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Convergence, data dependence and T-stability of AK iteration procedure for contractive like operators

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Abstract
In this paper, we prove the strong convergence of AK iteration procedure to a fixed point of a contractive like operator defined on an arbitrary nonempty closed convex subset of a normed linear space. Further, we study data dependence and T-stability of this procedure. Our results generalize the results that are available in the existing literature.

Keywords
Fixed point, contractive like operator, AK iteration procedure, Data dependence and $T-$stability.

AMS Subject Classification
47H10, 54H25.

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1. Introduction

Throughout this paper, let $(X,||.||)$ be a normed linear space and we denote it by X. Let K be a nonempty closed convex subset of X and $T:K\to K$ be a selfmap of K. We denote the set of all fixed points of T by $F(T)$.

Harder and Hicks [2] initiated the stability of general fixed point iteration procedure with respect to a selfmap $T:K\to K$ is as follows:

Definition 1.1. [2] *Let K be a nonempty closed convex subset of X and $T:K\to K$ be a selfmap. Let $x_0\in K$. Assume that the iteration procedure is defined by $x_{n+1}=f(T,x_n)$ for $n=0,1,\ldots$. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in K and set $\varepsilon_n=d(t_{n+1},f(T,t_n))$ for $n=0,1,\ldots$. Then the fixed point iteration procedure is said to be T-stable if $\lim\limits_{n\to\infty}\varepsilon_n=0$ if and only if $\lim\limits_{n\to\infty}t_n=p$.*

Definition 1.2. *Let $T,\tilde{T}:K\to K$ be two selfmaps. If there exists $\eta>0$ such that $||Tx-\tilde{T}x||\leq\eta$ for all $x\in K$ then we say that $\tilde{T}$ is an approximate operator of T with $\eta>0$.*

In 2016, Ullah and Arshad [4] introduced AK iteration procedure as follows:

$$\begin{cases} x_0\in K\\ z_n=T((1-\beta_n)x_n+\beta_nTx_n)\\ y_n=T((1-\alpha_n)z_n+\alpha_nTz_n)\\ x_{n+1}=Ty_n\end{cases}\tag{1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

Ullah and Arshad [4] proved the convergence, data dependence and T-stability of AK iteration procedure under certain assumptions on $\alpha_n's$ for contraction maps as follows.

Theorem 1.3. [4] Let K be a nonempty closed convex subset of a Banach space X and $T:K\to K$ be a contraction mapping. For $x_0\in K$, let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by AK iteration procedure with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0,1] satisfying $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

Theorem 1.4. [4] Let X,K,T be as in Theorem 1.3. Let $x_0\in K$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by AK iteration procedure with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{n=0}^{\infty}\}$

in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the AK iteration procedure is T-stable.

Theorem 1.5. [4] Let X, K, T be as in Theorem 1.3. Let $\tilde{T}$ be an approximate operator of a contraction map T with $\eta > 0$. For x_0 in K, let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1) for T and define an iterative sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ as follows.

$$\begin{cases} \tilde{x}_0 \in K \\ \tilde{z}_n = \tilde{T}((1-\beta_n)\tilde{x}_n + \beta_n \tilde{T}\tilde{x}_n) \\ \tilde{y}_n = \tilde{T}((1-\alpha_n)\tilde{z}_n + \alpha_n \tilde{T}\tilde{z}_n) \\ \tilde{x}_{n+1} = \tilde{T}\tilde{y}_n \end{cases} \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that (i) $\alpha_n \geq \frac{1}{2}$ for $n = 0,1,2...$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
If $Tp = p$, $\tilde{T}\tilde{p} = \tilde{p}$ and $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$ then $||p - \tilde{p}|| \leq \frac{9\eta}{1-\delta}$.

Ertürk [1] proved that convergence and data dependence for AK iteration procedure with certain assumptions on $\alpha_n' s$ and $\beta_n' s$ for the mapping $T : K \to K$ that satisfies the condition

$$||Tx - Ty|| \leq \delta ||x - y|| + 2\delta ||x - Tx|| \tag{3}$$

for all $x, y \in K$ and for some $0 < \delta < 1$.

Theorem 1.6. [1] Let $T : K \to K$ be an operator that satisfies the inequality (3). For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by (1) where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \beta_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point p of T.

Theorem 1.7. [1] Let $T : K \to K$ be an operator that satisfies the inequality (3), $\tilde{T} : K \to K$ be an approximate operator of T with $\eta > 0$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be the sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \beta_n = \infty$. For $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ and $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the sequences generated by (1) and (2) respectively. If $Tp = p$, $\tilde{T}\tilde{p} = \tilde{p}$ and $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$ then $||p - \tilde{p}|| \leq \frac{\eta}{1-\delta}$.

Based on the inequality (3), Imoru and Olatinwo [3] defined contractive like operator as follows:

Definition 1.8. *An operator $T : K \to K$ is called a contractive like operator if there exist a constant $\delta \in (0,1)$ and a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ satisfying*

$$||Tx - Ty|| \leq \delta ||x - y|| + \varphi(||x - Tx||) \tag{4}$$

for all $x, y \in K$.

