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Initial-value problems for nonlinear hybrid implicit Caputo fractional differential equations

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Abstract

In this study, we use the contraction mapping principle to obtain the existence, interval of existence and uniqueness of solutions for nonlinear hybrid implicit Caputo fractional differential equations. We also use the generalization of Gronwall's inequality to show the estimate of the solutions.

Keywords

Implicit fractional differential equations, Caputo fractional derivatives, fixed point theorems, existence, uniqueness.

AMS Subject Classification

34A12, 34K20, 45N05.

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Contents

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[7], [9]–[15] and the references therein.

Recently, Ahmad and Ntouyas [2] discussed the existence of solutions for the hybrid Hadamard differential equation

$$\begin{cases} {}^{H}D^{\alpha}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f(t,x(t)),\ t\in[1,T], \\ {}^{H}I^{\alpha}x(t)\big|_{t=1} = \eta, \end{cases}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of order $0<\alpha\leq 1$. By employing the Dhage fixed point theorem, the authors obtained existence results.

The implicit fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f\left(t,x(t),{}^{C}D^{\alpha}x(t)\right), \\ x(0)=x_0, \end{cases}$$

has been investigated in [7], where ${}^{C}D^{\alpha}$ is the standard Caputo fractional derivative of order $0<\alpha<1$. By using the contraction mapping principle, the existence, interval of existence and uniqueness of solutions has been established.

In this paper, we are interested in the analysis of qualitative theory of the problems of the existence, interval of existence and uniqueness of solutions to nonlinear hybrid implicit Caputo fractional differential equations. Inspired and motivated by the works mentioned above and the references in this paper, we concentrate on the existence, interval of existence and uniqueness of solutions for the nonlinear hybrid implicit Caputo fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f\left(t,x(t),{}^{C}D^{\alpha}\left(\dfrac{x(t)}{g(t,x(t))}\right)\right), \\ x(0)=\eta g(0,x(0)),\ \eta\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ and $g:[0,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ are nonlinear continuous functions and ${}^{C}D^{\alpha}$ denotes the Caputo derivative of order $0 < \alpha < 1$. To show the existence, interval of existence and uniqueness of solutions of (1.1), we transform (1.1) into an integral equation and then use the contraction mapping principle. Further, by the generalization of Gronwall's inequality we obtain the estimate of solutions of (1.1).

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus [6, 9, 13] which are used throughout this paper.

Definition 2.1 ([9]). *The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 ([9]). *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$^{C}D^{\alpha}x(t) = I^{n-\alpha}x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds,$$

where $n = [\alpha]+1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3 ([9]). *Let $\Re(\alpha) > 0$. Suppose $x \in C^{n-1}[0,+\infty)$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I^{\alpha}\, {}^{C}D^{\alpha}x\right)(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k.$$

In particular, when $0 < \Re(\alpha) < 1$, $\left(I^{\alpha}\, {}^{C}D^{\alpha}x\right)(t) = x(t) - x(0)$.

Lemma 2.4 ([9]). *For all $\alpha,\beta \in [0,\infty)$, Then*

$$\int_0^t (t-s)^{\beta-1}s^{\alpha-1}ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}t^{\alpha+\beta-1}.$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

Lemma 2.5 ([8]). *Let $x : [0,T] \to [0,\infty)$ be a real function and w is a nonnegative locally integrable function on $[0,T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$*

$$x(t) \le w(t) + a\int_0^t (t-s)^{-\alpha}x(s)ds.$$

Then, there exist a constant $K = K(\alpha)$ such that

$$x(t) \le w(t) + Ka\int_0^t (t-s)^{-\alpha}w(s)ds,$$

for every $t \in [0,T]$.

3. Main results

In this section, we give the equivalence of the initial value problem (1.1) and prove the existence, interval of existence, uniqueness and estimate of solutions of (1.1).

The proof of the following lemma is close to the proof of Lemma 6.2 given in [6].

Lemma 3.1. *If the functions $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ and $g : [0,T]\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ are continuous, then the initial value problem (1.1) is equivalent to nonlinear fractional Volterra integro-differential equation*

$$\begin{aligned} x(t) &= \eta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \times f\left(s,x(s),{}^{C}D^{\alpha}\left(\frac{x(s)}{g(s,x(s))}\right)\right)ds, \end{aligned}$$

for $t \in [0,T]$.

