Malaya Journal of Matematik, Vol. 7, No. 2, 345-352, 2019

https://doi.org/10.26637/MJM0702/0030

New results on distance degree sequences of graphs

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Abstract

The distance $d(u,v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The *distance degree sequence* (dds) of a vertex v in a graph G is a list of the number of vertices at distance 1, 2, ... , $e(v)$; in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the distance degree sequence of the vertex v_i in G where, d_{i_j} denotes the number of vertices at distance j from v_i. In this article we present results to find distance degree sequences of some of the derived graphs viz., the line graph, the sub-division graph, the total graph, the powers of a graph, the Mycieleskian of a graph etc.

Keywords

Distance Degree Sequence, Distance Degree Regular graph, Line graph, Sub-division graph, Power of a graph, Mycieleskian of a graph.

AMS Subject Classification

05C12, 05C38, 05C40, 05C76, 05C05.

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Article History: Received **12** January **2019**; Accepted **11** May **2019**

Contents

1. Introduction and Preliminaries

A *sequence* for a graph is an invariant consisting of a list of numbers rather than a single number or invariant or index pertaining to the graph. The major advantage of considering a sequence is the ease of handling the sequences to calculate as a single numerical invariant yet the weight or information carried by it is much higher compared to a single invariant. There are many sequences representing a graph available in literature viz., degree sequence, eccentric sequence, distance degree sequence, status sequence, path degree sequence, etc.

First sequence introduced onto a graph was its *degree sequence* [6]. The realizability of a degree sequence was addressed by Havel [11] and Hakimi [10] independently. The *eccentric sequences* were the first distance related sequences introduced for undirected graphs. Major contributions are due to Lesniak [19], Ostrand [21], Behzad and Simpson [1], Nandakumar [20]. Next distance based sequences were the *path degree sequences* and *distance degree sequences* studied by Randic [23]. Initially these sequences were defined to model a chemical structure by its molecular graph and study them to distinguish isomers. Kennedy and Quintas [18] examined how the distance degree sequences relate to embedding trees in lattice-graphs and other spaces. Such interplay has enriched the study of distance based sequences with the introduction of new ones. Bloom et al. [3], [2] continued the study of distance degree sequences by defining two classes of graphs namely *distance degree regular graphs (DDR)* and *distance degree injective (DDI) graphs*. Many researchers have contributed to the study of these graphs as they have peculiar properties and are extremes in terms of their structures. On one hand, DDI graphs are highly irregular, as in these graphs each vertex has a different distance degree sequence and on the other, DDR graphs are highly regular, as all the vertices have the same distance degree sequence. These two classes of graphs have numerous properties and are related to many graph parameters. Extensive research has been done in this area by Bloom et al.[3], [2], Kennedy et al. [18], Gargano et al. [7], Bussemaker et al. [5], Itagi Huilgol et al. [12], [16], [17], [15], [14], [13], Volf [25], Halberstam et al. [9], Martinez et al. [7], Quintas et al. [22], Slater [24], etc.

With such a rich history, the study of distance degree sequences for graphs has many persisting open problems. For many classes of graphs finding such sequences itself is challenging. In this paper we have found distance degree sequences of some of the derived graphs such as the line graph, the sub-division graph, the total graph, the powers of a graph, the Mycieleskian of a graph of a given graph with its distance degree sequence.

We will now define all the terms required. For all undefined terms we refer Buckley and Harary [4]. Let $G = (V, E)$ denote a graph with set of *vertices* V, whose cardinality is the order p and two element subsets of V, known as the *edges* forming E, whose cardinality is the size q of G.

Unless mentioned otherwise, in this article, by a *graph* we mean an undirected, finite graph without multiple edges and self-loops.

The *distance* $d(u,v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The *degree* of the vertex u is the number of vertices at distance one.

The sequence of numbers of vertices having $0,1,2,3,\ldots$ is called the *degree sequence*, which is the list of degrees of vertices of G arranged in non-decreasing order.

The *eccentricity* $e(v)$ of v is the distance of a farthest vertex from v.

