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Some new results on the connected sum of certain digital surfaces

Ismet Cinar[1]* and Ismet KARACA[2]

Abstract
In this paper, we construct some new digital surfaces from the topological sum of two digital surfaces. Also, we compute the digital simplicial homology groups of these digital surfaces. We calculate the Euler characteristics of certain digital connected surfaces. Moreover, we obtain some results of Euler characteristics of certain minimal simple closed surfaces.

Keywords
Digital surface, simplicial homology groups, connected sum, Euler characteristics.

AMS Subject Classification
55N35, 68R10.

[1,2] *Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey.*
*Corresponding author: [1] ismet__cinar@hotmail.com
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1. Introduction

Digital topology [24, 26] has been examined in image processing and medical research for several decades. The properties of digital objects are characterized with tools from topology by many researchers [4, 13, 14, 17].

Homology is a useful topological invariant that characterizes an object by its p-dimensional holes. It is a powerful tool for image processing since a general algorithm to determine whether two distinct objects having isomorphic homology groups can be presented. The first study in this area is given by Rosenfeld [25]. He introduces the concept of continuity of functions from a digital image to another digital image. Boxer [4, 5] establishes digital versions of a retraction and a homotopy by using the digital continuity of functions.

Arslan et al.[1] introduce the digital simplicial homology groups of n-dimensional digital images. In addition, they deal with a general algorithm for calculating digital homology groups of finite dimensional digital images. See [9, 11, 15, 16, 18, 19, 23] for more details.

Boxer et al.[8] investigate the simplicial homology groups of some minimal simple closed surfaces and show how to compute the Euler characteristics of several digital surfaces. Then Demir and Karaca [10] calculate the simplicial homology groups of digital surfaces MSS_6, $MSS_6 \sharp MSS_6$ and $MSS_{18} \sharp MSS_{18}$. They also present some basic properties of the squaring operations on digital images such as the Bockstein homomorphism, the Cartan formula and the Adem relations.

Karaca and Burak [22] define the simplicial cup product for digital images and use it to establish ring structure of digital cohomology. They introduce the relative cohomology groups of digital images.

This paper is organized as follows: Some basic notions are stated in Section 2. In the next section, we compute the simplicial homology groups of certain minimal simple closed surfaces. Finally, we get some results related to Euler characteristics for some digital connected surfaces.

2. Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space, where $\mathbb{Z}$ is the set of integers. A (binary) digital image is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ represents certain adjacency relations in the study of digital images.
Let u be a positive integer, $1 \leq u \leq n$. Let $p, q \in \mathbb{Z}^n$, $p \neq q$. We say that p and q are *c_u-adjacent* [6] if

• there are at most l indices i for which $|p_i - q_i| = 1$, and

• for all indices j such that $|p_j - q_j| \neq 1$, we have $p_j = q_j$.

The notation c_u is sometimes also understood as the number of points $q \in \mathbb{Z}^n$ that are c_u-adjacent to given point $p \in \mathbb{Z}^n$. E.g.,

• in $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency;

• in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;

• in $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

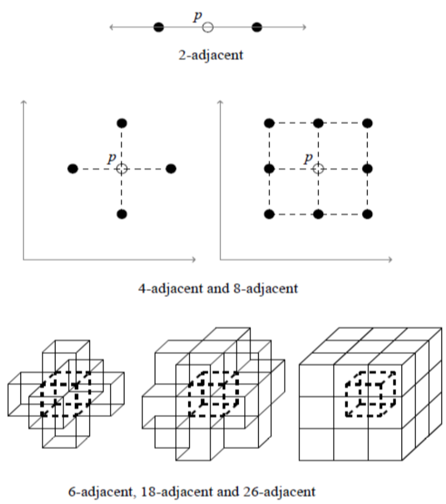

Figure 1. Examples of adjacency relations

Let $\kappa \in \{2,4,6,8,18,26\}$. A κ-*neighbor* [5] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ which is κ-adjacent to p. A digital image X is said to be κ-*connected* [21] if and only if for every pair of different points p and q in X, there is a sequence $\{p_0, p_1, ..., p_r\}$ of points of X such that $p = p_0$, $q = p_r$ and p_i and p_{i+1} are κ-adjacent where $i \in \{0,1,...,r-1\}$. A κ-component of a digital image X is a maximal κ-connected subset of X. Let $a, b \in \mathbb{Z}$ with $a < b$. A set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}$$

is called a *digital interval* [4].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency, respectively. Then a function $f : X \to Y$ is called (κ_0, κ_1)-*continuous* [4, 26] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is a digitally continuous. Similar notions are defined on discrete manifolds in [12]: Let D_1 and D_2 be two discrete manifolds and $f : D_1 \to D_2$ be a mapping. f is said to be an immersion from D_1 to D_2 or a gradually varied operator if x and y are adjacent in D_1 implies either $f(x) = f(y)$ or $f(x)$, $f(y)$ are adjacent in D_2.

A $(2,\kappa)$-continuous function $f : [0,m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$ is called a digital κ-*path* from x to y in a digital image X [5]. If $f(0) = f(m)$, then the κ-path is said to be *closed* and the function is called a κ-*loop*. If there exists a κ-path in X from x to y for every $x, y \in X$, then X is called a digital κ-*path connected* [5].

A *simple closed* κ-*curve* of $m \geq 4$ points in a digital image X is a sequence

$$\{f(0), f(1), ..., f(m-1)\}$$

of images of the κ-path $f : [0, m-1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ κ-adjacent if and only if $j = \pm(i+1) \bmod m$ [6].

A point $x \in X$ is called a κ-*corner* if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other [3]. In addition, the κ-corner x is called *simple* if y, z are not κ-corner and if x is the only point κ-adjacent to both y and z [2]. X is called a *generalized simple closed* κ-*curve* if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve [21].

