



Convergence analysis and approximate solution of fractional differential equations

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Abstract

This paper investigates the iterative solution of linear and nonlinear fractional partial differential equations using fractional Adomian decomposition method (ADM). We also establish uniqueness and convergence criteria for obtaining approximate solution. To illustrate applicability of present technique, solutions of some test problems and their graphical representation are done by Mathematica software.

Keywords

Fractional differential equation, Adomian Decomposition Method, Uniqueness, Convergence.

AMS Subject Classification

26A33, 33E12 34A08, 35R11.

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1. Introduction

In the last three decades, the topic of Fractional calculus has attracted researchers from various fields of science and engineering. The theory of derivatives of non-integer order was discovered by Leibniz in 1695 [17]. Leibniz's note led to the theory of fractional calculus, which was developed by Liouville, Grunwald, Letnikov and Riemann in 19th century [7, 20, 23, 25, 26]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The benefits of fractional derivative become deceptive in demonstrating mechanical and electrical properties of real materials, elaborating theory of fractals, theory of control of dynamical systems etc. On account of confounded nature the exact analytical solutions of most of fractional differential equations does not

exist, accordingly prominent consideration is given to get approximate solutions of these equations. The most commonly used methods to solve these equations are the Variational iteration method [15], Finite-difference method [32], Laplace transform method [28], Homotopy analysis method [18, 27], Homotopy-perturbation method [16], Laplace homotopy analysis method [21], Homotopy perturbation transform method [13], Fractional complex transformation [29–31], Feng's first integral method [33], the (G'/G) -expansion method [34]. etc. In recent years one of the most effective and accurate algorithm to obtain solution in terms of rapidly convergent series of nonlinear partial differential equations is suggested known as Adomian decomposition method [4]. Over the last 25 years the Adomian Decomposition Method is used to obtain a approximate solution of a wide class of partial differential equations. Nowadays, this method is an alternative tool for obtaining solution of several mathematical models involving higher order linear or nonlinear partial differential equations. Also ADM is most desirable method to obtain realistic solutions of highly complex real life problems such as delay differential equations [8, 12, 24] The main advantage of the method is that without linearization, perturbation or discretization it gives approximate or analytical solution to a large class of nonlinear equations [3, 9–11, 22].

The nonlinear evolution equations are mostly used as models to describe complex physical phenomenon in several branches of science and engineering. Qaseem and Kashif

have studied analytical solutions of fractional Walter’s B fluid with applications [19]. Atangana and Baleanu [5] presented a heat transfer model by implementing Caputo-Fabrizio space-fractional derivative. Hristov [14] has traced out the analytical solution for steady-state heat conduction using Caputo-Fabrizio space-fractional derivative with nonsingular fading memory. Kashif and Ilyas analysed the heat and mass transfer in the MHD flow of a generalized Casson fluid using Caputo-Fabrizio time-fractional derivative [1]

One of the fundamental problems for these models is to obtain their traveling wave solutions as well as solitary wave solutions. The first observation of a solitary wave was made in 1834 by the Scottish scientist and engineer John Scott Russell. Solitary wave is localized wave that propagates along one space direction only, which conserves speed and shape. Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytical formulae and serve as prototypical solutions that model physical localized waves. For many examples, localized initial data ultimately breaks up into a finite collection of solitary wave solutions [2]. This fact has been proved analytically by certain equations such as Korteweg-de-Vries equation (KdV), Schrodinger equation, Boussinesq equation etc. We will make an attempt to solve time fractional KdV equation.

We organize this paper as follows: In section 2, we define some basic preliminaries and properties of fractional calculus. Section 3, is developed for detailed analysis of fractional Adomian Decomposition Method. Section 4, we discuss convergence of fractional Adomian decomposition method. In Section 5, we present some examples to show the applicability and efficiency of the method and also their solutions are demonstrated with the help of Mathematica. Finally, we give our conclusions in Section 6.

2. Fractional operator with properties

This section is devoted to studying some basic definitions of fractional calculus.

Definition 2.1. For a real number $p > \alpha$, a function $f(t)$, $t > 0$, is belong to the space C_α , $\alpha \in R$, if $f(t) = t^p f_1(t)$, where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_α^m if and only if $f^{(m)}(t) \in C_\alpha$, $m \in N$.