We note that every contraction mapping is a contractive like operator. But a contractive like operator is not a contraction mapping (Example 2.5).

Remark 1.9. *The operator T that satisfies the inequality (3) is a contractive like operator with $\varphi(t) = 2\delta t$ for $t \geq 0$.*

In section 2, we prove the strong convergence of AK iteration procedure for contractive like operators defined on a nonempty closed convex subset of X and its convergence is independent of the choices of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ and provide an example in support of our result. In Section 3, we prove data dependence of AK iteration procedure for a contractive like operator for any choices of $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. Further, we show that the conditions on $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ of Theorem 1.5 and Theorem 1.7 are redundant. In Section 4, we prove that the AK iteration procedure is T-stable for contractive like operators for any choices of $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. Our results generalize the results of Ullah and Arshad [4].

2. Convergence of AK iteration procedure

Theorem 2.1. Let K be a nonempty closed convex subset of a normed linear space X and $T : K \to K$ be a contractive like operator. Assume that $F(T) \neq \emptyset$. For $x_0 \in K$, let the sequence $\{x_n\}_{n=0}^{\infty}$ be generated by AK iteration procedure with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point of T.

Proof. **Proof.** Since a contractive like operator has at most one fixed point and $F(T) \neq \emptyset$, we suppose that $F(T) = \{p\}$. We consider

$$\begin{aligned} ||x_{n+1} - p|| &= ||Ty_n - Tp|| \\ &\leq \delta ||y_n - p|| + \varphi(||p - Tp||) \\ &= \delta ||y_n - p|| \text{ for } n = 0,1,2... \text{ (since } \varphi(0) = 0) . \end{aligned}$$

Therefore

$$||x_{n+1} - p|| \leq \delta ||y_n - p||. \tag{5}$$

We now consider

$$\begin{aligned} ||y_n - p|| &= ||T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tp|| \\ &\leq \delta ||(1-\alpha_n)z_n + \alpha_n Tz_n - p|| + \varphi(||p - Tp||) \\ &\leq \delta [(1-\alpha_n)||z_n - p|| + \alpha_n ||Tz_n - Tp||] \\ &\leq \delta (1-\alpha_n)||z_n - p|| + \alpha_n \delta [\delta ||z_n - p|| \\ &\qquad + \varphi(||p - Tp||)] \\ &= \delta [1 - \alpha_n(1-\delta)]||z_n - p|| \end{aligned}$$

so that

$$||y_n - p|| \leq \delta ||z_n - p||. \tag{6}$$

Now we consider

$$\begin{aligned} ||z_n - p|| &= ||T((1-\beta_n)x_n + \beta_n Tx_n) - Tp|| \\ &\leq \delta ||(1-\beta_n)x_n + \beta_n Tx_n - p|| + \varphi(||p - Tp||) \\ &\leq \delta [(1-\beta_n)||x_n - p|| + \beta_n ||Tx_n - Tp||] \\ &\leq \delta [(1-\beta_n)||x_n - p|| + \beta_n \delta ||x_n - p|| \\ &\qquad + \beta_n \varphi(||p - Tp||)] \\ &= \delta [1 - \beta_n(1-\delta)]||x_n - p||. \end{aligned}$$

Hence

$$||z_n - p|| \leq \delta ||x_n - p||. \tag{7}$$

From (5), (6) and (7) we have
$||x_{n+1}-p|| \leq \delta^3||x_n-p||$
$\leq \delta^{3+3}||x_{n-1}-p||.$
On continuing this process, it follows that

$$||x_{n+1}-p|| \leq \delta^{3n+3}||x_0-p||.$$

As $0<\delta<1$, we have $\lim_{n\to\infty} x_n = p$. Thus the sequence $\{x_n\}_{n=0}^{\infty}$ generated by AK iteration procedure converges to the unique fixed point p of T. □

Note 2.2. *In the proof of Theorem 2.1, we did not use any conditions on the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0,1].*

Remark 2.3. As $||Tx-p|| \leq \delta||x-p||$ for all $x \in K$, T is continuous at the fixed point.

Remark 2.4. From (5), (6) and (7), we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = p$.

In the following example we show that Theorem 2.1 is independent of the choices of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ of [0,1].

Example 2.5. Let $X=R$ with the usual norm. Let $K=[1,3]$. Let $x_0 \in K$ be arbitrary. We define $\varphi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=\frac{3t^2}{4}$ for $t \geq 0$ so that φ is a strictly increasing continuous function with $\varphi(0)=0$. Now we define $T:[1,3]\to[1,3]$ by $Tx = 2+\frac{1}{x}$ so that T satisfies
$||Tx-Ty|| \leq \frac{2}{3}||x-y|| + \varphi(||x-Tx||)$ for all $x,y \in K$. Here we observe that T is not a contraction map.

We show that in all the possible cases of $\{\alpha_n\}$ and $\{\beta_n\}$, the AK iteration procedure converges to the unique fixed point $1+\sqrt{2}$ of T.

