Malaya Journal of Matematik, Vol. 7, No. 2, 309-313, 2019

https://doi.org/10.26637/MJM0702/0025

On different kinds of arcs in interval valued fuzzy graphs

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Abstract

Type I strong arcs, Type II strong arcs, left feeble arcs, right feeble arcs and weak arcs in an Interval Valued Fuzzy Graph (IVFG) are introduced in this paper. We obtain a characterization of weak arcs. If every two arcs are comparable in an IVFG, then it contains only α strong arcs and weak arcs. An arc in an IVFG is a weak arc if and only if it is the unique weakest arc of at least one cycle in it.

Keywords

Interval valued fuzzy graph, Arc, Strong arc, Comparable arcs.

AMS Subject Classification

05C72, 08A72, 03E72.

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Article History: Received **19** December **2018**; Accepted **09** March **2019**

Contents

1. Introduction

Graph theoretic terms used in this work are either standard or are explained as and when they first appear. We consider only simple graphs. That is, graphs with multiple edges and loops are not considered. The notion of interval-valued fuzzy set is introduced by Zadeh[15] as an extension of fuzzy set [14]. Fuzzy graph was defined by Rosenfeld [11].

Hongmei and Lianhua introduced interval - valued fuzzy graphs(IVFG) [4]. It is a generalization of fuzzy graphs developed in [7] and [11]. Penetration of interval valued fuzzy graphs in the the arena of algebraic structures can be seen in [5] and [2].

Definition 1.1. *Let $G^\star = (V,E)$ be a crisp graph.Then an interval - valued fuzzy graph (IVFG) G on $G^\star$ is defined as a pair $G = (A,B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an interval - valued fuzzy set [1] on V and $B = [\mu_B^-(xy), \mu_B^+(xy)]$ is an interval - valued fuzzy set on E such that $\mu_B^-(xy) \le \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) \le \min\{\mu_A^+(x), \mu_A^+(y)\}$ for all $xy \in E$.*

The study of IVFGs is growing fast and has so many applications. [1] defined some operations on IVFGs and investigated their properties. Regular and edge regular IVFGs were studied in [12]. Interval – valued fuzzy bridges and interval – valued fuzzy cutnodes were defined in [10].

In crisp graph theory, study of the nature of arcs is not very significant as all arcs are strong in the sense of [3]. But in fuzzy graph theory and interval - valued fuzzy graph theory, study of the various characteristics of different types of arcs is indispensable as it gives us a better idea of the structure of graphs. It helps us to study many of their properties. Bhutani and Rosenfeld[3] classfied arcs into strong and non strong arcs.

Arcs in intuitionistic fuzzy graphs were studied in [6] and [9]. In [8], four different types of arcs were introduced with a detailed study of them. Strong arcs are divided into α strong arcs and β strong arcs and non strong arcs are classified into δ arcs and δ^* arcs. But when we come to the case of IVFGs, these classifications are not sufficient. So in this paper, we define nine different types of arcs and make an earnest effort to study the various characteristics of these arcs.

The concepts such as strongest path, unique strongest

Table 1. Types of Strong Arcs

Name	Requirement
α^- strong	$\mu_{B^-}(u,v) > NCONN_{G-(u,v)}(u,v)$
α^+ strong	$\mu_{B^+}(u,v) > PCONN_{G-(u,v)}(u,v)$
α strong	α^- strong and α^+ strong
β^- strong	$\mu_{B^-}(u,v) = NCONN_{G-(u,v)}(u,v)$
β^+ strong	$\mu_{B^+}(u,v) = PCONN_{G-(u,v)}(u,v)$
β strong	β^- strong andβ^+ strong
$\alpha\beta$ strong	α^- strong and β^+ strong
$\beta\alpha$ strong	β^- strong and α^+ strong
Type I strong	α strong or β strong
Type II strong	$\alpha\beta$ strong or $\beta\alpha$ strong

path, bridge, cutnode, weakest arc, strength of the path P ($S_{\mu^-}(P)$, $S_{\mu^+}(P)$), maximum of the μ^{-+}-strength (($\mu_{B^{-+}})^\infty$, $NCONN_G(PCONN_G)$), etc. are in the sense of [10].

2. Types of Arcs in an IVFG

In this section, we define different types of arcs in IVFGs based on the concepts $NCONN_G(u,v)$ and $PCONN_G(u,v)$. Remember that $NCONN_{G-(u,v)}(u,v)$ and $PCONN_{G-(u,v)}(u,v)$ are respectively the μ^- and μ^+ strength of connectedness between u and v in the IVFG formed from G by removing the arc (u,v). Here we consider only connected IVFGs.