Definition 2.2. The Caputo derivative of fractional order α of a function $f(t)$, $f(t) \in C_{-1}^m$ is defined as follows

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(t - \tau)^{(1-m+\alpha)}} d\tau,$$

for $m - 1 < \alpha \leq m$, $m \in N, x > 0$,

Definition 2.3. The Riemann - Liouville fractional integral operator of a function $f \in C_\mu$, $\mu \geq -1$, is defined for $\alpha \geq 0$ by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, x > 0$$

$$J^0 f(x) = f(x).$$

Properties:

It is simple to prove the following properties of fractional derivatives and integrals that will be used in the analysis [23]

- (i) $D^\alpha J^\alpha f(t) = f(t)$,
- (ii) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{(\gamma+\alpha)}$.
- (iii) $D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{(\gamma-\alpha)}$

In the next section, we develop the Adomian decomposition method for fractional partial differential equation.

3. Analysis of Adomian Decomposition Method

We consider the following general fractional partial differential equation

$$L^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \tag{3.1}$$

$$m - 1 < \alpha \leq m, x > 0, t > 0.$$

where L is fractional order derivative, R is a linear differential operator, N is a nonlinear operator and g is source term. Let

$$L^\alpha = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}} \tag{3.2}$$

be the $(n\alpha)^{th}$ order fractional derivative then the corresponding $L^{-\alpha}$ operator will be written in the following form

$$J^\alpha = L^{-\alpha} = \frac{1}{\Gamma^n(\alpha + 1)} \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \dots \int_0^{\tau_{n-1}} (d\tau_1)^\alpha (d\tau_2)^\alpha (d\tau_3)^\alpha \dots (d\tau_n)^\alpha \tag{3.3}$$

and

$$\frac{1}{\Gamma(\alpha + 1)} \int_0^t (d\tau_n)^\alpha$$

is the Caputo integration.

Operating with the operator J^α on both sides of equation (3.1), we have

$$J^\alpha [L^\alpha u(x, t) + Ru(x, t) + Nu(x, t)] = J^\alpha g(x, t)$$

$$J^\alpha L^\alpha u(x, t) = -J^\alpha [Ru(x, t) + Nu(x, t)] + J^\alpha g(x, t) \tag{3.4}$$

Using property (ii) in equation (3.4), we get

$$u(x, t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x, 0)}{\partial t^k} \frac{t^k}{k!} - J^\alpha [Ru(x, t) + Nu(x, t)] + J^\alpha g(x, t), \tag{3.5}$$

for $m - 1 < \alpha \leq m$. Now, we decompose the unknown function $u(x, t)$ into sum of an infinite number of components given by the decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \tag{3.6}$$



The nonlinear terms $Nu(x, t)$ are decomposed in the following form:

$$Nu(x, t) = \sum_{n=0}^{\infty} A_n \tag{3.7}$$

where the Adomian polynomial can be determined as follows:

$$A_n = \frac{1}{n!} \left[\frac{d^n N}{d\lambda^n} \left(\sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0} \tag{3.8}$$

where A_n is called Adomian polynomial [6] and that can be easily calculated by Mathematica software.

Substituting the decomposition series (3.6) and (3.7) into both sides of equation (3.5) gives

$$\sum_{n=0}^{\infty} u_n(x, t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x, 0)}{\partial t^k} \frac{t^k}{k!} + J^\alpha g(x, t) + \sum_{n=0}^{\infty} u_n(x, t) + \sum_{n=0}^{\infty} A_n, \quad x > 0. \tag{3.9}$$

The components $u_n(x, t)$, $n \geq 0$ of the solution $u(x, t)$ can be recursively determined by using the relation as follows:

$$u_0(x, t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x, 0)}{\partial t^k} \frac{t^k}{k!} + J^\alpha g(x, t), \tag{3.10}$$

$$\begin{aligned} u_1(x, t) &= -J^\alpha (Ru_0 + A_0) \\ u_2(x, t) &= -J^\alpha (Ru_1 + A_1) \\ u_3(x, t) &= -J^\alpha (Ru_2 + A_2) \\ &\vdots \\ u_{n+1}(x, t) &= -J^\alpha (Ru_n + A_n) \end{aligned}$$

where each component can be determined by using the preceding components and we can obtain the solution in a series form by calculating the components $u_n(x, t)$, $n \geq 0$. Finally, we approximate the solution $u(x, t)$ by the truncated series.

$$\begin{aligned} \phi_N(x, t) &\cong \sum_{n=0}^{N-1} u_n(x, t) \\ \lim_{N \rightarrow \infty} \phi_N &= u(x, t) \end{aligned}$$

In the next section, we develop convergence of the fractional Adomian decomposition method.