The minimum of the eccentricities is the *radius*, $rad(G)$ and the maximum is the *diameter*, $diam(G)$ of G.

A graph is said to be *self centered* if all the vertices have the same eccentricity.

If $d(u,v) = e(u), (u \neq v)$, we say that v is an eccentric vertex of u.

The *eccentric sequence* of a connected graph G is a list of the eccentricities of its vertices arranged in non-decreasing order.

The *distance degree sequence* of a vertex is a generalization of its degree sequence. The distance degree sequence (dds) of a vertex v in a graph G is a list of the number of vertices at distance $1,2,\ldots,e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the distance degree sequence of a vertex v_i in G where d_{i_j} denotes the number of vertices at distance j from v_i. The $p-$tuple of distance degree sequences of the vertices of G with entries arranged in lexicographic order is the *distance degree sequence (DDS) of G*.

As an illustration consider the graph G as shown in Figure 1.

$DDS(G) = ((1,1,1,2),(1,2,2),(1,3,1),(1,2,1,1)^2)$.

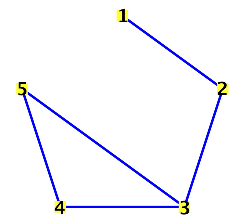

Figure 1. G

If we consider the 3-dimensional cube Q_3 as an example, we will get $DDS(Q_3) = (1,3,3,1)^8)$.

A graph G is said to be a *Distance degree regular (DDR) graph* if all the vertices of G have the same distance degree sequence. That is, for all vertices v of G, the distance degree sequence is $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$.

Note that a DDR graph has a single sequence with multiplicity p.

For example, the three dimensional cube Q_3 is a DDR graph having $(1,3,3,1)$ as the distance degree sequence of each of its vertex. Likewise, cycles, complete graphs are all DDR graphs.

In contrast to distance degree regular (DDR) graphs the *Distance Degree Injective (DDI) graphs* are the graphs with no two vertices having the same distance degree sequence (dds). These were also defined by Bloom et al. [3].

Note that the DDS of a DDI graph has multiplicity one for each of its sequence.

2. Powers of a graph

Here we give the distance degree sequences of powers of a graph and prove that powers of DDR graphs to be DDR.

Lemma 2.1. *In a graph G, for any vertex v, with distance degree sequence*
$dds_G(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{e(v)}(v))$,
the distance degree sequence of v in the k^{th} power of G, G^k, is given by $dds_{G^k}(v) =$
$(d_0(v), \sum_{i=1}^{k} d_i(v), \sum_{i=k+1}^{2k} d_i(v), \sum_{i=2k+1}^{3k} d_i(v), \ldots, \sum_{i=e(v)-(k-1)}^{e(v)} d_i(v))$.

Proof: Let G be a graph with a vertex v having distance degree sequence $(d_0(v), d_1(v), d_2(v), \ldots, d_{e(v)}(v))$. Let us denote the set of vertices at distance i from v as A_i, in G. So the distance degree sequence can be rewritten as follows: $dds_G(v) = (|A_0(v)|, |A_1(v)|, |A_2(v)|, \ldots, |A_p(v)|)$. In the k^{th} power of G, G^k, we know that all the vertices which are at distance less that or equal to k are adjacent to v. So, $A1_{G^k}(v) = \sum_{i=1}^{k} d_i(v)$. It is clear that the vertices which are at distance $k+1, k+2, \ldots, k+k = 2k$ in G, are at distance two from v in G^k. Hence $A2_{G^k}(v) = \sum_{i=k+1}^{2k} d_i(v)$. Similarly, we can get $A3_{G^k}(v) = \sum_{i=2k+1}^{3k} d_i(v)$. Continuing in this way, the last entry in the distance degree sequence of v in G^k is $\sum_{i=e(v)-(k-1)}^{e(v)} d_i(v) = \sum_{i=e(v)-(k-1)}^{e(v)} d_i(v)$. ■

Corollary 2.2. *Powers of distance degree regular graphs are distance degree regular.*

Remark 2.3. *If the eccentricity of a vertex v with distance degree sequence $dds_G(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{e(v)}(v))$ is m, then the eccentricity of v in G^k is $m-k+1$.*

3. Distance degree sequences of line graph of a graph

In this section, we consider line graphs of graphs and give their explicit distance degree sequences. First we consider line graph of a hypercube.

