: If $x\in\{(0,0)\}$ and $y\in\{(0,0)\}$, then $x=(0,0)$ $y=(0,0)$.*
$Tx=T(0,0)=\{(0,1)\}$ and $Ty=T(0,0)=\{(0,1)\}$.
So, $\mathscr{H}\ (Tx,Ty)$ = $\mathscr{H}\ (\{(0,1)\},\{(0,1)\})$ = $(0,0)$.
$D(x,Ty)$ = $D((0,0),T(0,0))=d((0,0),(0,1))$ = $(1,\frac{2}{3})$.
Hence, $S(x,Ty)=((1,\frac{2}{3}))$.
$D(y,Tx)$ = $D((0,0),T(0,0))$ = $d((0,0),(0,1))$ = $(1,\frac{2}{3})$.
Hence, $S(y,Tx)=((1,\frac{2}{3}))$.

$\lambda(S(x,Ty)+S(y,Tx)) = \lambda(2,\frac{4}{3})$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = $\lambda(2,\frac{4}{3}) \in P$., *for any* $\lambda \in [0,\frac{1}{2})$.
Hence, $\mathscr{H}$ $(Tx,Ty) \preceq \lambda(S(x,Ty)+S(y,Tx))$, *for any* $\lambda \in [0,\frac{1}{2})$.

Case 2*: If* $x \in \{(0,0)\}$ *and* $y \in \{(0,1)\}$, *then* $x = (0,0)$ $y = (0,1)$.
$Tx = T(0,0) = \{(0,1)\}$ *and* $Ty = T(0,1) = \{(0,1)\}$.
So, $\mathscr{H}$ (Tx,Ty) = $\mathscr{H}$ $(\{(0,1)\},\{(0,1)\})$ = $(0,0)$.
$D(x,Ty)$ = $D((0,0),T(0,1))$ = $d((0,0),(0,1))$ = $(1,\frac{2}{3})$.
Hence, $S(x,Ty) = ((1,\frac{2}{3}))$.
$D(y,Tx)$ = $D((0,1),T(0,0))$ = $d((0,1),(0,1))$ = $(0,0)$.
Hence, $S(y,Tx) = ((0,0))$.
$\lambda(S(x,Ty)+S(y,Tx)) = \lambda(1,\frac{2}{3})$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = $\lambda(1,\frac{2}{3})$., $\lambda \in [0,\frac{1}{2})$.

Case 3*: If* $x \in \{(0,0)\}$ *and* $y \in \{(1,0)\}$, *then* $x = (0,0)$ $y = (1,0)$.
$Tx = T(0,0) = \{(0,1)\}$ *and* $Ty = T(1,0) = \{(0,0)\}$.
So, $\mathscr{H}$ (Tx,Ty) = $\mathscr{H}$ $(\{(0,1)\},\{(0,0)\})$ = $(1,\frac{2}{3})$.
$D(x,Ty)$ = $D((0,0),T(1,0))$ = $d((0,0),(0,0))$ = $(0,0)$.
Hence, $S(x,Ty) = (0,0)$.
$D(y,Tx)$ = $D((1,0),T(0,0))$ = $d((1,0),(0,1))$ = $(\frac{7}{3},\frac{5}{3})$.
Hence, $S(y,Tx)$ = $(\frac{7}{3},\frac{5}{3})$.
$\lambda(S(x,Ty)+S(y,Tx)) = \lambda(\frac{7}{3},\frac{5}{3})$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = $(\lambda\frac{7}{3}-1,\lambda\frac{5}{3}-\frac{2}{3}) \in P$.
If $\lambda\frac{7}{3}-1 \geq 0$ *if* $\lambda\frac{7}{3} \geq 1$ *that is* $\lambda \geq \frac{3}{7}$ *and also if*
$\lambda\frac{5}{3}-\frac{2}{3} \geq 0$ *if* $\lambda\frac{5}{3} \geq \frac{2}{3}$ *that is* $\lambda \geq \frac{2}{5}$.
So if we take $\lambda = \frac{1}{4}$, *we get,*
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ $(Tx,Ty) \in P$, *for any* $\lambda = \frac{1}{4}$.
$\mathscr{H}$ $(Tx,Ty) \preceq \lambda(S(x,Ty)+S(y,Tx)), \forall\, x,y \in M$, *with* $\lambda = \frac{1}{4}$.