4. Convergence

The practical solution by fractional Adomian decomposition method is given by-

$$\begin{aligned} \phi_N(x, t) &= \sum_{n=0}^{N-1} u_n(x, t) \\ \lim_{N \rightarrow \infty} \phi_N &= u(x, t) \end{aligned}$$

The sufficient condition that guarantees existence of a unique solution is introduced in theorem (4.1) and convergence of the series solution is proved in theorem (4.2).

Theorem 4.1. (Uniqueness Theorem) We consider the following general time fractional partial differential equation

$$L^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \tag{4.1}$$

$m - 1 < \alpha \leq m$, $x > 0$, $t > 0$. where L is fractional order derivative, R is linear differential operator, N is nonlinear operator and g is source term, where $R(u)$ and $N(u)$ satisfy Lipschitz condition with constants L_1 and L_2 . Then equation (4.1) has a unique solution whenever $0 < k < 1$ for $k = \frac{(L_1+L_2)t^\alpha}{\Gamma(\alpha+1)}$.

Proof: Let X be the Banach space of all continuous functions on $I = [0, T]$ with the norm

$$\|u(t)\| = \max|u(t)|.$$

We define a mapping $F : X \rightarrow X$, where

$$F(u(t)) = \phi(t) + J^\alpha g(x, t) - J^\alpha [Ru(x, t)] - [J^\alpha Nu(x, t)] \tag{4.2}$$

Let $u, \bar{u} \in X$

$$\begin{aligned} \|Fu - F\bar{u}\| &= \max|Fu - F\bar{u}| \\ \|Fu - F\bar{u}\| &= \left| \phi(t) + J^\alpha g(x, t) - J^\alpha [Ru(x, t)] - [J^\alpha Nu(x, t)] - \phi(t) - J^\alpha g(x, t) + J^\alpha [R\bar{u}(x, t)] + [J^\alpha N\bar{u}(x, t)] \right| \\ \|Fu - F\bar{u}\| &= \left| J^\alpha [Ru(x, t) - R\bar{u}(x, t)] + J^\alpha [Nu(x, t) - N\bar{u}(x, t)] \right| \end{aligned}$$

Now suppose $R(u)$ and $N(u)$ satisfy Lipschitz condition with constants L_1 and L_2 .

Therefore

$$\begin{aligned} \|Fu - F\bar{u}\| &\leq \max \left[J^\alpha |Ru(x, t) - R\bar{u}(x, t)| + J^\alpha |Nu(x, t) - N\bar{u}(x, t)| \right] \\ \|Fu - F\bar{u}\| &\leq \max \left[L_1 J^\alpha |u(x, t) - \bar{u}(x, t)| + L_2 J^\alpha |u(x, t) - \bar{u}(x, t)| \right] \\ \|Fu - F\bar{u}\| &\leq (L_1 + L_2) \|u(x, t) - \bar{u}(x, t)\| \frac{t^\alpha}{\Gamma(\alpha + 1)} \\ \|Fu - F\bar{u}\| &\leq k \|u(x, t) - \bar{u}(x, t)\| \text{ where } k = \frac{(L_1 + L_2)t^\alpha}{\Gamma(\alpha + 1)} \end{aligned}$$

Theorem 4.2. (Convergence Theorem) The solution $u = \sum_{i=0}^{\infty} u_i(x, t)$ of equation (4.1) using ADM converges if $0 < k < 1$ and $\|u_i\| < \infty$ where $k = \frac{(L_1+L_2)t^\alpha}{\Gamma(\alpha+1)}$.