Theorem 3.1. *The line graph of the $n-$dimensional hypercube Q_n is a distance degree regular graph with distance degree sequence $(1, 2(n-1), {}^{(n-1)}C_1[2n-3], {}^{(n-1)}C_2[2n-5], \ldots, {}^{(n-1)}C_{k-1}[2n-2k+1], \ldots, 3(n-1), 1)$.*

Proof: Consider the hypercube graph of dimension n. Since Q_n is a distance transitive graph, its line graph is vertex transitive and hence is distance degree regular graph. The distance degree sequence is given as follows:
Let a and b be two adjacent vertices of Q_n then $(a-b)$ is a vertex in $L(Q_n)$. By distance transitivity of Q_n, we know that there are ${}^nC_d$ vertices at distance d from every vertex u of Q_n. Hence at each distance d in the line graph of Q_n, $L(Q_n)$, there are ${}^{(n-1)}C_{(d-1)}[(n-d+1)+(n-d)] = {}^{(n-1)}C_{(d-1)}[2n-2d+1]$ for each vertex $(a-b)$ in $L(Q_n)$. ■

Next we consider line graphs of trees. For this we use the definition of *edge degree ed(e)* as defined in [4].

Theorem 3.2. *Let T be a tree and let u and v be any two adjacent vertices in T with respective distance degree sequences $dds_T(u) = (d_0(u), d_1(u), d_2(u), \ldots, d_{ecc(u)}(u))$, $dds_T(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{ecc(v)}(v))$, where $ecc(u)$ and $ecc(v)$ represent the eccentricity of u and v respectively. Then the distance degree sequence of the vertex $e = uv$ in the line graph of T, $L(T)$, is $dds_{L(T)}(e) = (d'_0(e), d'_1(e), d'_2(e), \ldots, d'_{ecc(e)}(e))$, where $ecc(e)$ is the eccentricity of e in $L(T)$ is $min(ecc_T(u), ecc_T(v))$, with each entry given as $d'_0(e) = 1$, $d'_1(e) = d_1(u) + d_1(v) - 2$, $d'_2(e) = d_2(u) + d_2(v) - d'_1(e)$, $d'_3(e) = d_3(u) + d_3(v) - d'_2(e)$, $\ldots$ $d'_{k+1}(e) = d_{k+1}(u) + d_{k+1}(v) - d'_k(e)$, for $2 \le e \le ecc(e)$.*

Proof: Let T be a tree with two adjacent vertices u and v, with respective distance degree sequences given as
$dds_T(u) = (d_0(u), d_1(u), d_2(u), \ldots, d_{ecc(u)}(u))$,
$dds_T(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{ecc(v)}(v))$.
Here $ecc(u)$ and $ecc(v)$ are the eccentricities of u and v respectively. Then $e = uv$ is a vertex in the line graph of T, $L(T)$. Let the distance degree sequence of e in $L(T)$ be written as
$dds_{L(T)}(e) = (d'_0(e), d'_1(e), d'_2(e), \ldots, d'_{ecc(e)}(e))$. It is clear that the eccentricity of e in $L(T)$ is the minimum of the eccentricities of its end vertices, that is,
$ecc_{L(T)}(e) = min\{ecc_T(u), ecc_T(v)\}$. Here we find the values of each entry of the $dds_{L(T)}(e)$. Since, T is simple, undirected, we get $d'_0(e) = 1$. In the line graph of T, $e = uv$ has $deg_{L(T)}(e) = ed(e) = deg_T(u) + deg_T(v) - 2$, as the edge is incident with neighbours of both u and v. At distance two from e in $L(T)$ are the edges incident with the neighbours of e, that is, $d'_2(e)$ is the distinct number of second neighbors of u and v. Hence, $d'_2(e)$ in $L(T) = d_2(u) + d_2(v) - ed(e)$ as the number of second neighbours of v are neighbors of u and vice-versa. But $d_1(u) + d_1(v) - 2 = d'_1(e)$. Hence $d'_2(e) = d_2(u) + d_2(v) - d'_1(e)$. Now for any k, $2 \le k \le ecc_{L(T)}(e) - 1$ we can write $d'_{k+1}(e) = d_{k+1}(u) + d_{k+1}(v) - d'_k(e)$, as the $(k+1)^{th}$ neighbors of e in $L(T)$ are $(k+1)^{th}$ distinct neighbors of both u and v. ■