An arc of a fuzzy graph is strong if its weight is greater than or equal to the strength of connectedness of its end nodes in the fuzzy graph formed by removing it.[3] Analogous to this we define a strong arc in an IVFG as follows.

Definition 2.1. *If $\mu_{B^-}(u,v) \geq NCONN_{G-(u,v)}(u,v)$ and $\mu_{B^+}(u,v) \geq PCONN_{G-(u,v)}(u,v)$, then an arc (u,v) in an IVFG G is called a strong arc.*

According to the definition 2.1, strong arcs can be classified into four. They are α strong arcs, β strong arcs, $\alpha\beta$ strong arcs and $\beta\alpha$ strong arcs. See the table 1 for various types of strong arcs (u,v) in an IVFG, G and their requirements to be of that type. Analogous to the definition of δ arc in [8], we now define a δ arc or a weak arc in an IVFG.

Definition 2.2. *An arc (u,v) in G is called δ^- arc if $\mu_{B^-}(u,v) < NCONN_{G-(u,v)}(u,v)$ and it is called δ^+ arc if $\mu_{B^+}(u,v) < PCONN_{G-(u,v)}(u,v)$. The arc (u,v) in G is called δ arc or a weak arc if it is both δ^- arc and δ^+ arc.*

See the table 2 for the weak arcs based on the definition 2.2.

Definition 2.3. *An arc (u,v) in an IVFG, G is called a strong arc if it is either Type I strong or Type II strong. If (u,v) is strong, we say that u and v are strong neighbours.*

Definition 2.4. *A path comprising only combinations of strong arcs is said to be a strong path in an IVFG G. Particularly, it is said to be an α strong path if its component arcs are all α strong and is said to be a β strong path if its component arcs are all β strong.*

Table 2. Types of Weak Arcs

Name	Requirement
α^- strong	$\mu_{B^-}(u,v) > NCONN_{G-(u,v)}(u,v)$
$\alpha\delta$	$\alpha^- strong$ and δ^+
$\beta\delta$	$\beta^- strong$ and δ^+
right feeble	$\alpha\delta$ or $\beta\delta$
$\delta\alpha$	δ^- and $\alpha^+ strong$
$\delta\beta$	δ^- and $\beta^+ strong$
left feeble	$\delta\alpha$ or $\delta\beta$

Definition 2.5. *Two arcs e_1 and e_2 are said to be comparable if their membership degrees are such that either $\mu_{B^-}(e_1) > \mu_{B^-}(e_2)$and $\mu_{B^+}(e_1) > \mu_{B^+}(e_2)$ or $\mu_{B^-}(e_1) < \mu_{B^-}(e_2)$and $\mu_{B^+}(e_1) < \mu_{B^+}(e_2)$*

Definition 2.6. *Two arcs e_1 and e_2 are said to be equal if their membership degrees are equal. That is if $\mu_{B^-}(e_1) = \mu_{B^-}(e_2)$ and $\mu_{B^+}(e_1) = \mu_{B^+}(e_2)$*

Based on the discussion we had just now, we have the following theorem. We state it without proof.

Theorem 2.7. *Let G be an IVFG such that every two arcs are either comparable or equal. Then G contains only combinations of Type I strong arcs and weak arcs.*

Example 2.8. *In figure 1, membership degrees of every arc is such that it satisfies the conditions of the above theorem. Using the above definitions we can clearly see that (a,b) and (a,d) are β strong, (b,c) and (c,d) are α strong and (b,d) is a δ arc.*

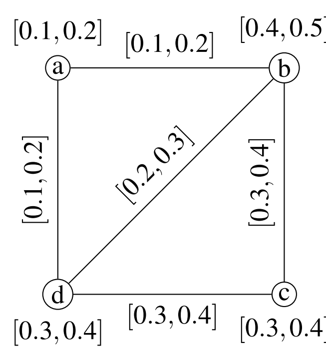

Figure 1. Illustration of the theorem 2.7

The converse of theorem 2.7 is not true. This is clear from the example 2.9. For the IVFG G given in Figure 2 any membership degrees can be given to nodes so that it satisfies the condition of IVFG.