Case 4*: If* $x \in \{(1,0)\}$ *and* $y \in \{(0,1)\}$, *then* $x = (1,0)$ $y = (0,1)$.
$Tx = T(1,0) = \{(0,0)\}$ *and* $Ty = T(0,1) = \{(0,1)\}$.
So, $\mathscr{H}$ (Tx,Ty) = $\mathscr{H}$ $(\{(0,0)\},\{(0,1)\})$ = $(1,\frac{2}{3})$.
$D(x,Ty)$ = $D((1,0),T(0,1))$ = $d((1,0),(0,1))$ = $(\frac{7}{3},\frac{5}{3})$.
Hence, $S(x,Ty)$ = $(\frac{7}{3},\frac{5}{3})$.
$D(y,Tx)$ = $D((0,1),T(1,0))$ = $d((0,1),(0,0))$ = $(1,\frac{2}{3})$.
Hence, $S(y,Tx) = (1,\frac{2}{3})$.
$\lambda(S(x,Ty)+S(y,Tx)) = \lambda(\frac{10}{3},\frac{7}{3})$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = $(\lambda\frac{10}{3}-1,\lambda\frac{7}{3}-\frac{2}{3}) \in P$.
If $\lambda\frac{10}{3}-1 \geq 0$ *if* $\lambda\frac{10}{3} \geq 1$ *that is* $\lambda \geq \frac{3}{10}$ *and also if*
$\lambda\frac{7}{3}-\frac{2}{3} \geq 0$ *if* $\lambda\frac{7}{3} \geq \frac{2}{3}$ *that is* $\lambda \geq \frac{2}{7}$
So if we take $\lambda = \frac{1}{4}$, *we get,*
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ $(Tx,Ty) \in P$, *for any* $\lambda = \frac{1}{4}$.
$\mathscr{H}$ $(Tx,Ty) \preceq \lambda(S(x,Ty)+S(y,Tx)), \forall\, x,y \in M$, *with* $\lambda = \frac{1}{4}$.

Case 5*: If* $x \in \{(0,1)\}$ *and* $y \in \{(0,1)\}$, *then* $x = (0,1)$ $y = (0,1)$.
$Tx = T(0,0) = \{(0,1)\}$ *and* $Ty = T(0,1) = \{(0,1)\}$.
So, $\mathscr{H}$ (Tx,Ty) = $\mathscr{H}$ $(\{(0,1)\},\{(0,1)\})$ = $(0,0)$.
$D(x,Ty)$ = $D((0,1),T(0,1))$ = $d((0,1),(0,1))$ = $(0,0)$.
Hence, $S(x,Ty) = (0,0)$.
$D(y,Tx)$ = $D((0,1),T(0,1))$ = $d((0,1),(0,1))$ = $(0,0)$.
Hence, $S(y,Tx) = ((0,0))$.
$\lambda(S(x,Ty)+S(y,Tx)) = (0,0)$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = *(0,0)* $\in P$, *for any* $\lambda \in [0,\frac{1}{2})$.
$\mathscr{H}$ $(Tx,Ty) \preceq \lambda(S(x,Ty)+S(y,Tx))$, $\forall\, x,y \in M$, *with* $\lambda \in [0,\frac{1}{2})$.