Proof: Let S_n be the partial sum of series.

$$S_n = \sum_{i=0}^n u_i(x, t)$$



We shall prove that $\{S_n\}$ is a Cauchy sequence in Banach space X. Consider

$$\begin{aligned} \|S_{n+p} - S_n\| &= \max \left| \sum_{i=n+1}^{n+p} u_i(x,t) \right| \\ \|S_{n+p} - S_n\| &= \max \left| J^\alpha (R(S_{n+p-1}) - R(S_{n-1})) \right. \\ &\quad \left. + J^\alpha (N(S_{n+p-1}) - N(S_{n-1})) \right| \\ \|S_{n+p} - S_n\| &\leq \max \left| J^\alpha (R(S_{n+p-1}) - R(S_{n-1})) \right| \\ &\quad + \max \left| J^\alpha (N(S_{n+p-1}) - N(S_{n-1})) \right| \end{aligned}$$

Again by Lipschitz condition

$$\begin{aligned} \|S_{n+p} - S_n\| &\leq L_1 \|J^\alpha (S_{n+p-1} - S_{n-1})\| \\ &\quad + L_2 \|J^\alpha (S_{n+p-1} - S_{n-1})\| \\ \|S_{n+p} - S_n\| &\leq (L_1 + L_2) \| (S_{n+p-1} - S_{n-1}) \| \frac{t^\alpha}{\Gamma(\alpha + 1)} \end{aligned}$$

Therefore

$$\|S_{n+p} - S_n\| \leq k \| (S_{n+p-1} - S_{n-1}) \| \text{ where } k = \frac{(L_1 + L_2)t^\alpha}{\Gamma(\alpha + 1)}$$

Similarly we have

$$\|S_{n+p-1} - S_{n-1}\| \leq k \| (S_{n+p-2} - S_{n-2}) \|$$

So

$$\begin{aligned} \|S_{n+p} - S_n\| &\leq k^2 \| (S_{n+p-2} - S_{n-2}) \| \\ \|S_{n+p} - S_n\| &\leq k^n \| (S_p - S_0) \| \end{aligned}$$

Now for $n > m$, we have

$$\begin{aligned} \|S_n - S_m\| &\leq \| (S_{m+1} - S_m) \| + \dots + \| (S_n - S_{n-1}) \| \\ \|S_n - S_m\| &\leq \frac{k^m \|u_1\|}{1 - k}. \end{aligned}$$

Since $u(x,t)$ is bounded. So as $n \rightarrow \infty, \|S_n - S_m\| \rightarrow 0$. Hence S_n is a Cauchy sequence in X. Therefore the series is convergent.

In the next section, we illustrate some examples and their solutions are represented graphically by mathematica software.

5. Applications

FADM for Time Fractional Nonlinear Kortweg- deVries (KdV)Equations

Consider the following time fractional nonlinear Kortweg- deVries (KdV) equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + au(x,t) \frac{\partial u(x,t)}{\partial x} + b \frac{\partial^3 u(x,t)}{\partial x^3} = 0, u(x,0) = f(x).$$

(5.1)

Where a, b are constants. In the operator form , KdV equation becomes

$$L_t^\alpha u(x,t) = -bu_{xxx} - auu_x \tag{5.2}$$

Therefore, by FADM from equation (3.9) we can write

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) - bJ^\alpha \left[\sum_{n=0}^{\infty} u_n(x,t) \right]_{xxx} - aJ^\alpha \left[\sum_{n=0}^{\infty} A_n \right]$$

The typical approach of Adomian Decomposition Method is the introduction of the recursive relation

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ u_{k+1}(x,t) &= -bJ^\alpha \left[u_k(x,t) \right]_{xxx} - aJ^\alpha \left[A_k \right], k \geq 0. \end{aligned} \tag{5.3}$$

The remaining components $u_n, n \geq 1$ is successively determined and the series solution is obtained.

Example 5.1. Consider the following time fractional partial differential equation

$$\begin{aligned} u_t^\alpha - 6uu_x + u_{xxx} &= 0, 0 < x < \pi, 0 < \alpha \leq 1, t > 0 \\ u(x,0) &= 6x \end{aligned}$$

Therefore , by FADM we can write

$$\sum_{n=0}^{\infty} u_n(x,t) = 6x - J^\alpha \left[\sum_{n=0}^{\infty} u_n(x,t) \right]_{xxx} + 6J^\alpha \left[\sum_{n=0}^{\infty} A_n \right]$$

By using ADM, we have following recursive relation

$$\begin{aligned} u_0(x,t) &= 6x \\ u_{k+1}(x,t) &= -J^\alpha \left[u_k(x,t) \right]_{xxx} + 6J^\alpha \left[A_k \right], k \geq 0. \end{aligned}$$