Case (i): We take $\alpha_n = \frac{n+1}{n+2}$ and $\beta_n = \frac{1}{n^2+1}$ so that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty$.
So for any $x_0 \in [1,3]$, $z_n = T((1-\beta_n)x_n + \beta_n Tx_n)$
$= T((1-\frac{1}{n^2+1})x_n + \frac{1}{n^2+1}(2+\frac{1}{x_n}))$
$= \frac{(2x_n^2+x_n)n^2+5x_n+2}{n^2x_n^2+(2x_n+1)}.$
Therefore $y_n = T((1-\alpha_n)z_n + \alpha_n Tz_n)$
$= T((1-\frac{n+1}{n+2})z_n + \frac{n+1}{n+2}(2+\frac{1}{z_n}))$
$= T(\frac{z_n^2+n(2z_n+1)+2z_n+1}{(n+2)z_n})$
$= \frac{(5z_n+2)n+2z_n^2+6z_n+2}{(2z_n+1)n+z_n^2+2z_n+1}.$
Therefore $x_{n+1} = Ty_n = \frac{(12z_n+5)n+5z_n^2+14z_n+5}{(5z_n+2)n+2z_n^2+6z_n+2}$
$= \frac{5z_n^2+(12n+14)z_n+5n+5}{2z_n^2+(5n+6)z_n+2n+2} = \frac{N}{D},$
where $N = (29x_n^4+12x_n^3)n^5 + (53x_n^4+34x_n^3+5x_n^2)n^4 + (128x_n^3+82x_n^2+12x_n)n^3 + (246x_n^3+184x_n^2+34x_n)n^2 + (140x_n^2+128x_n+29)n + (285x_n^2+246x_n+53)$ and $D = (12x_n^4+5x_n^3)n^5 + (22x_n^4+14x_n^3+2x_n^2)n^4 + (53x_n^3+34x_n^2+5x_n)n^3 + (102x_n^3+76x_n^2+14x_n)n^2 + (58x_n^2+53x_n+12)n + (118x_n^2+102x_n+22)$.
Therefore $x_{n+1} = \frac{29x_n+12}{12x_n+5} + A_n$ where
$A_n = \frac{(-2x_n^5+3x_n^4+4x_n^3+x_n^2)n^4+(-x_n^4+2x_n^3+x_n^2)n^3+(-6x_n^4+10x_n^3+10x_n^2+2x_n)n^2}{(12x_n+5)D}$
$+ \frac{(-2x_n^3+3x_n^2+4x_n+1)n+(-2x_n^3+3x_n^2+4x_n+1)}{(12x_n+5)D}.$
It is easy to see that $|A_n| \leq \frac{846n^4+144n^3+852n^2+94n+94}{289n^5+646n^4+1564n^3+3264n^2+2091n+4114}$ so that $\lim_{n\to\infty} A_n = 0$. Therefore
$x_{n+1}-(1+\sqrt{2}) = \frac{29x_n+12}{12x_n+5} - (1+\sqrt{2}) + A_n$
$= \frac{(17-12\sqrt{2})x_n+(7-5\sqrt{2})}{12x_n+5} + A_n$
$= \frac{(17-12\sqrt{2})}{12x_n+5}[x_n + \frac{7-5\sqrt{2}}{17-12\sqrt{2}}] + A_n.$
Hence $|x_{n+1}-(1+\sqrt{2})| \leq \frac{17-2\sqrt{2}}{17}|x_n-(1+\sqrt{2})| + |A_n|$ for $n=0,1,2...$ so that
$\limsup|x_{n+1}-(1+\sqrt{2})| \leq \frac{17-12\sqrt{2}}{17}\limsup|x_n-(1+\sqrt{2})|$ and hence $\limsup|x_n-(1+\sqrt{2})| \leq 0$, i.e., $\lim_{n\to\infty} x_n = 1+\sqrt{2}$.