Corollary 3.3. *For two trees G and H having the same distance degree sequence, and $L(G)$ and $L(H)$ having the same distance degree sequence, then there exists a one-to-one mapping corresponding to the distance degree sequences of adjacent vertices of G and H.*

Proof:
Let G and H be two trees with the same distance degree sequence. That is, $DDS(G) = DDS(H)$. Let $e = uv$ be an edge in G with its end vertices having
$dds_G(u) = (d_0(u), d_1(u), d_2(u), \ldots, d_{ecc(u)}(u))$, and
$dds_G(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{ecc(v)}(v))$, where $ecc(u)$ and $ecc(v)$ represent the eccentricity of u and v respectively. Then by by the Theorem 3.2 the $dds_{L(G)}(e)$ is given by
$(d'_0(e), d'_1(e), d'_2(e), \ldots, d'_{ecc(e)}(e))$, where $ecc(e)$ is the eccentricity of e in $L(G)$ is $min(ecc_G(u), ecc_G(v))$, with each entry given as $d'_0(e) = 1$, $d'_1(e) = d_1(u) + d_1(v) - 2$, $d'_2(e) = d_2(u) + d_2(v) - d'_1(e)$, $d'_3(e) = d_3(u) + d_3(v) - d'_2(e)$, $\ldots$ $d'_{k+1}(e) = d_{k+1}(u) + d_{k+1}(v) - d'_k(e)$, for $2 \le e \le ecc_{L(G)}(e)$
where $ecc_{L(G)}(e) = min\{ecc_G(u), ecc_G(v)\}$.
Let $f = st$ be an edge in H with respective distance degree sequences $dds_H(s) = (d_0(s), d_1(s), d_2(s), \ldots, d_{ecc(s)}(s))$, $dds_H(t) = (d_0(t), d_1(t), d_2(t), \ldots, d_{ecc(t)}(t))$, where $ecc(s)$ and $ecc(t)$ represent the eccentricity of s and t respectively. Similarly, the $dds_{L(H)}(f)$ is given by
$(d'_0(f), d'_1(f), d'_2(f), \ldots, d'_{ecc(f)}(f))$, where $ecc(f)$ is the eccentricity of e in $L(H)$ is $min(ecc_H(s), ecc_H(t))$, with each entry given as
$d'_0(f) = 1$, $d'_1(f) = d_1(s) + d_1(t) - 2$, $d'_2(f) = d_2(s) + d_2(t) - d'_1(f)$, $d'_3(f) = d_3(s) + d_3(t) - d'_2(f)$, $\ldots$ $d'_{k+1}(f) = d_{k+1}(s) + d_{k+1}(t) - d'_k(f)$, for $2 \le e \le ecc_{L(H)}(f)$ where $ecc_{L(H)}(f) = min\{ecc_H(s), ecc_H(t)\}$.

By hypothesis we know that $L(G)$ and $L(H)$ have the same distance degree sequence. Hence for the edge e in G, there will be an edge in H, say f, such that $dds_{L(G)}(e) =$