Let (X, κ) be a κ-connected digital image in $\mathbb{Z}^n$, $n \geq 3$, where κ is an adjacent relation for the members of X. We can write following:

$$|X|^x = N^*_{3^n-1}(x) \cap X$$

where $N^*_{3^n-1}(x) = \{x' \in \mathbb{Z}^n : x \text{ and } x' \text{ are } 3^n - 1 \text{ adjacent}\}$ [20].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency, respectively. Then a function $f : X \to Y$ is called (κ_0, κ_1) *isomorphism* [7] if f is (κ_0, κ_1)-continuous, bijective and the inverse of f is (κ_1, κ_0)-continuous.

Definition 2.1. *[20] Let $c^* := \{x_0, x_1, ..., x_n\}$ be a closed κ-curve in $\mathbb{Z}^2$, where $\{\kappa, \bar{\kappa}\} = \{4, 8\}$. A point x of the complement $\bar{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be in the* interior *of c^* if it belongs to the bounded $\bar{\kappa}$-connected component of $\bar{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.*

Definition 2.2. *[20] Let (X, κ) be a digital image in Z^n, $n \geq 3$ and $\bar{X} = \mathbb{Z}^n - X$. Then X is called a* closed κ-surface *if it satisfies the following.*

(1) *In case that $(\kappa, \bar{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from Definition 2.1 with $\kappa \neq 3^n - 2^n - 1$ and $\bar{\kappa}$ is the adjacency on $\bar{X}$, then*

(a) *for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;*

(b) *$|\bar{X}|^x$ has exactly two $\bar{\kappa}$-components $\bar{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and*

(c) *for any point $y \in N_\kappa(x) \cap X$, $N_{\bar{\kappa}}(y) \cap C^{xx} \neq 0$ and $N_{\bar{\kappa}}(y) \cap D^{xx} \neq 0$, where $N_\kappa(x)$ means the κ-neighbors of x.*

Furthermore, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) *In that case $(\kappa, \bar{\kappa}) = (3^n - 2^n - 1, 2n)$,*

(a) *X is κ-connected,*

(b) *for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.*

Further, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

Hereafter, we denote by MSS_κ *the minimal simple closed* κ*-surface* in $\mathbb{Z}^n$ for $n \geq 3$. In addition we recall the following notations introduced in [20]:
• $MSS_6 \approx_{(6,6)} (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4 \times \{0,2\})$, where MSC_4 is 4-isomorphic to the set $\{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}$.
• $MSS_6' \approx_{(6,6)} X \times [0,1]_{\mathbb{Z}}$, where $X = \{(0,0),(1,1),(0,1),(1,0)\}$.
• $MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8 \times \{0,2\})$, where MSC_8 is 8-isomorphic to the set $\{(0,0),(-1,1),(-2,0),(-2,-1),(-1,-2),(0,-1)\}$.
• $MSS_{18}' \approx_{(18,18)} (MSC_8' \times \{1\}) \cup (Int(MSC_8' \times \{0,2\})$, where MSC_8' is 8-isomorphic to the set $\{(0,0),(-1,1),(-2,0),(-1,-1)\}$.

The digital images MSC_4^*, $MSC_8'^*$ and MSC_8^* which come from the minimal simple closed curves MSC_4, MSC_8' and MSC_8 in $\mathbb{Z}^2$, respectively, play important roles in establishing a connected sum of closed κ-surfaces [20]:
• $MSC_4^* = MSC_4 \cup Int(MSC_4)$,
• $MSC_8'^* = MSC_8' \cup Int(MSC_8')$,
• $MSC_4^* = MSC_8 \cup Int(MSC_8)$.
The digital images MSC_{18}^* and MSC_6^* are in $\mathbb{Z}^3$. They are obtained from the minimal simple closed curves MSC_8 and MSC_4 in $\mathbb{Z}^2$, respectively, and essentially used in generating the notion of connected sum [20],
• $MSS_6^* = MSS_6 \cup Int(MSS_6)$ where
$MSS_6 \approx_{(6,6)} (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0,2\})$
and MSC_4 is 4-isomorphic to the set
$\{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}$.
• $MSS_{18}^* = MSS_{18} \cup Int(MSS_{18})$ where
$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$
and MSC_8 is 8-isomorphic to the set
$\{(0,0),(-1,1),(-2,0),(-2,-1),(-1,-2),(0,-1)\}$.

Definition 2.3. *[20] Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A_{\kappa_0}' \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that $A_{\kappa_0}' \approx_{(\kappa_0,8)} Int(MSC_8^*)$, $A_{\kappa_0}' \approx_{(\kappa_0,4)} Int(MSC_4^*)$ or $A_{\kappa_0}' \approx_{(\kappa_0,8)} Int(MSC_8'^*)$. Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0,κ_1)-isomorphism and let*

$$S_{\kappa_1}' = S_{\kappa_1} - f(A_{\kappa_0}') \text{ and } S_{\kappa_0}' = S_{\kappa_0} - A_{\kappa_0}'.$$

Then the connected sum*, denoted by $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S_{\kappa_0} \cup S_{\kappa_1} / \sim$, where $i : A_{\kappa_0} - A_{\kappa_0}' \to S_{\kappa_0}'$ is the including map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} - A_{\kappa_0}'$.*

Example 2.4. *Let MSS_6 and MSS_6' be the minimal simple closed 6-surfaces. Then the connected sums of $MSS_6' \sharp MSS_6'$ and $MSS_6' \sharp MSS_6$ are as follows:*

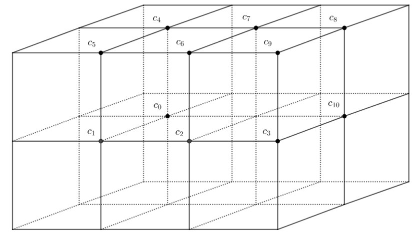

Figure 2. $MSS_6' \sharp MSS_6'$

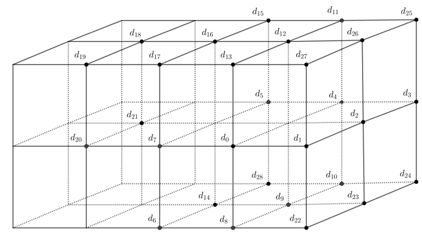

Figure 3. $MSS_6' \sharp MSS_6$

Let S_{κ_1}, S_{κ_2} and S_{κ_3} be disjoint digital minimal simple surfaces. Then we have the following properties:
1) The digital topological sum is commutative:

$$S_{\kappa_1} \sharp S_{\kappa_2} \approx S_{\kappa_2} \sharp S_{\kappa_1}.$$

For example, let MSS_{18} and MSS_{18}' be a minimal simple closed surfaces.