Case (ii) : We take $\alpha_n = \frac{1}{2^n}$ and $\beta_n = \frac{1}{3^n}$ so that $\sum_{n=0}^{\infty}\alpha_n < \infty$, $\sum_{n=0}^{\infty}\beta_n < \infty$ and $\sum_{n=0}^{\infty}\alpha_n\beta_n < \infty$.
Therefore for any $x_0 \in [1,3]$, we have
$z_n = \frac{3^n(2x_n^2+x_n)+2(-x_n^2+2x_n+1)}{3^nx_n^2+(-x_n^2+2x_n+1)}$, $y_n = \frac{2^n(2z_n^2+z_n)+2(-z_n^2+2z_n+1)}{2^nz_n^2+(-z_n^2+2z_n+1)}$ and $x_{n+1} = \frac{2^n(5z_n^2+2z_n)+5(-z_n^2+2z_n+1)}{2^n(2z_n^2+z_n)+2(-z_n^2+2z_n+1)}$.
It is easy to write $x_{n+1} = \frac{12x_n+5}{5x_n+2} + B_n$ where $\lim_{n\to\infty} B_n = 0$.
Therefore $(x_{n+1}-(1+\sqrt{2})) = \frac{7-5\sqrt{2}}{5x_n+2}(x_n + \frac{3-2\sqrt{2}}{7-5\sqrt{2}}) + B_n$ so that $|x_{n+1}-(1+\sqrt{2})| \leq \frac{7-5\sqrt{2}}{7}|x_n-(1+\sqrt{2})| + |B_n|$ for $n=0,1,2...$. By applying limit superior on both sides, we have $\limsup|x_{n+1}-(1+\sqrt{2})| \leq \frac{7-5\sqrt{2}}{7}\limsup|x_n-(1+\sqrt{2})| + \limsup|B_n|$. Hence $\limsup|x_n-(1+\sqrt{2})| \leq 0$, i.e., $\lim_{n\to\infty} x_n = 1+\sqrt{2}$.
Case (iii) : We take $\alpha_0 = \beta_0 = 0$, $\alpha_n = \frac{1}{\sqrt{n}}$ $\beta_n = \frac{1}{\sqrt{n}}$ for $n=1,2,...$ so that $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=0}^{\infty}\beta_n = \infty$, $\sum_{n=0}^{\infty}\alpha_n\beta_n = \infty$. For any $x_0 \in [1,3]$, $z_n = \frac{\sqrt{n}(2x_n^2+x_n)+2(-x_n^2+2x_n+1)}{\sqrt{n}x_n^2+(-x_n^2+2x_n+1)}$,
$y_n = \frac{\sqrt{n}(2z_n^2+z_n)+2(-z_n^2+2z_n+1)}{\sqrt{n}z_n^2+(-z_n^2+2z_n+1)}$ and $x_{n+1} = \frac{z_n^2(5\sqrt{n}-5)+z_n(2\sqrt{n}+10)+5}{z_n^2(2\sqrt{n}-2)+z_n(\sqrt{n}+4)+2}$.
It is easy to write $x_{n+1} = \frac{12x_n+5}{5x_n+2} + C_n$ where $\lim_{n\to\infty} C_n = 0$ and by proceeding as in *Case* (ii), the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point $1+\sqrt{2}$ of T.

Remark 2.6. It is easy to see that Theorem 1.3 follows as a corollary to Theorem 2.1. Since the map T that is defined in Example 2.5 is a contractive like operator but not a contraction map, it follows that Theorem 2.1 generalizes Theorem 1.3. Also, we note that the assumption $\sum_{n=0}^{\infty}\alpha_n = \infty$ of Theorem 1.3 is redundant.

Remark 2.7. By Remark 1.9, Theorem 1.6 follows as a corollary to Theorem 2.1 and the condition $\sum_{n=0}^{\infty}\beta_n = \infty$ of Theorem 1.6 is redundant.

3. Data dependence

Theorem 3.1. Let X,K,T be as in Theorem 2.1 and $\tilde{T}$ be an approximate operator of T with $\eta > 0$. Let $x_0, \tilde{x}_0 \in K$. Let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence generated by AK iteration procedure (1) with respect to T and and $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the sequence defined by (2). If $Tp = p$, $\tilde{T}\tilde{p} = \tilde{p}$ and $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$ then $||p-\tilde{p}|| \leq \frac{\delta^3\eta+2\delta^2\eta+\delta\eta+\eta}{1-\delta^3}$.

Proof. **Proof.** By Theorem 2.1, we have $\lim_{n\to\infty} x_n = p$ with $Tp = p$.
We consider

$$\begin{aligned}||x_{n+1}-\tilde{x}_{n+1}|| &= ||Ty_n - \tilde{T}\tilde{y}_n||\\ &\leq ||Ty_n - T\tilde{y}_n|| + ||T\tilde{y}_n - \tilde{T}\tilde{y}_n||\\ &\leq \delta||y_n-\tilde{y}_n|| + \varphi(||y_n - Ty_n||)+\eta.\end{aligned}$$

We now consider

$$\begin{aligned}||y_n-\tilde{y}_n|| &= ||T((1-\alpha_n)z_n+\alpha_nTz_n) - \tilde{T}((1-\alpha_n)\tilde{z}_n+\alpha_n\tilde{T}\tilde{z}_n)||\\ &\leq ||T((1-\alpha_n)z_n+\alpha_nTz_n) - T((1-\alpha_n)\tilde{z}_n+\alpha_n\tilde{T}\tilde{z}_n)||\\ &\quad + ||T((1-\alpha_n)\tilde{z}_n+\alpha_n\tilde{T}\tilde{z}_n) - \tilde{T}((1-\alpha_n)\tilde{z}_n+\alpha_n\tilde{T}\tilde{z}_n)||\\ &\leq \delta||(1-\alpha_n)(z_n-\tilde{z}_n)+\alpha_n(Tz_n-\tilde{T}\tilde{z}_n)||\\ &\quad +\varphi(||(1-\alpha_n)z_n+\alpha_nTz_n-y_n||)+\eta\\ &\leq \delta[(1-\alpha_n)||z_n-\tilde{z}_n||+\alpha_n||Tz_n-\tilde{T}\tilde{z}_n||]\\ &\quad +\varphi(||(1-\alpha_n)z_n+\alpha_nTz_n-y_n||)+\eta\\ &\leq \delta[(1-\alpha_n)||z_n-\tilde{z}_n||+\alpha_n(\delta||z_n-\tilde{z}_n||\\ &\quad +\varphi(||z_n-Tz_n||)+\eta)]+\varphi(||(1-\alpha_n)z_n\\ &\quad +\alpha_nTz_n-y_n||)+\eta\\ &\leq \delta[(1-\alpha_n(1-\delta))||z_n-\tilde{z}_n||\\ &\quad +\alpha_n\varphi(||z_n-Tz_n||)+\alpha_n\eta]\\ &\quad +\varphi(||(1-\alpha_n)z_n+\alpha_nTz_n-y_n||)+\eta\\ &\leq \delta||z_n-\tilde{z}_n||+\delta\alpha_n\varphi(||z_n-Tz_n||)+\delta\alpha_n\eta\\ &\quad +\varphi(||(1-\alpha_n)z_n+\alpha_nTz_n-y_n||)+\eta\\ &= \delta||z_n-\tilde{z}_n||+\alpha_n\delta\eta+A_n+\eta, \text{ where}\end{aligned}$$