Theorem 3.2. *Let $T > 0$. Assume that the continuous functions $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ and $g : [0,T]\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ satisfy the following conditions*

(H1) There exists $M_f \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \le M_f,$$

for all $u,v \in \mathbb{R}$ and $t \in [0,T]$.

(H2) There exists $M_g \in \mathbb{R}^+$ such that

$$|g(t,u)| \le M_g,$$

for all $u \in \mathbb{R}$ and $t \in [0,T]$.

(H3) There exist $K_1, K_3 \in \mathbb{R}^+$, $K_2 \in (0,1)$ with $K_3|\eta| \in (0,1)$ such that

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \le K_1|u-\tilde{u}| + K_2|v-\tilde{v}|,$$

and

$$|g(t,u) - g(t,\tilde{u})| \le K_3|u-\tilde{u}|,$$

for all $u,v,\tilde{u},\tilde{v} \in \mathbb{R}$ and $t \in [0,T]$.
Let

$$0 < b < \min\left\{T, \left(\frac{(1-K_3|\eta|)(1-K_2)\Gamma(\alpha+1)}{K_3(1-K_2)M_f + K_1M_g}\right)^{\frac{1}{\alpha}}\right\},$$

then (1.1) has a unique solution $x \in C([0,b],\mathbb{R})$.

Proof. Let

$$^{C}D^{\alpha}\left(\frac{x(t)}{g(t,x(t))}\right) = z_x(t),\ x(0) = \eta g(0,x(0)),$$

then by Lemma 3.1,

$$x(t) = \eta g(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z_x(s)ds,$$

where

$$z_x(t) = f(t, \eta g(t, x(t)) + g(t, x(t)) I^{\alpha} z_x(t), z_x(t)).$$

That is $x(t) = \eta g(t, x(t)) + g(t, x(t)) I^{\alpha} z_x(t)$. Define the mapping $P : C([0,b], \mathbb{R}) \to C([0,b], \mathbb{R})$ as follows

$$(Px)(t) = \eta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s) \, ds.$$

It is clear that the fixed points of P are solutions of (1.1). Let $x, y \in C([0,b], \mathbb{R})$, then we have

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&= \left| \eta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s) \, ds \right. \\
&\left. - \eta g(t, y(t)) - \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_y(s) \, ds \right| \\
&\le |\eta| \, |g(t, x(t)) - g(t, y(t))| \\
&+ |g(t, x(t)) - g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s)| \, ds \\
&+ |g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| z_x(s) - z_y(s) \right| ds \\
&\le K_3 |\eta| \, |x(t) - y(t)| \\
&+ K_3 |x(t) - y(t)| \frac{M_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&+ \frac{M_g}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| z_x(s) - z_y(s) \right| ds, \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\left| z_x(t) - z_y(t) \right| &\le \left| f(t, x(t), z_x(t)) - f(t, x(t), z_y(t)) \right| \\
&\le K_1 |x(t) - y(t)| + K_2 \left| z_x(t) - z_y(t) \right| \\
&\le \frac{K_1}{1 - K_2} |x(t) - y(t)|. \qquad (3.2)
\end{aligned}$$

By replacing (3.2) in the inequality (3.1), we get

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\le K_3 |\eta| \, |x(t) - y(t)| \\
&+ K_3 |x(t) - y(t)| \frac{M_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&+ \frac{M_g}{\Gamma(\alpha)} \frac{K_1}{1 - K_2} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| \, ds \\
&\le K_3 \left(|\eta| + \frac{M_f t^{\alpha}}{\Gamma(\alpha+1)} \right) \|x - y\| \\
&+ \frac{M_g}{\Gamma(\alpha)} \frac{K_1}{1 - K_2} \left(\int_0^t (t-s)^{\alpha-1} ds \right) \|x - y\| \\
&\le \left(K_3 |\eta| + \left(K_3 M_f + \frac{K_1 M_g}{1 - K_2} \right) \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right) \|x - y\|.
\end{aligned}$$

Since $t \in [0,b]$, then

$$\|Px - Py\| \le \beta \|x - y\|, \; 0 < \beta < 1,$$

where

$$\beta = K_3 |\eta| + \frac{K_3 (1 - K_2) M_f + K_1 M_g}{1 - K_2} \frac{b^{\alpha}}{\Gamma(\alpha+1)}.$$