$dds_{L(H)}(f)$. As $DDS(G) = DDS(H)$, for every vertex u in G, there exists a vertex in H having the same distance degree sequence as of u. We pick this vertex to be s in H, that is $dds_G(u) = dds_H(s)$. Since $dds_{L(G)}(e) = dds_{L(H)}(f)$ we get
$(d_0^{'}(e), d_1^{'}(e), d_2^{'}(e), \ldots, d_{ecc(e)}^{'}(e)) =$
$(d_0^{'}(f), d_1^{'}(f), d_2^{'}(f), \ldots, d_{ecc(f)}^{'}(f))$.
We have $d_i(u) = d_i(s)$ for $1 \le i \le d_{ecc(s)}(s) = d_{ecc(u)}(u)$ as $dds_G(u) = dds_H(s)$ by choice. Hence $d_0^{'}(e) = d_0^{'}(f) = 1$ $d_1^{'}(e) = d_1^{'}(f)$ then $d_1(u) + d_1(v) - 2 = d_1(s) + d_1(t) - 2$.
But $d_i(u) = d_i(s)$, for $1 \le i \le d_{ecc(s)}(s) = d_{ecc(u)}(u)$ as $dds_G(u) = dds_H(s)$ implying that $d_1(v) = d_1(t)$.
Similarly $d_2^{'}(e) = d_2^{'}(f)$ implies
$d_2(u) + d_2(v) - d_1^{'}(e) = d_2(s) + d_2(t) - d_1^{'}(f)$
implies $d_2(v) = d_2(t)$.Therefore $d_k^{'}(e) = d_k^{'}(f)$
implying that
$d_k(u) + d_{k+1}(v) - d_{k-1}^{'}(e) = d_k(s) + d_k(t) - d_{k-1}^{'}(f)$
Hence, $d_k(v) = d_k(t)$ for any
$1 \le k \le d_{ecc(v)}(v) = d_{ecc(t)}(t)$ implying that
$dds_G(v) = dds_H(t)$. Hence Proved. ■

Illustration: Now we consider two trees T_1 and T_2 having the same distance degree sequences given below

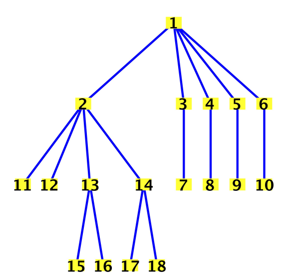

Figure 2. T_1

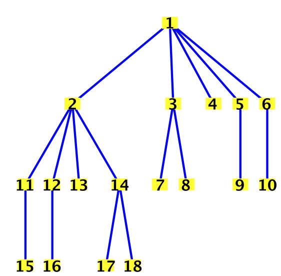

Figure 3. T_2

The distance degree sequences of the vertices of T_1 are a given below.
$dds(1) = (1,5,8,4)$, $dds(2) = (1,5,8,4)$,
$dds(3) = (1,2,4,7,4)$, $dds(4) = (1,2,4,7,4)$,
$dds(5) = (1,2,4,7,4)$, $dds(6) = (1,2,4,7,4)$,
$dds(7) = (1,1,1,4,7,4)$, $dds(8) = (1,1,1,4,7,4)$,
$dds(9) = (1,1,1,4,7,4)$, $dds(10) = (1,1,1,4,7,4)$,
$dds(11) = (1,1,4,8,4)$, $dds(12) = (1,1,4,8,4)$,
$dds(13) = (1,3,4,6,4)$, $dds(14) = (1,3,4,6,4)$,
$dds(15) = (1,1,2,4,6,4)$,$dds(16) = (1,1,2,4,6,4)$,
$dds(17) = (1,1,2,4,6,4)$, $dds(18) = (1,1,2,4,6,4)$.

Note that the edge set of T_1 is $\{(1,2),(1,3),(1,4),$ $(1,5),(1,6),(2,11),(2,12),(2,13),(2,14),(3,7),(4,8),$ $(5,9),(6,10),(13,15),(13,16),(14,17),(14,18)\}$
Hence the respective distance degree sequences of vertices forming edges can be considered as adjacent sequences.

It is clear that T_2 is not isomorphic to T_1, but has the same distance degree sequence as that of T_1. Also note that $L(T_1)$ and $L(T_2)$ have the same distance degree sequence. Now with the labels of vertices we can see that the edge set of T_2 is not equal to that of T_1, but the end vertices of each edge have the same pair of distance degree sequences as in the case of edges of T_1.

Note: If adjacent vertex distance degree sequences are defined as the edge distance degree sequences (edds) then the above corollary can be rephrased as follows.