Using equation(5.3) , we have

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ &= 6x \\ u_1(x,t) &= -J^\alpha \left[u_0(x,t) \right]_{xxx} + 6J^\alpha A_0, \\ A_0 &= u_0 D_x u_0, \\ &= 6x D_x 6x, \\ &= 6^2 x, \\ u_1(x,t) &= 6J^\alpha 6^2 x \\ u_1(x,t) &= 6^3 x \frac{t^\alpha}{\Gamma(\alpha + 1)}, \end{aligned}$$



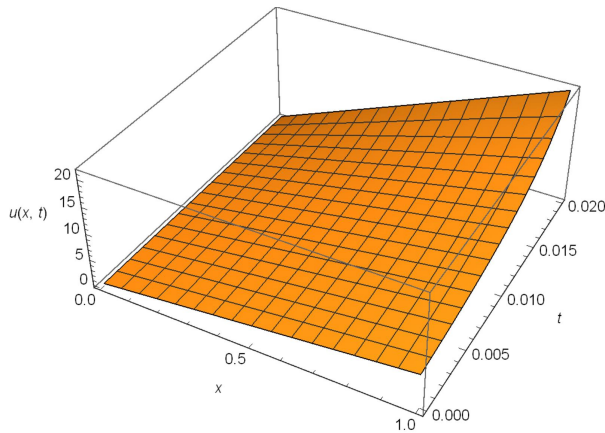


Figure 1. The exact solution of Ex.5.1

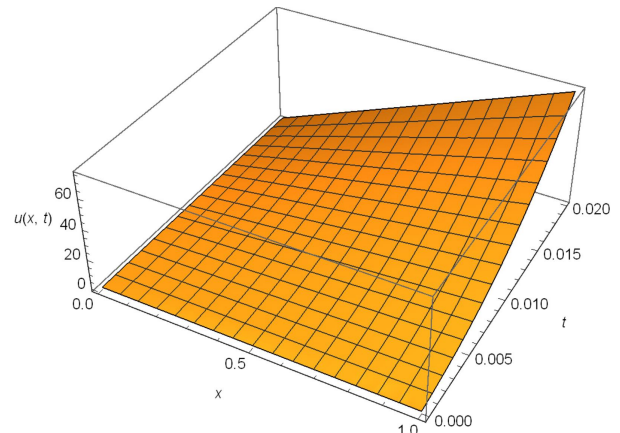


Figure 2. Apprx. soln. for $\alpha = 0.8$ of Ex.5.1

$$u_2(x,t) = -J^\alpha \left[u_1(x,t) \right]_{xxx} + 6J^\alpha A_1,$$

$$A_1 = u_1 D_x u_0 + u_0 D_x u_1,$$

$$= 6^3 x \frac{t^\alpha}{\Gamma(\alpha+1)} 6 + 6x6^3 \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$= 26^4 x \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = 2x6^5 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = -J^\alpha \left[u_2(x,t) \right]_{xxx} + 6J^\alpha A_2$$

$$A_2 = u_2 D_x u_0 + u_1 D_x u_1 + u_0 D_x u_2$$

$$A_2 = 4(6^5)x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 6^6 x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}$$

$$u_3(x,t) = 4(6^7)x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (6^7)x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}$$

Therefore, the series solution for the IBVP is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$

Substituting values of components in above equation, we get

$$u(x,t) = 6^3 x \frac{t^\alpha}{\Gamma(\alpha+1)} + 2x6^5 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 4(6^7)x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$+ (6^7)x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + \dots$$

This solution can be written as follows For $\alpha = 1$, the exact solution of the original IBVP is given by

$$u(x,t) = \frac{6x}{1-36t}, \quad |36t| < 1.$$

The graphical representation of the solution is given in Figure 1 and 2.