Example 2.9. *In the IVFG G given in Figure 2, (a,b) (a,e) and (b,e) are δ arcs and (c,d) and (c,e) are β strong arcs and all other arcs are α strong. But not every two arcs are comparable or equal. For example, (a,f) and (e,f) are neither comparable nor equal.*

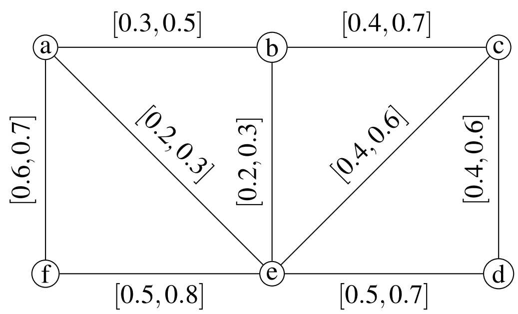

Figure 2. Illustration of the theorem 2.7

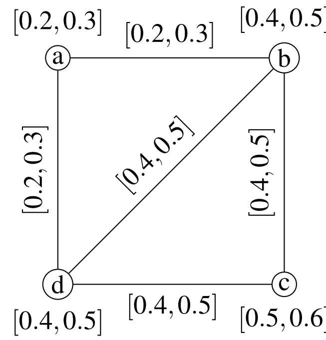

Figure 3. Illustration of the converse of the theorem 3.1

Remark 2.10. *Let G be an IVFG such that every two arcs are either comparable or equal. Then by theorem 2.7, G ontains only α, β, and δ arcs as defined in [8]. So all the theorems in [13] and [8] hold for G.*

3. Main Results

Theorem 3.1. *Let G be an IVFG. If every two arcs of G are equal,then all the arcs of G will be β strong.*

Proof. Let G be an IVFG such that every two arcs are equal. Then $[\mu_{B^-}, \mu_{B^+}]$ is a constant for every arc of G, and let $[\mu_{B^-}, \mu_{B^+}](e) = [c_1, c_2]$ for every arc $e = (u, v)$ of G where c_1 and c_2 are constants. Let u and v be any two nodes of G. Then for every $u-v$ path P of G, $S_{\mu^-}(P) = c_1$ and $S_{\mu^+}(P) = c_2$. Then by the definition of the μ^- and μ^+ strength of connectedness,

$$NCONN_G(u, v) = c_1,$$

a constant and

$$PCONN_G(u, v) = c_2,$$

a constant. Again, since $S_{\mu^-}(P) = c_1$ and $S_{\mu^+}(P) = c_2$, where P is any $u-v$ path in G, we have,

$$NCONN_{G-(u,v)}(u, v) = c_1$$

and

$$PCONN_{G-(u,v)}(u, v) = c_2.$$

Thus, we have,

$$\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$$

and

$$\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v).$$

Hence, arc (u, v) is β strong. Since u and v are arbitrary, all the arcs of G will be β strong. □

The converse of theorem 3.1 is not true. This is clear from the example 3.2.

Example 3.2. *In the IVFG G given in Figure 3, all arcs are β strong arcs. But not every two arcs are equal. For example, (a,b) and (b,c) are not equal.*

We can observe that β strong arcs are of three types.
Type I

$$\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$$

and

$$\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v),$$

and $NCONN_{G-(u,v)}(u, v)$ and $PCONN_{G-(u,v)}(u, v)$ correspond to same arcs of the same path.
Type II

$$\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$$

and

$$\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v),$$

but $NCONN_{G-(u,v)}(u, v) and PCONN_{G-(u,v)}(u, v)$ correspond to two different arcs of the same path.
Type III

$$\mu_{B^-}(u, v) = NCONN_{G-(u,v)}(u, v)$$

and

$$\mu_{B^+}(u, v) = PCONN_{G-(u,v)}(u, v),$$

but $NCONN_{G-(u,v)}(u, v)$ and $PCONN_{G-(u,v)}(u, v)$ correspond to two different paths.

The theorem 3.3 characterizes weak arcs.

Theorem 3.3. *Let $G = (A, B)$ be an IVFG and (u, v) be an arc in G. Then (u, v) is a weak arc if and only if it is the unique weakest arc of at least one cycle in G.*

Proof. Let $G = (A, B)$ be an IVFG. Let (u, v) be a *weak arc* in G. Then by definition 2.2, $\mu_{B^-}(u, v) < NCONN_{G-(u,v)}(u, v)$ and $\mu_{B^+}(u, v) < PCONN_{G-(u,v)}(u, v)$. i.e., There exists at least one path P joining u and v and not containing the arc (u, v) such that $S_{\mu^-}(P) > \mu_{B^-}(u, v)$ and $S_{\mu^+}(P) > \mu_{B^+}(u, v)$. This path together with the arc (u, v) forms a cycle in which (u, v) is the unique weakest arc.