Corollary 3.4. *Two trees G and H with the same DDS and LDDS have the same EDDS.*

Next we consider a result to find the distance degree sequences of a line graph of a graph. In this result we use some of the notation as below: Let $N_i(v)$ denote the i^{th} neighborhood of a vertex v in G. This can be viewed as a spanning tree rooted at the vertex v. Let $E_i(u)$ and $E_i(v)$ denote the edge set at distance i in the spanning tree rooted at u and v respectively. Let $E\langle A_i\rangle$ and $E\langle B_i\rangle$ be the edges in vertex induced subgraph containing the vertices at distance i from the vertices u and v respectively. The edges induced in G between $(i-1)^{th}$ neighborhood vertices and i^{th} neighborhood vertices without the respective spanning tree edges rooted at that vertex is given by $E_{i-1,i} = E_{i-1,i}(u) \cup E_{i-1,i}(v)$.

Theorem 3.5. *Let G be simple connected graph, let u and v be any two adjacent vertices in G with respective distance degree sequences*
$dds_G(u) = (d_0(u), d_1(u), d_2(u), \ldots, d_{ecc(u)}(u))$,
$dds_G(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{ecc(v)}(v))$,
where $ecc(u)$ and $ecc(v)$ represent the eccentricity of u and v respectively. Then the distance degree sequence of the vertex $e = uv$ in the line graph of G, $L(G)$ is $dds_{L(G)}(e) = (d_0^{''}(e), d_1^{''}(e), d_2^{''}(e), \ldots, d_{ecc(e)}^{''}(e))$, where $ecc(e)$ is the eccentricity of e in $L(G)$, and $d_0^{''}(e) = 1$, $d_1^{''}(e) = |\{E_1(u) \cup E_1(v)\} \setminus e|$,

$$d_2^{''}(e) = \left| \begin{array}{c} \{\{E_2(u) \cup E_2(v)\} \cup \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \cup E_{1,2}\} \\ \setminus \{E_1(u) \cup E_1(v)\} \setminus e \end{array} \right|,$$

$$d_3^{''}(e) = \left| \begin{array}{c} \{\{E_3(u) \cup E_3(v)\} \cup \{E\langle A_2\rangle \cup E\langle B_2\rangle \cup E_{2,3}\} \\ \setminus \{E_2(u) \cup E_2(v)\} \setminus \{E_1(u) \cup E_1(v)\} \setminus \\ \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \setminus E_{1,2} \setminus e \end{array} \right|,$$

$$\vdots$$

$$d_{k+1}^{''}(e) = \left| \begin{array}{c} \{\{E_{k+1}(u) \cup E_{k+1}(v)\} \cup E\langle A_k\rangle \cup E\langle B_k\rangle \cup E_{(k-1),k}\} \\ \setminus \cup_{i=1}^{k-1} \{E_i(u) \cup E_i(v)\} \setminus \\ \cup_{i=1}^{k-1} \{E\langle A_i\rangle \cup E\langle B_i\rangle \setminus \cup_{i=2}^{k-1} E_{(i-1),i}\} \setminus e \end{array} \right|,$$

for $2 \le k \le ecc(e)$.

Proof: Let G be any simple connected graph with two adjacent vertices u and v, with respective distance degree sequences given as in the statement of the theorem. Then

$e = uv$ is a vertex in the line graph $L(G)$ of G. We know that distance degree sequence of any vertex can be obtained from a spanning tree rooted at that vertex. If G is a tree then the same degree sequence holds good, otherwise the only changes will be in the respective k^{th} neighborhood edges and the edges induced in G between $(k-1)^{th}$ neighborhood vertices and k^{th} neighborhood vertices denoted by $E_{k-1,k}$, which can be given as follows: $dds_{L(G)}(e) = (d_0''(e), d_1''(e), d_2''(e), \ldots, d_{ecc(e)}''(e))$. Each entry is explicitly given as