Example 5.2. Consider the following time fractional partial differential equation

$$u_t^\alpha - 6uu_x + u_{xxx} = 0, \quad 0 < x < \pi, \quad 0 < \alpha \leq 1, t > 0$$

$$u(x,0) = \frac{1}{6}(x-1).$$

Using equation(5.3) , we have

$$u_0(x,t) = u(x,0)$$

$$= \frac{1}{6}(x-1),$$

$$u_1(x,t) = -J^\alpha \left[u_0(x,t) \right]_{xxx} + 6J^\alpha A_0,$$

$$A_0 = u_0 D_x u_0$$

$$= \frac{1}{6}(x-1) D_x \frac{1}{6}(x-1),$$

$$= \frac{1}{6^2}(x-1),$$

$$u_1(x,t) = 6J^\alpha \frac{1}{6^2}(x-1),$$

$$u_1(x,t) = \frac{1}{6}(x-1) \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = -J^\alpha \left[u_1(x,t) \right]_{xxx} + 6J^\alpha A_1$$

$$A_1 = u_1 D_x u_0 + u_0 D_x u_1$$

$$A_1 = \frac{t^\alpha}{\Gamma(\alpha+1)} \frac{1}{6}(x-1) \frac{1}{6} + \frac{1}{6}(x-1) \frac{1}{6} \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$= \frac{2}{6^2}(x-1) \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = \frac{2}{6}(x-1) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$



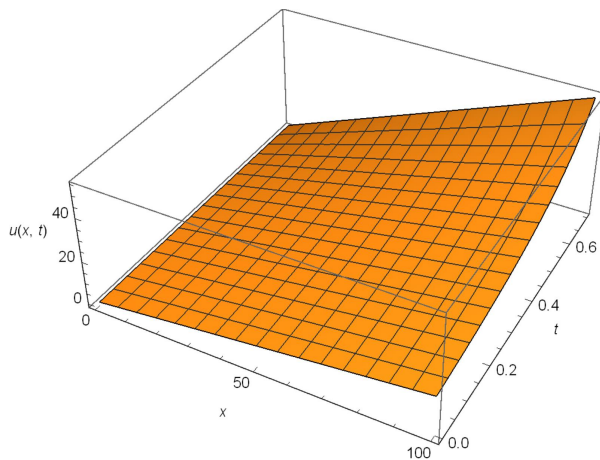


Figure 3. The exact solution of Ex.5.2

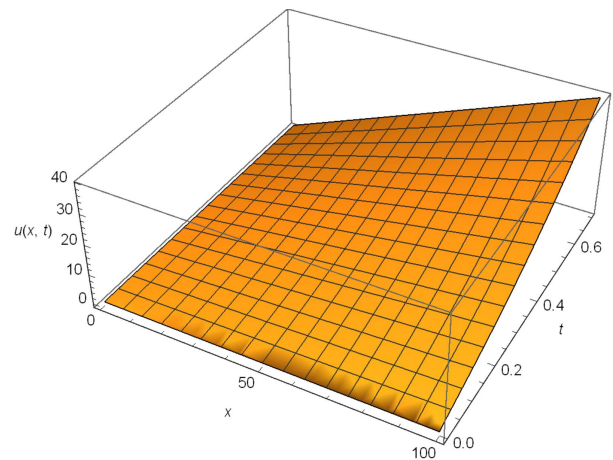


Figure 4. Apprx. soln. for $\alpha = 0.8$ of Ex.5.2

$$u_3(x,t) = -J^\alpha \left[u_2(x,t) \right]_{xxx} + 6J^\alpha A_2$$

$$A_2 = u_2 D_x u_0 + u_1 D_x u_1 + u_0 D_x u_2$$

$$A_2 = \frac{4}{6^2}(x-1) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(\alpha+1)} \frac{1}{6^2}(x-1)$$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)} \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_3(x,t) = \frac{4(x-1)}{6} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \frac{(x-1)}{6}$$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)} \frac{t^\alpha}{\Gamma(\alpha+1)^2}$$

Therefore, the series solution for the IBVP is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$

Substituting values of components in above equation, we get

$$u(x,t) = \frac{1}{6}(x-1) \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2}{6}(x-1) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+ \frac{4(x-1)}{6} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots$$

This solution can be written as follows For $\alpha = 1$, the exact solution of the original IBVP of is given by

$$u(x,t) = \frac{x-1}{6(1-t)}, |t| < 1.$$

The graphical representation of the solution is given in Figure 3 and 4.

6. Conclusions:

In this paper ADM has been successfully applied to find the solution of time fractional partial differential equations. We have also developed the algorithm for convergence of ADM. The applicability and efficiency of ADM is illustrated by obtaining the solutions of several examples. It is worth mentioning that

the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods. Hence ADM is very powerful and efficient in finding solutions for wide class of nonlinear fractional partial differential equations.

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