$A_n = \delta\alpha_n\varphi(||z_n-Tz_n||)+\varphi(||(1-\alpha_n)z_n+\alpha_nTz_n-y_n||)$.
Now, we have

$$\begin{aligned}||z_n-\tilde{z}_n|| &= ||T((1-\beta_n)x_n+\beta_nTx_n) - \tilde{T}((1-\beta_n)\tilde{x}_n+\beta_n\tilde{T}\tilde{x}_n)||\\ &\leq ||T((1-\beta_n)x_n+\beta_nTx_n) - T((1-\beta_n)\tilde{x}_n+\beta_n\tilde{T}\tilde{x}_n)||\\ &\quad + ||T((1-\beta_n)\tilde{x}_n+\beta_n\tilde{T}\tilde{x}_n) - \tilde{T}((1-\beta_n)\tilde{x}_n+\beta_n\tilde{T}\tilde{x}_n)||\\ &\leq \delta||(1-\beta_n)(x_n-\tilde{x}_n)+\beta_n(Tx_n-\tilde{T}\tilde{x}_n)||\\ &\quad +\varphi(||(1-\beta_n)x_n+\beta_nTx_n-z_n||)+\eta\\ &\leq \delta[(1-\beta_n)||x_n-\tilde{x}_n||+\beta_n(\delta||x_n-\tilde{x}_n||\\ &\quad +\varphi(||x_n-Tx_n||)+\eta)]\\ &\quad +\varphi(||(1-\beta_n)x_n+\beta_nTx_n-z_n||)+\eta\\ &\leq (1-\beta_n)\delta||x_n-\tilde{x}_n||+\beta_n\delta^2||x_n-\tilde{x}_n||\\ &\quad +\beta_n\delta\varphi(||x_n-Tx_n||)+\beta_n\eta\delta+\\ &\quad \varphi(||(1-\beta_n)x_n+\beta_nTx_n-z_n||)+\eta\\ &= (1-\beta_n(1-\delta))\delta||x_n-\tilde{x}_n||+\beta_n\delta\varphi(||x_n-Tx_n||)\\ &\quad +\beta_n\delta\eta+\varphi(||(1-\beta_n)x_n+\beta_nTx_n-z_n||)+\eta\\ &\leq \delta||x_n-\tilde{x}_n||+B_n+\beta_n\delta\eta+\eta, \text{ where}\end{aligned}$$

$B_n = \beta_n\delta\varphi(||x_n-Tx_n||)+\varphi(||(1-\beta_n)x_n+\beta_nTx_n-z_n||)$ so that $||y_n-\tilde{y}_n|| \leq \delta^2||x_n-\tilde{x}_n||+B_n\delta+\beta_n\delta^2\eta+\eta\delta+\alpha_n\eta\delta+A_n+\eta$.
Therefore

$$||x_{n+1}-\tilde{x}_{n+1}|| \leq \delta^3||x_n-\tilde{x}_n||+B_n\delta^2+\beta_n\delta^3\eta+\alpha_n\delta^2\eta$$
$$+\delta^2\eta+\delta A_n+\delta\eta+\varphi(||y_n-Ty_n||)+\eta. \quad (8)$$

By Remark 2.3, Remark 2.4 and by using continuity of φ, we have $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 0$ and $\lim_{n\to\infty}\varphi(||y_n-Ty_n||) = 0$.
By applying limit superior on both sides of (8), we have
$\limsup||x_{n+1}-\tilde{x}_{n+1}|| \leq \delta^3\limsup||x_n-\tilde{x}_n||+\delta^3\eta+\delta^2\eta+\delta\eta+\eta\delta^2+\eta$ so that
$\limsup||x_n-\tilde{x}_n|| \leq \frac{\delta^3\eta+2\delta^2\eta+\delta\eta+\eta}{1-\delta^3}$.
Since $\lim_{n\to\infty}\tilde{x}_n = \tilde{p}$, we have $||p-\tilde{p}|| \leq \frac{\delta^3\eta+2\delta^2\eta+\delta\eta+\eta}{1-\delta^3}$. □

Remark 3.2. Here we note that Theorem 1.5 follows as a corollary to Theorem 3.1 and the conditions (i) $\alpha_n \geq \frac{1}{2}$ and (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ of Theorem 1.5 are redundant. Also, we observe that the estimate $\frac{\delta^3\eta+2\delta^2\eta+\delta\eta+\eta}{1-\delta^3}$ of Theorem 3.1 is much more sharper than the estimate $\frac{9\eta}{1-\delta}$ of Theorem 1.5.