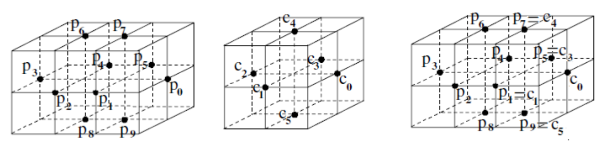

Figure 4. MSS_{18}, MSS_{18}' and $MSS_{18} \sharp MSS_{18}'$

From the above figures, $MSS_{18} \sharp MSS_{18}' \approx MSS_{18}' \sharp MSS_{18}$.
2) The digital topological sum is associative:

$$(S_{\kappa_1} \sharp S_{\kappa_2}) \sharp S_{\kappa_3} \approx S_{\kappa_1} \sharp (S_{\kappa_2} \sharp S_{\kappa_3}).$$

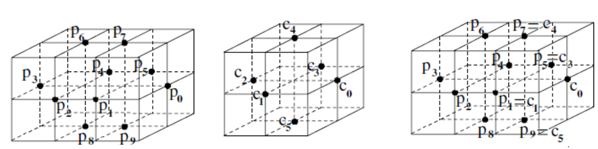

Figure 5. MSS_{18}, MSS_{18}' and $(MSS_{18} \sharp MSS_{18}') \sharp MSS_{18}'$

As a result,
$(MSS_{18} \sharp MSS_{18}') \sharp MSS_{18}' \approx MSS_{18} \sharp (MSS_{18}' \sharp MSS_{18}')$.

Corollary 2.5. 1) $MSS_{18} \sharp MSS_{18}' \approx MSS_{18}$.
2) $MSS_6' \sharp MSS_6 \approx MSS_6 \sharp MSS_6'$.

Proof. 1) By the commutativity of a connected sum, we have $MSS'_{18}\sharp MSS_{18} \approx MSS_{18}\sharp MSS'_{18}$. Since $MSS'_{18}\sharp MSS_{18} \approx MSS_{18}$, the result holds.
2) Since a connected sum of two digital surfaces is commutative, it can be seen that $MSS'_6\sharp MSS_6 \approx MSS_6\sharp MSS'_6$. $\square$

Let S be a set of nonempty subset of a digital image (X,κ). Then the members of S are called the simplex the *simplexes* of (X,κ) if the following two statements hold:

(i) If p and q are two distinct points of S, then p and q are κ-adjacent,

(ii) If $s \in S$ and $\varnothing \neq t \subset s$, then $t \in S$ [27].

If the cardinality of s is equal to $n+1$, then s is called a *n-simplex*. If s' is a nonempty proper subset of s, then s' is a face of s.
Let (X,κ) be a finite collection of digital m-simplices, $0 \leq m < d$ for some non-negative integer d. Then (X,κ) is called *a finite digital simplicial complex* if the following hold:

(i) If P belongs to X, then every face of P also belongs to X, and

(ii) If P and Q in X, then $P \cap Q$ is either empty or a common face of P and Q [1].

$C_q^\kappa(X)$ is a free abelian group with basis all digital (κ,q)-simplicies in X [8].
Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ defined by

$$\partial_q(< p_0,...,p_q >) = \begin{cases} \sum(-1)^i < p_0,...,\hat{p}_i,...,p_q >, & q \leq m \\ 0, & q > m \end{cases}$$

is called a *boundary homomorphism* where $\hat{p}_i$ means deleting the point p_i. Then for all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$ [1].
Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} C_{m-1}^\kappa(X) \xrightarrow{\partial_{m-1}} ... \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a chain complex [1].
Let (X,κ) be a digital simplicial complex of dimension m.

• $Z_q^\kappa(X) = Ker\partial_q$ is called the group of digital simplicial q-cycles.

• $B_q^\kappa(X) = Im\partial_{q+1}$ is called the group of digital simplicial q-boundaries.

Note that $B_q^\kappa(X) \subset Z_q^\kappa(X) \subset C_q^\kappa(X)$ for each $q \geq 0$ and hence we can consider the quotients

$$H_q^\kappa(X) := \frac{Z_q^\kappa(X)}{B_q^\kappa(X)}$$

called the *q-th simplicial homology group* of a digital simplicial complex (X,κ) [1].
If $f : X \to Y$ is a digital (κ_0,κ_1)-isomorphism, then for all $q \leq m$ [1]

$$H_q^{\kappa_0}(X) \cong H_q^{\kappa_1}(Y).$$

Theorem 2.6. *[8] Let (X,κ) be a directed digital simplicial complex of dimension m.*
(1) *$H_q^\kappa(X)$ is a finitely generated abelian group for every $q \geq 0$.*
(2) *$H_q^\kappa(X)$ is a trivial group for every $q > m$.*
(3) *$H_q^\kappa(X)$ is a free abelian group, possibly zero.*

Theorem 2.7. *[20] $MSS'_{18}\sharp MSS'_{18} \approx MSS'_{18}$ via $A_{18} \approx_{(18,8).h} MSC'^*_{18}$, where A_{18} is a subset of MSS'_{18}. $MSC'^*_{18} = MSC'_{18} \cup Int(MSC'_{18})$, where MSC'_{18} is any set which is 8-homeomorphic to the set $\{(0,0),(-1,1),(-2,0),(-1,-1)\}$.*

Let (X,κ) be a digital image of dimension m, and for each $q \geq 0$, let α_q be the number of digital (κ,q)-simplexes in X. The Euler characteristics of X [8], denoted by $\chi(X,\kappa)$, is defined by

$$\chi(X,\kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q.$$

If (X,κ) is a digital image of dimension m, then [8]

$$\chi(X,\kappa) = \sum_{q=0}^{m} (-1)^q \mathrm{rank} H_q^\kappa(X).$$

Boxer et al.[8] show that the Euler characteristics of the digital surface MSS'_{18} is equal to 2.