Case 6*: If* $x \in \{(1,0)\}$ *and* $y \in \{(1,0)\}$, *then* $x = (1,0)$ $y = (1,0)$.
$Tx = T(1,0) = \{(0,0)\}$ *and* $Ty = T(1,0) = \{(0,0)\}$.
So, $\mathscr{H}$ (Tx,Ty) = $\mathscr{H}$ $(\{(0,0)\},\{(0,0)\})$ = $(0,0)$.
$D(x,Ty)$ = $D((1,0),T(1,0))$ = $d((1,0),(0,0))$ = $(\frac{4}{3},1)$.
Hence, $S(x,Ty) = (\frac{4}{3},1)$.
$D(y,Tx)$ = $D((1,0),T(1,0))$ = $d((1,0),(0,0))$ = $(\frac{4}{3},1)$.
Hence, $S(y,Tx)$ = $(\frac{4}{3},1)$.
$\lambda(S(x,Ty)+S(y,Tx))$ = λ $(\frac{8}{3},1)$.
$\lambda(S(x,Ty)+S(y,Tx))$ - $\mathscr{H}$ (Tx,Ty) = λ $(\frac{8}{3},1) \in P$, *for any* $\lambda \in [0,\frac{1}{2})$.
$\mathscr{H}$ $(Tx,Ty) \preceq \lambda(S(x,Ty)+S(y,Tx))$, $\forall\, x,y \in M$, *with* $\lambda \in [0,\frac{1}{2})$.

Theorem 3.7. *Let* (M,d) *be a complete cone metric space. Let* $\mathscr{A}$ *be a nonempty collection of all nonempty closed subsets of M and* $T: M \to \mathscr{A}$ *be the set valued map. Consider an H-cone metric with respect to d* $\mathscr{H}: \mathscr{A} \times \mathscr{A} \to E$ *satisfying Defn. 2.8. Then if T satisfies the contraction condition* $\mathscr{H}$ $(Tx,Ty) \preceq \{a_1S(x,Tx)+a_2S(y,Ty)+a_3d(x,y)\}$, $\forall\, x,y \in M$. $a_i \geq 0\, \forall, i = 1,2,3$ *and* $a_1+a_2+a_3 < 1$. *Then T has a fixed point.*

Proof : Suppose that $\varepsilon_n \in$ E and $\varepsilon_n \gg \theta$, such that $\varepsilon_n \to \theta$ as $n \to \infty$.
Let $x_0 \in$ M be arbitrary and fixed.
Then $T(x_0) \in \mathscr{A}$. Let $x_1 \in T(x_0)$, be such that $\|d(x_0,x_1)\| = inf\{\|d(x_0,z)\|, \forall\, z \in Tx_0\}$.
Then $S(x_0,Tx_0) = d(x_0,x_1)$.
Let $x_2 \in T(x_1)$, such that $\|d(x_1,x_2)\|$=inf $\{\|d(x_1,z)\|, \forall z \in Tx_1\}$.
Then we have $S(x_1,Tx_1)$=$d(x_1,x_2)$.
Inductively we have for $x_{n+1} \in Tx_n$, $S(x_n,Tx_n)$=$d(x_n,x_{n+1})$.
Therefore, $d(x_n,x_{n+1}) \preceq \mathscr{H}$ (Tx_{n-1},Tx_n)+ε_n.
$\preceq \{a_1S(x_{n-1},Tx_{n-1})+a_2S(x_n,Tx_n)+a_3d(x_{n-1},x_n)\}$+$\varepsilon_n$.
$\preceq \{a_1d(x_{n-1},x_n)+a_2d(x_n,x_{n+1})+a_3d(x_{n-1},x_n)\}$+$\varepsilon_n$.
$(1-a_2)$ $d(x_n,x_{n+1}) \preceq (a_1+a_3)d(x_{n-1},x_n)$+$\varepsilon_n$.
$d(x_n,x_{n+1}) \preceq \frac{(a_1+a_3)}{(1-a_2)}$ $d(x_{n-1},x_n)$+$\frac{\varepsilon_n}{(1-a_2)}$.
So we have, $\preceq \frac{(a_1+a_3)}{(1-a_2)}$ $[\frac{(a_1+a_3)}{(1-a_2)}$ $d(x_{n-2},x_{n-1})$+$\frac{\varepsilon_{n-1}}{(1-a_2)}]$+$\frac{\varepsilon_n}{(1-a_2)}$.
$\preceq (\frac{a_1+a_3}{1-a_2})^2$ $d(x_{n-2},x_{n-1})$+$\frac{a_1+a_3}{(1-a_2)^2}$ ε_{n-1}+$\frac{1}{(1-a_2)}$ ε_n.
$\preceq (\frac{a_1+a_3}{1-a_2})^3$ $d(x_{n-3},x_{n-2})$+$\frac{(a_1+a_3)^2}{(1-a_2)^3}$ ε_{n-2}+$\frac{(a_1+a_3)}{(1-a_2)^2}$ ε_{n-1}+$\frac{1}{(1-a_2)}$ ε_n.
. . . .