Corollary 3.3. *In addition to the hypotheses of Theorem 3.1, if $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ then $||p-\tilde{p}|| \leq \frac{\eta}{1-\delta}$.*

Proof. **Proof.** From inequality (8) of Theorem 3.1, we have $\limsup||x_{n+1}-\tilde{x}_{n+1}|| \leq \delta^3\limsup||x_n-\tilde{x}_n||+\delta^2\eta+\delta\eta+\eta$ so that $\limsup||x_n-\tilde{x}_n|| \leq \frac{\eta(\delta^2+\delta+1)}{1-\delta^3} = \frac{\eta}{1-\delta}$.
Hence $||p-\tilde{p}|| \leq \frac{\eta}{1-\delta}$. □

Note 3.4. *By Remark 1.9, clearly Theorem 1.7 follows as a corollary to Corollary 3.3. From Corollary 3.3, We observe that the assumption $\sum_{n=0}^{\infty}\alpha_n = \infty$ or $\sum_{n=0}^{\infty}\beta_n = \infty$ of Theorem 1.7 is redundant.*

The following is an example in support of Theorem 3.1.

Example 3.5. Let X, K and T be as in Example 2.5.
We define $\tilde{T}: [1,3] \to [1,3]$ by $\tilde{T}(x) = \frac{5x+3}{3x}$ for $x \in [1,3]$.
Then $\tilde{T}$ is an approximate operator of T with $\eta = \frac{1}{3}$.
We take $\alpha_n = \frac{1}{2^n}$ and $\beta_n = \frac{1}{3^n}$ for $n = 0,1,2...$ so that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$.
Let $\tilde{x}_0 \in K$ be arbitrary. Then $\tilde{z}_n = \frac{(15\tilde{x}_n^2+9\tilde{x}_n)3^n-15\tilde{x}_n^2+25\tilde{x}_n+15}{9\tilde{x}_n^2 3^n-9\tilde{x}_n^2+15\tilde{x}_n+9}$,
$\tilde{y}_n = \frac{(15\tilde{z}_n^2+9\tilde{z}_n)2^n-15\tilde{z}_n^2+25\tilde{z}_n+15}{9\tilde{z}_n^2 2^n-9\tilde{z}_n^2+15\tilde{z}_n+9}$ and
$\tilde{x}_{n+1} = \frac{102(2^n-1)\tilde{z}_n^2+(45\times 2^n+170)\tilde{z}_n+102}{45(2^n-1)\tilde{z}_n^2+(27\times 2^n+75)\tilde{z}_n+45}$.
By substituting the values of $\tilde{z}_n$ in $\tilde{x}_{n+1}$,
we write $\tilde{x}_{n+1} = \frac{215\tilde{x}_n+102}{102\tilde{x}_n+45}+A_n$ for some sequence $\{A_n\}$ converges to 0. Therefore

$$\begin{aligned}\tilde{x}_{n+1}-\frac{5+\sqrt{61}}{6} &= \frac{130-17\sqrt{61}}{102\tilde{x}_n+45}\left(\tilde{x}_n+\frac{129-15\sqrt{61}}{260-34\sqrt{61}}\right)+A_n\\ &= \frac{130-17\sqrt{61}}{102\tilde{x}_n+45}\left(\tilde{x}_n-\frac{5+\sqrt{61}}{6}\right)+A_n.\end{aligned}$$

Hence
$|\tilde{x}_{n+1}-\frac{5+\sqrt{61}}{6}| \leq \frac{130-17\sqrt{61}}{147}|\tilde{x}_n-\frac{5+\sqrt{61}}{6}|+|A_n|$
for $n = 0,1,2...$.
Now by applying limit superior on both sides, we have
$\lim_{n\to\infty}\tilde{x}_n = \frac{5+\sqrt{61}}{6}$.
Here we observe that $p = 1+\sqrt{2}$ and $\tilde{p} = \frac{5+\sqrt{61}}{6}$ are the fixed points of T and $\tilde{T}$ respectively and
$|p-\tilde{p}| = \frac{1+6\sqrt{2}-\sqrt{61}}{6} < \frac{77}{57} = \frac{\delta^3\eta+2\delta^2\eta+\delta\eta+\eta}{1-\delta^3}$.

In the following, we give justification for the assumption $\tilde{T}\tilde{p} = \tilde{p}$, $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$ of Theorem 3.1. For this purpose, we show that the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ of Theorem 3.1 need not be convergent. Further, we show that even if it is convergent its limit need not be a fixed point of $\tilde{T}$.

Example 3.6. Let X, K and T be as in Example 2.5.
We define $\tilde{T} : [1,3] \to [1,3]$ by

$$\tilde{T}x = \begin{cases} \frac{5}{2} & \text{if} \quad x \in [1, 1+\sqrt{2}] \\ \frac{23}{10} & \text{if} \quad x \in (1+\sqrt{2}, 3] \end{cases}$$

so that $\tilde{T}$ is an approximate operator of T with $\eta = \frac{1}{2}$.
Case (i) : In this case, we show that for any $\tilde{x}_0 \in [1,3]$ the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ of Theorem 3.1 does not converges.