Theorem 2.8. *[8] If $(X,\kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y,\kappa_1) \subset \mathbb{Z}^{n_1}$ are (κ_0,κ_1)-isomorphic, then*

$$\chi(X,\kappa_0) = \chi(Y,\kappa_1).$$

Theorem 2.9. *[8] The digital simplicial homology groups of MSS'_{18} are*

$$H_n^6(MSS'_{18}) = \begin{cases} \mathbb{Z}, & n = 0,2 \\ 0, & n \neq 0,2. \end{cases}$$

3. Main Results

Theorem 3.1. *Let $MSS'_6\sharp MSS'_6$ be a connected sum of the digital minimal simple surface MSS'_6 with itself. Then we have*

$H_0^6(MSS'_6\sharp MSS'_6) \cong \mathbb{Z}$, $H_1^6(MSS'_6\sharp MSS'_6) \cong \mathbb{Z}^6$, and for $n \neq 0,1$, $H_n^6(MSS'_6\sharp MSS'_6) = \{0\}$.

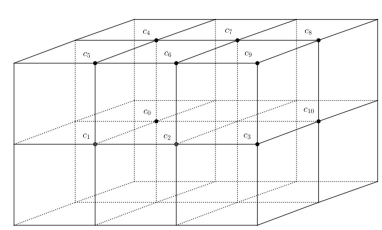

Figure 6. $MSS'_6\sharp MSS'_6$

Proof. Let

$$\begin{aligned} MSS'_6\sharp MSS'_6 = \{ & c_0=(0,0,0), c_1=(1,0,0), c_2=(1,1,0),\\ & c_3=(1,2,0), c_4=(0,0,1), c_5=(1,0,1),\\ & c_6=(1,1,1), c_7=(0,1,1), c_8=(0,2,1),\\ & c_9=(1,2,1), c_{10}=(0,2,0)\}. \end{aligned}$$

$MSS'_6\sharp MSS'_6$ can be directed by the following ordering:
$c_1 < c_6 < c_8 < c_3 < c_0 < c_5 < c_4 < c_7 < c_{10} < c_9 < c_2$.
We get the simplicial chain complexes listed below:
$C_0^6(MSS'_6\sharp MSS'_6)$ has for a basis $\{< c_0 >, < c_1 >, < c_2 >, ..., < c_{10} >\}$,
$C_1^6(MSS'_6\sharp MSS'_6)$ has for a basis
$\{< c_1c_0 >, < c_0c_4 >, < c_1c_2 >, < c_1c_5 >, < c_6c_2 >, < c_3c_2 >, < c_3c_9 >, < c_3c_{10} >, < c_5c_4 >, < c_4c_7 >, < c_6c_5 >, < c_8c_7 >, < c_8c_9 >, < c_8c_{10} >, < c_6c_7 >, < c_6c_9 >\}$.
Hence we obtain the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS'_6\sharp MSS'_6) \xrightarrow{\partial_1} C_0^6(MSS'_6\sharp MSS'_6) \xrightarrow{\partial_0} 0.$$

From Theorem 2.6, $H_n^6(MSS'_6\sharp MSS'_6)$ is a trivial group for all $n \geq 2$.
By the short sequence, we get

$$\text{Im } \partial_2 = 0 \quad \text{and} \quad \text{Ker } \partial_0 \cong \mathbb{Z}^{11}.$$

Now we can find the image of ∂_1. Let
$\partial_1(\alpha_1 < c_1c_0 > +\alpha_2 < c_0c_4 > +\alpha_3 < c_1c_2 > +\alpha_4 < c_1c_5 > +\alpha_5 < c_6c_2 > +\alpha_6 < c_3c_2 > +\alpha_7 < c_3c_9 > +\alpha_8 < c_3c_{10} > +\alpha_9 < c_5c_4 > +\alpha_{10} < c_4c_7 > +\alpha_{11} < c_6c_5 > +\alpha_{12} < c_8c_7 > +\alpha_{13} < c_8c_9 > +\alpha_{14} < c_8c_{10} > +\alpha_{15} < c_6c_7 > +\alpha_{16} < c_6c_9 >)$
$= \alpha_1 < c_0 > -\alpha_1 < c_1 > +\alpha_2 < c_4 > -\alpha_2 < c_0 > + \alpha_3 < c_2 > -\alpha_3 < c_1 > +\alpha_4 < c_5 > -\alpha_4 < c_1 > + \alpha_5 < c_2 > -\alpha_5 < c_6 > +\alpha_6 < c_2 > -\alpha_6 < c_3 > +\alpha_7 < c_9 > -\alpha_7 < c_3 > +\alpha_8 < c_{10} > -\alpha_8 < c_3 > +\alpha_9 < c_4 > - \alpha_9 < c_5 > +\alpha_{10} < c_7 > -\alpha_{10} < c_4 > +\alpha_{11} < c_5 > - \alpha_{11} < c_6 > +\alpha_{12} < c_7 > -\alpha_{12} < c_8 > +\alpha_{13} < c_9 > - \alpha_{13} < c_8 > +\alpha_{14} < c_{10} > -\alpha_{14} < c_8 > +\alpha_{15} < c_7 > - \alpha_{15} < c_6 > +\alpha_{16} < c_9 > -\alpha_{16} < c_6 >$
$= (\alpha_1-\alpha_2) < c_0 > +(-\alpha_1-\alpha_3-\alpha_4) < c_1 > + (\alpha_3+\alpha_5+\alpha_6) < c_2 > +(-\alpha_6-\alpha_7-\alpha_8) < c_3 > + (\alpha_2+\alpha_9-\alpha_{10}) < c_4 > +(\alpha_4-\alpha_9+\alpha_{11}) < c_5 > + (-\alpha_5-\alpha_{11}-\alpha_{15}-\alpha_{16}) < c_6 > +(\alpha_{10}+\alpha_{12}+\alpha_{15}) < c_7 > +(-\alpha_{12}-\alpha_{13}-\alpha_{14}) < c_8 > +(\alpha_7+\alpha_{13}+\alpha_{16}) < c_9 > + (\alpha_8+\alpha_{14}) < c_{10} >$.
If