Conversely, let (u, v) be the unique weakest arc of a cycle C in G. Let P be the $u-v$ path in C not containing the arc (u, v). Then,

$$\mu_{B^-}(u, v) < S_{\mu^-}(P) \text{ and } \mu_{B^+}(u, v) < S_{\mu^+}(P). \tag{3.1}$$

Suppose (u,v) is not a weak arc in G. Then we have by definition 2.2,

$$\mu_{B^-}(u,v) \geq NCONN_{G-(u,v)}(u,v) \tag{3.2}$$

and

$$\mu_{B^+}(u,v) \geq PCONN_{G-(u,v)}(u,v). \tag{3.3}$$

Also, we have

$$S_{\mu^-}(P) \leq NCONN_{G-(u,v)}(u,v) \tag{3.4}$$

and

$$S_{\mu^+}(P) \leq PCONN_{G-(u,v)}(u,v). \tag{3.5}$$

Hence, we get,

$$\mu_{B^-}(u,v) \geq S_{\mu^-}(P) \text{ and } \mu_{B^+}(u,v) \geq S_{\mu^+}(P) \tag{3.6}$$

which contradicts (3.1).

Therefore, (u,v) is a weak arc in G □

Theorem 3.4. *Let G be an IVFG such that every two arcs are comparable. Then G contains only α strong arcs and weak arcs.*

Proof. Let $G=(A,B)$ be an IVFG on $G^*=(V,E)$ such that every two arcs are comparable. If G^* has no cycles, then G^* is a tree and every arc of G^* is a bridge. Hence every arc of G is also a bridge and its removal changes the lower and upper membership degrees of the corresponding end vertices to 0. Hence we can see that if (u,v) is any such arc, it satisfies the definition of α strong arc and hence we can conclude that every arc of G is an α strong arc if G^* has no cycles.

Now suppose that G^* has cycles. If (u,v) is an arc of G such that it does not belongs to any of the cycles, then (u,v) is a bridge and arguing as above, (u,v) is α strong. Now let (u,v) belongs to atleast one cycle of G. Since every two arcs of G are comparable, every cycle of G has a unique weakest arc. By theorem 3.3, this unique weakest arc will be a *weak* arc. It remains to show that all the remaining arcs are α strong. For that let (u,v) be an arc of G such that (u,v) is not the unique weakest arc of any of the cycles of G. To show that (u,v) is α strong it is enough to prove that $\mu_{B^-}(u,v) > NCONN_{G-(u,v)}(u,v)$ & $\mu_{B^+}(u,v) > PCONN_{G-(u,v)}(u,v)$. In $G-uv$, there are as many $u-v$ paths as there are cycles in G,containing (u,v) and these paths are obtained by deleting (u,v) from the corresponding cycle. Suppose they are n in number. Let it be $P_1,P_2,\ldots,P_n$. Then clearly, $\mu_{B^-}(u,v) > S_{\mu^-}(P_i)$ & $\mu_{B^+}(u,v) > S_{\mu^+}(P_i)$ $\forall i=1,2,\ldots,n$ and hence $\mu_{B^-}(u,v) > NCONN_{G-(u,v)}(u,v)$ & $\mu_{B^+}(u,v) > PCONN_{G-(u,v)}(u,v)$. Thus (u,v) is α strong. □

The converse of theorem 3.4 is not true. This is clear from the example 3.5.

Example 3.5. *In the IVFG G given in Figure 4, (a,b) and (b,c) are δ arcs and (a,d),(c,d) and (b,d) are α strong arcs. Thus we can see that G does not contain any β strong arcs, but every two arcs are not comparable.*

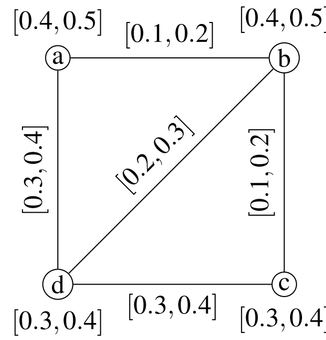

Figure 4. Illustration of the converse of the theorem 3.4

4. Conclusion

Many fuzzy graph theoretic terms are yet to be properly generalized in the case of interval valued fuzzy graphs. In this paper, we made an attempt to study about some types of arcs in IVFGs. We have defined Type I strong arcs, Type II strong arcs, *left feeble* arcs, *right feeble* arcs and weak arcs in IVFG.We have shown that if every two arcs of an IVFG are equal, then all the arcs of G will be β strong. We also have shown that an arc in an IVFG is a *weak arc* if and only if it is the unique weakest arc of at least one cycle in it. Finally, we have proved that if every two arcs are comparable in an IVFG, then it contains only α strong arcs and *weak* arcs. There is huge scope in continuing with the study based on the various types of arcs defined in this paper.

Acknowledgment

The work of the first author is supported by the University Grants Commission of India under the Faculty Improvement Pogramme.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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