. . . .
$\preceq (\frac{a_1+a_3}{1-a_2})^n\ d(x_0,x_1) + \frac{(a_1+a_3)^{n-1}}{(1-a_2)^n}\ \varepsilon_1 + \frac{(a_1+a_3)^{n-2}}{(1-a_2)^{n-1}}\ \varepsilon_2 + \frac{(a_1+a_3)^{n-3}}{(1-a_2)^{n-2}}\ \varepsilon_3$
$+ \ldots\ldots\ldots\ldots + \frac{(a_1+a_3)^2}{(1-a_2)^3}\ \varepsilon_{n-2} + \frac{a_1+a_3}{(1-a_2)^2}\ \varepsilon_{n-1} + \frac{1}{(1-a_2)}\ \varepsilon_n.$
Therefore, we have,
$d(x_n,x_{n+1}) \preceq \alpha^n\ d(x_0,x_1) + \sum_{r=1}^{n} \frac{(a_1+a_3)^{n-r}}{(1-a_2)^{n+1-r}}\ \varepsilon_r$. Where $\alpha = \frac{a_1+a_3}{1-a_2} < 1$.
For $m \geq n$, we have,
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1} d(x_j,x_{j+1})$.
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1} [\alpha^j\ d(x_0,x_1) + \sum_{r=1}^{j} \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j+1-r}}\ \varepsilon_r]$.
$d(x_n,x_m) \preceq \sum_{j=n}^{m-1} \alpha^j\ d(x_0,x_1) + \sum_{j=n}^{m-1} \sum_{r=1}^{j} \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j+1-r}}\ \varepsilon_r$.
Taking limit n $\to \infty$, we have,
$\sum_{j=n}^{m-1} \alpha^j\ d(x_0,x_1) \to \theta$ and $\sum_{j=n}^{m-1} \sum_{r=1}^{j} \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j+1-r}}\ \varepsilon_r \to \theta$.
Let $c \in IntP$, then there exist a natural number N such that,
$\sum_{j=n}^{m-1} \alpha^j\ d(x_0,x_1) \ll \frac{c}{2}$ and $\sum_{j=n}^{m-1} \sum_{r=1}^{j} \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j+1-r}}\ \varepsilon_r \ll \frac{c}{2}$, for all $n > N$.
Therefore, we have,
$d(x_n,x_m) \ll c$. Hence, $\{x_n\}$ is a Cauchy sequence.
Since, (M,d) is complete, $\{x_n\}$ is convergent. Let us suppose that $x_n \to x^*$ in M. We claim that x^* is the fixed point of T i.e., $x^* \in Tx^*$.
suppose that, $x_1 \in Tx^*$, such that, $\|d(x^*,x_1)\|$=inf $\{\|d(x^*,z)\| : z \in Tx^*\}$.
Now, for $x_n \in Tx_{n-1}$, $\exists\ x_1 \in Tx^*$, such that,
$d(x_n,x_1) \preceq \mathscr{H}\ (Tx_{n-1},Tx^*)$+$\varepsilon_n$.
$\preceq [a_1S(x_{n-1},Tx_{n-1}) + a_2S(x^*,Tx^*) + a_3d(x_{n-1},x^*)]$+$\varepsilon_n$.
$\preceq [a_1d(x_{n-1},x_n) + a_2d(x^*,x_1) + a_3d(x_{n-1},x^*)]$+$\varepsilon_n$.
Taking $n \to \infty$, we get,
$d(x^*,x_1) \preceq a_2d(x^*,x_1)$. Since, $a_2 < 1$. $d(x^*,x_1) = \theta$.
$x^* = x_1 \in Tx^*$. Hence, $x^* \in Tx^*$.

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