$$\alpha_1-\alpha_2 = a_1,$$

$$-\alpha_1-\alpha_3-\alpha_4 = a_2,$$

$$\alpha_3+\alpha_5+\alpha_6 = a_3,$$

$$-\alpha_6-\alpha_7-\alpha_8 = a_4,$$

$$\alpha_2+\alpha_9-\alpha_{10} = a_5,$$

$$\alpha_4-\alpha_9+\alpha_{11} = a_6,$$

$$-\alpha_5-\alpha_{11}-\alpha_{15}-\alpha_{16} = a_7,$$

$$\alpha_{10}+\alpha_{12}+\alpha_{15} = a_8,$$

$$-\alpha_{12}-\alpha_{13}-\alpha_{14} = a_9,$$

$$\alpha_7+\alpha_{13}+\alpha_{16} = a_{10},$$

$$\alpha_8+\alpha_{14} = a_{11},$$

then $a_1+a_2+a_3+a_4+a_5+a_6+a_8+a_9+a_{10}+a_{11} = -a_7$.
Thus, we obtain Im $\partial_1 \cong \mathbb{Z}^{10}$.
In the next step, we shall find the kernel of ∂_1. Let
$\partial_1(\alpha_1 < c_1c_0 > +\alpha_2 < c_0c_4 > +\alpha_3 < c_1c_2 > +\alpha_4 < c_1c_5 > +\alpha_5 < c_6c_2 > +\alpha_6 < c_3c_2 > +\alpha_7 < c_3c_9 > +\alpha_8 < c_3c_{10} > +\alpha_9 < c_5c_4 > +\alpha_{10} < c_4c_7 > +\alpha_{11} < c_6c_5 > +\alpha_{12} < c_8c_7 > +\alpha_{13} < c_8c_9 > +\alpha_{14} < c_8c_{10} > +\alpha_{15} < c_6c_7 > +\alpha_{16} < c_6c_7 >) = 0$.
From the above equation, we have the following equations:

$$\alpha_1-\alpha_2 = 0$$

$$-\alpha_1-\alpha_3-\alpha_4 = 0,$$

$$\alpha_3+\alpha_5+\alpha_6 = 0,$$

$$-\alpha_6-\alpha_7-\alpha_8 = 0,$$

$$\alpha_2+\alpha_9-\alpha_{10} = 0,$$

$$\alpha_4-\alpha_9+\alpha_{11} = 0,$$

$$-\alpha_5-\alpha_{11}-\alpha_{15}-\alpha_{16} = 0,$$

$$\alpha_{10}+\alpha_{12}+\alpha_{15} = 0,$$

$$-\alpha_{12}-\alpha_{13}-\alpha_{14} = 0,$$

$$\alpha_7+\alpha_{13}+\alpha_{16} = 0,$$

$$\alpha_8+\alpha_{14} = 0.$$

By arranging these equations we obtain
$Z_1^6(MSS'_6\sharp MSS'_6) = \{\alpha_1 < c_1c_0 > +\alpha_2 < c_0c_4 > +\alpha_3 < c_1c_2 > +(-\alpha_1-\alpha_4) < c_1c_5 > +\alpha_5 < c_6c_2 > +(-\alpha_3-\alpha_5) < c_3c_2 > +\alpha_7 < c_3c_9 > +(\alpha_3+\alpha_5-\alpha_7) < c_3c_{10} > +\alpha_9 < c_5c_4 > +(\alpha_1+\alpha_9) < c_4c_7 > +(\alpha_9+\alpha_1+\alpha_3) < c_6c_5 > +\alpha_{12} < c_8c_7 > +(\alpha_3+\alpha_5-\alpha_7-\alpha_{12}) < c_8c_9 > +(-\alpha_3-\alpha_5+\alpha_7) < c_8c_{10} > +(-\alpha_1-\alpha_9-\alpha_{12}) < c_6c_7 > +(-\alpha_3-\alpha_5+\alpha_{12}) < c_6c_7 > |\alpha_i \in \mathbb{Z}, i = 1,3,5,7,9,12\} \cong \mathbb{Z}^6$.
As a result, we have $H_0^6(MSS'_6\sharp MSS'_6) \cong \mathbb{Z}$ and
$H_1^6(MSS'_6\sharp MSS'_6) \cong \mathbb{Z}^6$. □

Proposition 3.2. *The Euler characteristics of $MSS'_6\sharp MSS_6$ is equal to -22.*