Let $\tilde{x}_0$ be an arbitrary point in $[1,3]$, and let $\alpha_n = \frac{1}{2^n}$ and $\beta_n = \frac{1}{3^n}$ for $n = 0,1,2\ldots$.
Sub case(i) : We show that $\tilde{x}_n = \frac{5}{2}$ for some $n \geq 1$ implies that $\tilde{x}_{n+1} = \frac{23}{10}$.
Let $\tilde{x}_n = \frac{5}{2}$ for some $n \geq 1$ so that
$\tilde{z}_n = \tilde{T}((1-\frac{1}{3^n})\frac{5}{2} + \frac{1}{3^n}\frac{23}{10}) = \tilde{T}(\frac{5}{2} - \frac{1}{5(3^n)})$.
Since $\frac{5}{2} - \frac{1}{5(3^n)} > 1+\sqrt{2}$, we have $\tilde{z}_n = \frac{23}{10}$,
$\tilde{y}_n = \tilde{T}((1-\frac{1}{2^n})\frac{23}{10} + \frac{1}{2^n}\frac{5}{2}) = \tilde{T}(\frac{23}{10} + \frac{1}{5(2^n)})$.
Since $\frac{23}{10} + \frac{1}{5(2^n)} < 1+\sqrt{2}$, we have $\tilde{y}_n = \frac{5}{2}$ and hence $\tilde{x}_{n+1} = \frac{23}{10}$.
Sub case (ii) : We show that $\tilde{x}_n = \frac{23}{10}$ for some $n \geq 2$ implies that $\tilde{x}_{n+1} = \frac{5}{2}$.
Let $\tilde{x}_n = \frac{23}{10}$ for some $n \geq 2$ so that
$\tilde{z}_n = \tilde{T}((1-\frac{1}{3^n})\frac{23}{10} + \frac{1}{3^n}\frac{5}{2}) = \tilde{T}(\frac{23}{10} + \frac{1}{5(3^n)})$.
Since $\frac{23}{10} + \frac{1}{5(3^n)} < 1+\sqrt{2}$ for $n \geq 2$, we have $\tilde{z}_n = \frac{5}{2}$ and
$\tilde{y}_n = \tilde{T}((1-\frac{1}{2^n})\frac{5}{2} + \frac{1}{2^n}\frac{23}{10}) = \tilde{T}(\frac{5}{2} - \frac{1}{5(2^n)})$.
Since $\frac{5}{2} - \frac{1}{5(2^n)} > 1+\sqrt{2}$ for $n \geq 2$, we have $\tilde{y}_n = \frac{23}{10}$ and hence $\tilde{x}_{n+1} = \tilde{T}\tilde{y}_n = \frac{5}{2}$.
Here, we observe that $\tilde{x}_2 = \frac{23}{10}$. Hence, for $n \geq 2$

$$\tilde{x}_n = \begin{cases} \frac{23}{10} & \text{if} \quad n \text{ is even} \\ \frac{5}{2} & \text{if} \quad n \text{ is odd} \end{cases}$$

which is an oscillating sequence and hence $\{\tilde{x}_n\}_{n=0}^{\infty}$ is not convergent.
Case (ii) : In this case, we show that the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ of Theorem 3.1 converges but its limit need not be a fixed point of $\tilde{T}$.
Here we take $\alpha_n = \beta_n = \frac{1}{2}$ for $n = 0,1,2\ldots$ and $\tilde{x}_0 = \frac{5}{2}$.
We show that $\tilde{x}_n = \frac{23}{10}$ for $n = 1,2,3\ldots$ by induction on n.
Since $\tilde{x}_0 = \frac{5}{2}$, we have $\tilde{z}_0 = \tilde{T}(\frac{\tilde{x}_0 + \tilde{T}\tilde{x}_0}{2}) = \tilde{T}(\frac{12}{5}) = \frac{5}{2}$,
$\tilde{y}_0 = \tilde{T}(\frac{\tilde{z}_0 + \tilde{T}\tilde{z}_0}{2}) = \tilde{T}(\frac{12}{5}) = \frac{5}{2}$ and $\tilde{x}_1 = \tilde{T}\tilde{y}_0 = \frac{23}{10}$.
We assume that $\tilde{x}_n = \frac{23}{10}$ for some $n \geq 1$ so that
$\tilde{z}_n = \tilde{T}(\frac{\tilde{x}_n + \tilde{T}\tilde{x}_n}{2}) = \tilde{T}(\frac{12}{5}) = \frac{5}{2}$,
$\tilde{y}_n = \tilde{T}(\frac{\tilde{z}_n + \tilde{T}\tilde{z}_n}{2}) = \tilde{T}(\frac{12}{5}) = \frac{5}{2}$ and $\tilde{x}_{n+1} = \tilde{T}\tilde{y}_n = \frac{23}{10}$.
Therefore by induction hypothesis $\tilde{x}_n = \frac{23}{10}$ for $n = 1,2,\ldots$ and hence $\lim_{n\to\infty} \tilde{x}_n = \frac{23}{10}$ which is not a fixed point $\tilde{T}$.