That is to say the mapping P is a contraction in $C([0,b], \mathbb{R})$. Hence P has a unique fixed point $x \in C([0,b], \mathbb{R})$. Therefore, (1.1) has a unique solution. □

Theorem 3.3. *Assume that $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy (H2) and (H3). If x is a solution of (1.1), then*

$$\begin{aligned}
|x(t)| \le &\frac{(1 - K_2)(1 - K_3 |\eta|) \Gamma(\alpha+1) + K_1 K M_g T^{\alpha}}{(1 - K_2)(1 - K_3 |\eta|)^2 \Gamma(\alpha+1)} \\
&\times \left(|\eta| Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1 - K_2) \Gamma(\alpha+1)} \right),
\end{aligned}$$

where $Q_1 = \sup_{t \in [0,T]} |g(t,0)|$, $Q_2 = \sup_{t \in [0,T]} |f(t,0,0)|$ and $K \in \mathbb{R}^+$ is a constant.

Proof. Let

$${}^{C}D^{\alpha} \left(\frac{x(t)}{g(t, x(t))} \right) = z_x(t), \; x(0) = \eta g(0, x(0)).$$

By Lemma 3.1, $x(t) = \eta g(t, x(t)) + g(t, x(t)) I^{\alpha} z_x(t)$. Then by (H2) and (H3), for any $t \in [0,T]$ we have

$$\begin{aligned}
|x(t)| &\le |\eta| \, |g(t, x(t))| + |g(t, x(t))| I^{\alpha} |z_x(t)| \\
&\le |\eta| \left(|g(t, x(t)) - g(t,0)| + |g(t,0)| \right) \\
&+ M_g I^{\alpha} |z_x(t)| \\
&\le |\eta| \left(Q_1 + K_3 |x(t)| \right) + M_g I^{\alpha} |z_x(t)|,
\end{aligned}$$

where $Q_1 = \sup_{t \in [0,T]} |g(t,0)|$. On the other hand, for any $t \in [0,T]$ we get

$$\begin{aligned}
|z_x(t)| &= |f(t, x(t), z_x(t))| \\
&\le |f(t, x(t), z_x(t)) - f(t,0,0)| + |f(t,0,0)| \\
&\le K_1 |x(t)| + K_2 |z_x(t)| + |f(t,0,0)| \\
&\le \frac{K_1}{1 - K_2} |x(t)| + \frac{Q_2}{1 - K_2},
\end{aligned}$$

where $Q_2 = \sup_{t \in [0,T]} |f(t,0,0)|$. Therefore

$$\begin{aligned}
|x(t)| &\le |\eta| \left(Q_1 + K_3 |x(t)| \right) \\
&+ M_g I^{\alpha} \left(\frac{Q_2}{1 - K_2} + \frac{K_1}{1 - K_2} |x(t)| \right).
\end{aligned}$$

Thus

$$\begin{aligned}
&(1 - K_3 |\eta|) |x(t)| \\
&\le |\eta| Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1 - K_2) \Gamma(\alpha+1)} \\
&+ \frac{K_1 M_g}{(1 - K_2)(1 - K_3 |\eta|)} I^{\alpha} \left\{ (1 - K_3 |\eta|) |x(t)| \right\}.
\end{aligned}$$

By Lemma 2.5, there is a constant $K = K(\alpha)$ such that

$$\begin{aligned}
&(1-K_3|\eta|)|x(t)| \\
&\leq |\eta|Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1-K_2)\Gamma(\alpha+1)} \\
&+ \frac{K_1 K M_g T^{\alpha}}{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1)} \\
&\times \left(|\eta|Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1-K_2)\Gamma(\alpha+1)}\right) \\
&\leq \frac{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1) + K_1 K M_g T^{\alpha}}{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1)} \\
&\times \left(|\eta|Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1-K_2)\Gamma(\alpha+1)}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
|x(t)| \leq\ & \frac{(1-K_2)(1-K_3|\eta|)\Gamma(\alpha+1) + K_1 K M_g T^{\alpha}}{(1-K_2)(1-K_3|\eta|)^2\Gamma(\alpha+1)} \\
&\times \left(|\eta|Q_1 + \frac{M_g Q_2 T^{\alpha}}{(1-K_2)\Gamma(\alpha+1)}\right).
\end{aligned}$$

This completes the proof. □

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