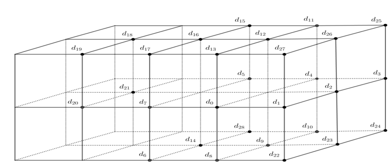

Figure 7. $MSS'_6\sharp MSS_6$

Proof. We can direct $MSS_6'\sharp MSS_6$ by the ordering $d_{20} < d_7 < d_0 < d_1 < d_2 < d_3 < d_4 < d_5 < d_{28} < d_{11} < d_{25} < d_{26} < d_{27} < d_{13} < d_{17} < d_{16} < d_{18} < d_{19} < d_{21} < d_6 < d_8 < d_9 < d_{10} < d_{12} < d_{14} < d_{15} < d_{22} < d_{23} < d_{24}$.
It is clear from the Figure 7 that $MSS_6'\sharp MSS_6$ has 29 zero-simplex:

$$\{< d_0 >, < d_1 >, < d_2 >, ..., < d_{28} >\}.$$

Moreover, this surface has 51 one-simplex:
$\{< d_7d_0 >, < d_0d_{13} >, < d_0d_1 >, < d_0d_8 >, < d_1d_2 >,$
$< d_1d_{22} >, < d_1d_{27} >, < d_2d_3 >, < d_2d_{26} >, < d_2d_{23} >,$
$< d_3d_4 >, < d_3d_{25} >, < d_3d_{24} >, < d_4d_5 >, < d_4d_{11} >,$
$< d_4d_{10} >, < d_5d_{15} >, < d_5d_{28} >, < d_6d_8 >, < d_6d_{14} >,$
$< d_7d_6 >, < d_7d_{17} >, < d_{20}d_7 >, < d_8d_9 >, < d_8d_{22} >,$
$< d_9d_{10} >, < d_9d_{14} >, < d_9d_{23} >, < d_{10}d_{24} >, < d_{28}d_{10} >,$
$< d_{11}d_{12} >, < d_{11}d_{15} >, < d_{11}d_{25} >, < d_{13}d_{12} >,$
$< d_{16}d_{12} >, < d_{26}d_{12} >, < d_{28}d_{14} >, < d_{16}d_{15} >,$
$< d_{17}d_{16} >, < d_{16}d_{18} >, < d_{17}d_{19} >, < d_{18}d_{19} >,$
$< d_{18}d_{21} >, < d_{20}d_{19} >, < d_{22}d_{23} >, < d_{23}d_{24} >,$
$< d_{25}d_{26} >, < d_{26}d_{27} >, < d_{13}d_{17} >, < d_{27}d_{13} >,$
$< d_{20}d_{21} >\}$.
By the definition of Euler characteristics, α_q is a number of digital $(6,q)$-simplexes in $MSS_6'\sharp MSS_6$.
$\chi(MSS_6'\sharp MSS_6, 6) = \sum_{q=0}^{m}(-1)^q\alpha_q$
$= (-1)^0.29 + (-1)^1.51 + (-1)^2.0 + ... = -22.$ □

Theorem 3.3. *Let $MSS_6'\sharp MSS_6$ be a connected sum of the digital minimal simple surface MSS_6' with MSS_6. Then the digital simplicial homology groups of $MSS_6'\sharp MSS_6$ are as follows:*
$H_0^6(MSS_6'\sharp MSS_6) \cong \mathbb{Z}$, $H_1^6(MSS_6'\sharp MSS_6) \cong \mathbb{Z}^{23}$, *and for* $n \neq 0,1$, $H_n^6(MSS_6'\sharp MSS_6) = \{0\}$.

Proof. Let $MSS_6'\sharp MSS_6 = \{d_0 = (1,2,0), d_1 = (1,3,0),$
$d_2 = (0,3,0), d_3 = (-1,3,0), d_4 = (-1,2,0), d_5 = (-1,1,0),$
$d_6 = (1,1,-1), d_7 = (1,1,0), d_8 = (1,2,-1), d_9 = (0,2,-1),$
$d_{10} = (-1,2,-1), d_{11} = (-1,2,1), d_{12} = (0,2,1),$
$d_{13} = (1,2,1), d_{14} = (0,1,-1), d_{15} = (-1,1,1),$
$d_{16} = (0,1,1), d_{17} = (1,1,1), d_{18} = (0,0,1), d_{19} = (1,0,1),$
$d_{20} = (1,0,0), d_{21} = (0,0,0), d_{22} = (1,3,-1),$
$d_{23} = (0,3,-1), d_{24} = (1,3,-1), d_{25} = (-1,3,1),$
$d_{26} = (0,3,1), d_{27} = (1,3,1), d_{28} = (1,1,-1)\}$.
We can direct $MSS_6'\sharp MSS_6$ by the ordering
$d_{20} < d_7 < d_0 < d_1 < d_2 < d_3 < d_4 < d_5 < d_{28} < d_{11} < d_{25} < d_{26} < d_{27} < d_{13} < d_{17} < d_{16} < d_{18} < d_{19} < d_{21} < d_6 < d_8 < d_9 < d_{10} < d_{12} < d_{14} < d_{15} < d_{22} < d_{23} < d_{24}$.

$C_0^6(MSS_6'\sharp MSS_6)$ and $C_1^6(MSS_6'\sharp MSS_6)$ are free abelian groups with basis
$\{< d_0 >, < d_1 >, < d_2 >, ..., < d_{28} >\}$ and
$\{< d_7d_0 >, < d_0d_{13} >, < d_0d_1 >, < d_0d_8 >, < d_1d_2 >,$
$< d_1d_{22} >, < d_1d_{27} >, < d_2d_3 >, < d_2d_{26} >, < d_2d_{23} >,$
$< d_3d_4 >, < d_3d_{25} >, < d_3d_{24} >, < d_4d_5 >, < d_4d_{11} >,$
$< d_4d_{10} >, < d_5d_{15} >, < d_5d_{28} >, < d_6d_8 >, < d_6d_{14} >,$
$< d_7d_6 >, < d_7d_{17} >, < d_{20}d_7 >, < d_8d_9 >, < d_8d_{22} >,$
$< d_9d_{10} >, < d_9d_{14} >, < d_9d_{23} >, < d_{10}d_{24} >, < d_{28}d_{10} >,$
$< d_{11}d_{12} >, < d_{11}d_{15} >, < d_{11}d_{25} >, < d_{13}d_{12} >,$
$< d_{16}d_{12} >, < d_{26}d_{12} >, < d_{28}d_{14} >, < d_{16}d_{15} >,$
$< d_{17}d_{16} >, < d_{16}d_{18} >, < d_{17}d_{19} >, < d_{18}d_{19} >,$
$< d_{18}d_{21} >, < d_{20}d_{19} >, < d_{22}d_{23} >, < d_{23}d_{24} >,$
$< d_{25}d_{26} >, < d_{26}d_{27} >, < d_{13}d_{17} >, < d_{27}d_{13} >,$
$< d_{20}d_{21} >\}$, respectively.
Therefore we get the following short sequence