4. T-Stability

Theorem 4.1. Let X, K, T be as in Theorem 2.1, and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be arbitrary sequences in $[0,1]$. Then the AK iteration procedure is $T-$stable.

Proof. **Proof.** By Theorem 2.1 for any $x_0 \in K$, the AK iteration procedure $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p (say) of T in K and it is unique.

Let $\{s_n\}_{n=0}^{\infty}$ be an arbitrary sequence in K and
$\varepsilon_n = ||s_{n+1} - f(T,s_n)||$ where $f(T,s_n) = Tv_n$,
$v_n = T((1-\alpha_n)u_n + \alpha_n Tu_n)$ and $u_n = T((1-\beta_n)s_n + \beta_n Ts_n)$ for $n = 0,1,2\ldots$.
First we consider

$$\begin{aligned} ||f(T,s_n) - p|| &= ||Tv_n - Tp|| \\ &\leq \delta||v_n - p|| + \varphi(||p - Tp||) \\ &= \delta||v_n - p|| \\ &= \delta||T((1-\alpha_n)u_n + \alpha_n Tu_n) - Tp|| \\ &\leq \delta^2||(1-\alpha_n)u_n + \alpha_n Tu_n - p|| \\ &\quad + \delta\varphi(||p - Tp||) \\ &\leq \delta^2[(1-\alpha_n)||u_n - p|| + \alpha_n||Tu_n - Tp||] \\ &\leq \delta^2[(1-\alpha_n)||u_n - p|| + \alpha_n\delta||u_n - p|| \\ &\quad + \alpha_n\varphi(||p - Tp||)] \\ &= \delta^2[1 - \alpha_n(1-\delta)]||u_n - p|| \\ &\leq \delta^2||u_n - p|| \\ &= \delta^2||T((1-\beta_n)s_n + \beta_n Ts_n) - Tp|| \\ &\leq \delta^3||(1-\beta_n)s_n + \beta_n Ts_n - p|| \\ &\quad + \delta^2\varphi(||p - Tp||) \\ &\leq \delta^3[(1-\beta_n)||s_n - p|| + \beta_n||Ts_n - Tp||] \\ &\leq \delta^3[(1-\beta_n)||s_n - p|| \\ &\quad + \beta_n(\delta||s_n - p|| + \varphi(||p - Tp||))] \\ &= \delta^3(1 - \beta_n(1-\delta))||s_n - p|| \\ &\leq \delta^3||s_n - p||. \end{aligned}$$

Therefore

$$||f(T,s_n) - p|| \leq \delta^3||s_n - p|| \tag{9}$$

We assume that $\lim_{n\to\infty} \varepsilon_n = 0$.
From the inequality (9), we have

$$\begin{aligned} ||s_{n+1} - p|| &\leq ||s_{n+1} - f(T,s_n)|| + ||f(T,s_n) - p|| \\ &\leq \varepsilon_n + \delta^3||s_n - p|| \text{ for } n = 0,1,2\ldots . \end{aligned}$$

By applying limit superior on both sides, we have

$$\begin{aligned} \limsup ||s_{n+1} - p|| &\leq \limsup \varepsilon_n + \delta^3 \limsup ||s_n - p|| \\ &= \delta^3 \limsup ||s_n - p|| \text{ so that} \end{aligned}$$

$\limsup ||s_n - p|| \leq 0$ and hence $\lim_{n\to\infty} s_n = p$.
Conversely, we assume that $\lim_{n\to\infty} s_n = p$.
From (9), we have

$$\begin{aligned} \varepsilon_n &= ||s_{n+1} - f(T,s_n)|| \leq ||s_{n+1} - p|| + ||f(T,s_n) - p|| \\ &\leq ||s_{n+1} - p|| + \delta^3||s_n - p|| \text{ for } n = 0,1,2\ldots . \end{aligned}$$

By applying limit superior on both sides, we have $\limsup \varepsilon_n \leq \limsup ||s_{n+1} - p|| + \delta^3 \limsup ||s_n - p|| = 0$ so that $\lim_{n\to\infty} \varepsilon_n = 0$.
Thus the AK iteration procedure is $T-$stable.

□

Remark 4.2. Here we note that Theorem 1.4 follows as a corollary to Theorem 4.1 and the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$ of Theorem 1.4 is redundant.

References

[1] M. Ert*ü*rk, Convergence and data dependence results for *AK* iterative algorithm, *Celal Bayar University Journal of Science*, 13(3), (2017), 755-759.

[2] A. M. Harder, T. L. Hicks, Stability results for fixed point iteration procedures, *Math. Japonica*, 33 (1988) 693-706.

[3] C. O. Imoru and M. O. Olatinwo, On the stability of Picard and Mann iteration processes, *Carpathain J. Math.*, 19(2),(2003), 155-160.

[4] Kifayat Ullah and Muhammad Arshad, On different results for new three step iteration process in Banach spaces, *Springer Plus*, 5:1616 (2016),DOI 10.1186/s40064-016-3056-x,15 pages.

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