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6'\sharp MSS_6) \xrightarrow{\partial_1} C_0^6(MSS_6'\sharp MSS_6) \xrightarrow{\partial_0} 0$$

By Theorem 2.6, $H_n^6(MSS_6'\sharp MSS_6)$ is a trivial group for all $n \geq 2$.
From the short sequence, we obtain

$$\text{Im}\,\partial_2 = 0 \quad \text{and} \quad \text{Ker}\,\partial_0 \cong \mathbb{Z}^{29}.$$

We shall calculate the image of ∂_1:
$\partial_1(a_1 < d_7d_0 > +a_2 < d_0d_{13} > +a_3 < d_0d_1 > +a_4 < d_0d_8 >$
$+a_5 < d_1d_2 > +a_6 < d_1d_{22} > +a_7 < d_1d_{27} > +a_8 < d_2d_3 >$
$+a_9 < d_2d_{26} > +a_{10} < d_2d_{23} > +a_{11} < d_3d_4 > +$
$a_{12} < d_3d_{25} > +a_{13} < d_3d_{24} > +a_{14} < d_4d_5 > +$
$a_{15} < d_4d_{11} > +a_{16} < d_4d_{10} > +a_{17} < d_5d_{15} > +$
$a_{18} < d_5d_{28} > +a_{19} < d_6d_8 > +a_{20} < d_6d_{14} > +$
$a_{21} < d_7d_6 > +a_{22} < d_7d_{17} > +a_{23} < d_{20}d_7 > +a_{24} < d_8d_9 >$
$+a_{25} < d_8d_{22} > +a_{26} < d_9d_{10} > +a_{27} < d_9d_{14} > +$
$a_{28} < d_9d_{23} > +a_{29} < d_{10}d_{24} > +a_{30} < d_{28}d_{10} > +$
$a_{31} < d_{11}d_{12} > +a_{32} < d_{11}d_{15} > +a_{33} < d_{11}d_{25} > +$
$a_{34} < d_{13}d_{12} > +a_{35} < d_{16}d_{12} > +a_{36} < d_{26}d_{12} > +$
$a_{37} < d_{28}d_{14} > +a_{38} < d_{16}d_{15} > +a_{39} < d_{17}d_{16} > +$
$a_{40} < d_{16}d_{18} > +a_{41} < d_{17}d_{19} > +a_{42} < d_{18}d_{19} > +$
$a_{43} < d_{18}d_{21} > +a_{44} < d_{20}d_{19} > +a_{45} < d_{22}d_{23} > +$
$a_{46} < d_{23}d_{24} > +a_{47} < d_{25}d_{26} > +a_{48} < d_{26}d_{27} > +$
$a_{49} < d_{13}d_{17} > +a_{50} < d_{27}d_{13} > +a_{51} < d_{20}d_{21} >)$

$= \{(a_1 - a_2 - a_3 - a_4) < d_0 > +(a_3 - a_5 - a_6 - a_7) < d_1 >$
$+(a_5 - a_8 - a_9 - a_{10}) < d_2 > +(a_8 - a_{11} - a_{12} - a_{13}) < d_3 >$
$+(a_{11} - a_{14} - a_{15} - a_{16}) < d_4 > +(a_{14} - a_{17} - a_{18}) < d_5 >$
$+(-a_{19} - a_{20} + a_{21}) < d_6 > +(-a_1 - a_{21} - a_{22} + a_{23}) < d_7 >$
$+(a_4 + a_{19} - a_{24} - a_{25}) < d_8 > +$
$(a_{24} - a_{26} - a_{27} - a_{28}) < d_9 > +$
$(a_{16} + a_{26} - a_{29} + a_{30}) < d_{10} > +$
$(a_{15} - a_{31} - a_{32} - a_{33}) < d_{11} > +$
$(a_{31} - a_{34} + a_{35} + a_{36}) < d_{12} > +$
$(a_2 - a_{34} - a_{49} + a_{50}) < d_{13} > +(a_{20} + a_{27} + a_{37}) < d_{14} >$
$+(a_{17} + a_{32} + a_{38}) < d_{15} > +(-a_{35} - a_{38} + a_{39} - a_{40}) < d_{16} >$
$+(a_{22} - a_{39} - a_{41} + a_{49}) < d_{17} > +(a_{40} - a_{42} - a_{43}) < d_{18} >$
$+(a_{41} + a_{42} + a_{44}) < d_{19} > +(-a_{23} - a_{44} - a_{51}) < d_{20} > +$
$(a_{43} + a_{51}) < d_{21} > +(a_6 + a_{25} - a_{45}) < d_{22} > +$
$(a_{10} + a_{28} + a_{45} - a_{46}) < d_{23} > +(a_{13} + a_{29} + a_{46}) < d_{24} >$
$+(a_{12} + a_{33} - a_{47}) < d_{25} > +(a_9 - a_{36} + a_{47} - a_{48}) < d_{26} >$
$+(a_7 + a_{48} - a_{50}) < d_{27} > +(a_{18} - a_{30} - a_{37}) < d_{28} > .\}$

If we use the MATLAB program for 29×51 size matrix, then we will calculate the rank of the matrix. Its rank is 28. Namely,

$(a_1-a_2-a_3-a_4) < d_0 > +(a_3-a_5-a_6-a_7) < d_1 > +$
$(a_5-a_8-a_9-a_{10}) < d_2 > +(a_8-a_{11}-a_{12}-a_{13}) < d_3 >$
$+(a_{11}-a_{14}-a_{15}-a_{16}) < d_4 > +(a_{14}-a_{17}-a_{18}) < d_5 >$
$+(-a_{19}-a_{20}+a_{21}) < d_6 > +(-a_1-a_{21}-a_{22}+a_{23}) < d_7 >$
$+(a_4+a_{19}-a_{24}-a_{25}) < d_8 > +$
$(a_{24}-a_{26}-a_{27}-a_{28}) < d_9 > +$
$(a_{16}+a_{26}-a_{29}+a_{30}) < d_{10} > +$
$(a_{15}-a_{31}-a_{32}-a_{33}) < d_{11} > +$
$(a_{31}-a_{34}+a_{35}+a_{36}) < d_{12} > +$
$(a_2-a_{34}-a_{49}+a_{50}) < d_{13} > +(a_{20}+a_{27}+a_{37}) < d_{14} >$
$+(a_{17}+a_{32}+a_{38}) < d_{15} > +(-a_{35}-a_{38}+a_{39}-a_{40}) < d_{16} >$
$+(a_{22}-a_{39}-a_{41}+a_{49}) < d_{17} > +(a_{40}-a_{42}-a_{43}) < d_{18} >$
$+(a_{41}+a_{42}+a_{44}) < d_{19} > +(-a_{23}-a_{44}-a_{51}) < d_{20} > +$
$(a_{43}+a_{51}) < d_{21} > +(a_6+a_{25}-a_{45}) < d_{22} > +$
$(a_{10}+a_{28}+a_{45}-a_{46}) < d_{23} > +(a_{13}+a_{29}+a_{46}) < d_{24} >$
$+(a_{12}+a_{33}-a_{47}) < d_{25} > +(a_9-a_{36}+a_{47}-a_{48}) < d_{26} >$
$+(a_7+a_{48}-a_{50}) < d_{27} >= -(a_{18}-a_{30}-a_{37}) < d_{28} >$.
Consequently, we have $\operatorname{Im}\partial_1 \cong \mathbb{Z}^{28}$.
To determine the $H_1^6(MSS_6'\sharp MSS_6)$ we can use Proposition 3.2. We know that

$$\chi(MSS_6'\sharp MSS_6, 6) = -22.$$

From the definition of the Euler characteristics, the following holds:

$$\begin{aligned}\chi(MSS_6'\sharp MSS_6, 6) &= \textstyle\sum_{q=0}^{n}(-1)^q \text{rank } H_q^6(MSS_6'\sharp MSS_6)\\ -22 &= (-1)^0.1+(-1)^1 \text{rank } H_1^6(MSS_6'\sharp MSS_6)\\ &\quad +(-1)^2.0+\dots\end{aligned}$$

Hence we obtain $\text{rank}(H_1^6(MSS_6'\sharp MSS_6)) = 23$ which gives us

$$H_1^6(MSS_6'\sharp MSS_6) \cong \mathbb{Z}^{23}.$$

□

Corollary 3.4. *The Euler characteristics of a digital surface $MSS_{18}'\sharp MSS_{18}'$ is equal to 2.*

Proof. From Theorem 2.7 and Theorem 2.8, we have

$$\chi(MSS_{18}'\sharp MSS_{18}') = \chi(MSS_{18}').$$

By [9, Example 4.4], the result holds. □

Corollary 3.5. *Let $MSS_{18}'\sharp MSS_{18}'$ be a connected sum of the digital minimal simple surface MSS_{18}' with itself. Then its homology groups are*

$H_0^6(MSS_{18}'\sharp MSS_{18}') \cong \mathbb{Z}$, $H_2^6(MSS_{18}'\sharp MSS_{18}') \cong \mathbb{Z}$, *and for* $n \neq 0,2$, $H_n^6(MSS_{18}'\sharp MSS_{18}') = \{0\}$.

Proof. By Theorem 2.7, we have $MSS_{18}'\sharp MSS_{18}' \approx MSS_{18}'$. From Theorem 2.8 and Theorem 2.9, we have the result. □

Theorem 3.6. *The digital simplicial homology groups of $MSS_6\sharp MSS_6'$ are*

$$H_n^6(MSS_6\sharp MSS_6') = \begin{cases} \mathbb{Z}, & n=0\\ \mathbb{Z}^{23}, & n=1\\ 0, & n\neq 0,1.\end{cases}$$

Proof. We have $MSS_6'\sharp MSS_6 \approx MSS_6\sharp MSS_6'$ by Corollary 2.5. If two digital images are isomorphic to each other, then they have the same digital homology groups. Consequently the result holds from the isomorphism. □

Corollary 3.7. $\chi(MSS_6'\sharp MSS_6') = -5$.

Proof. By Theorem 3.1 and the definition of Euler characteristics, we obtain

$$\begin{aligned}\chi(MSS_6'\sharp MSS_6') &= \sum_{q=0}^{n}(-1)^q \text{rank } H_q^6(MSS_6'\sharp MSS_6')\\ &= (-1)^0.1+(-1)^1.6+(-1)^2.0+\dots\\ &= -5.\end{aligned}$$

□

4. Conclusion

In this paper, we investigate some topological properties of certain digital surfaces such as the digital simplicial homology groups and the Euler characteristics. We give the digital homology groups of these surfaces. Finally, we calculate their Euler characteristics.

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