$d_0''(e) = 1, d_1''(e) = |\{\{E_1(u) \cup E_1(v)\} \setminus e\}|$

$$d_2''(e) = \left| \begin{array}{c} \{\{\{E_2(u) \cup E_2(v)\} \cup \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \cup E_{1,2}\} \\ \setminus \{E_1(u) \cup E_1(v)\} \setminus e\} \end{array} \right|,$$

$$d_3''(e) = \left| \begin{array}{c} \{\{E_3(u) \cup E_3(v)\} \cup \{E\langle A_2\rangle \cup E\langle B_2\rangle \cup E_{2,3}\} \\ \setminus \{E_2(u) \cup E_2(v)\} \setminus \{E_1(u) \cup E_1(v)\} \setminus \\ \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \setminus E_{1,2} \setminus e \end{array} \right|,$$

$\vdots$

$d_{k+1}''(e) =$

$$\left| \begin{array}{c} \{\{E_{k+1}(u) \cup E_{k+1}(v)\} \cup E\langle A_k\rangle \cup E\langle B_k\rangle \cup E_{(k-1),k}\} \\ \setminus \cup_{i=1}^{k-1} \{E_i(u) \cup E_i(v)\} \setminus \\ \cup_{i=1}^{k-1} \{E\langle A_i\rangle \cup E\langle B_i\rangle \setminus \cup_{i=2}^{k-1} E_{(i-1),i}\} \setminus e \end{array} \right|,$$

for $2 \leq k \leq ecc(e)$. ■

We will illustrate the above with the following example.

Example: Consider G to be the Petersen graph.

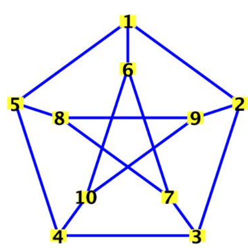

Figure 4. The Petersen Graph (G)

Consider the edge $(2,9)$ in G then $(2,9)$ is a vertex in $L(G)$. Pertesen Graph is a distance degree regular graph with distance degree sequence $(1,3,6)$.

Hence, $dds_G(2) = (1,3,6) = dds_G(9)$. Consider the spanning trees rooted at 2 and 9 in G.

$dds_{L(G)}(e) = (d_0''(e), d_1''(e), d_2''(e), \ldots, d_{ecc(e)}''(e))$.

$d_0''(e) = d_0'(e) = 1,$

$d_1''(e) = |\{E_1(u) \cup E_1(v)\} \setminus e|$

$d_1''(e) = |\{\{1,2\},\{2,3\},\{9,8\},\{9,10\}\}| = 4$

$$d_2''(e) = \left| \begin{array}{c} \{\{E_2(u) \cup E_2(v)\} \cup \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \cup E_{1,2}\} \\ \setminus \{E_1(u) \cup E_1(v)\} \setminus e \end{array} \right|$$

$d_2''(e) =$
$|\{\{5,8\},\{8,7\},\{4,10\},\{10,6\},\{1,5\},\{1,6\},\{4,3\},\{3,7\}\}|$

$d_2''(e) = 8$

$$d_3''(e) = \left| \begin{array}{c} \{\{E_3(u) \cup E_3(v)\} \cup \{E\langle A_2\rangle \cup E\langle B_2\rangle\} \cup E_{2,3}\} \\ \setminus \{E_2(u) \cup E_2(v)\} \setminus \{E_1(u) \cup E_1(v)\} \\ \setminus \{E\langle A_1\rangle \cup E\langle B_1\rangle\} \setminus E_{1,2} \\ \setminus e \end{array} \right|$$

$d_3''(e) = |\{\{4,5\},\{6,7\}\}|$

$d_3''(e) = 2$

Thus we get $dds_{L(G)}(e) = (1,4,8,2)$. Applying this procedure on each edge of G we can see that $L(G)$ is a distance degree regular graph with $DDS = (1,4,8,2)^{10}$.

Corollary 3.6. *If G and H are any two graphs with $DDS(L(G)) \neq DDS(L(H))$ then the one-to-one correspondence between adjacent vertex distance degree sequences cannot be established.*

Proof Let $e = uv \in E(G)$ and $f = st \in E(H)$ such that $dds_{L(G)}(e) \neq dds_{L(H)}(f)$. Using Theorem 3.2, Theorem 3.5 the proof follows. ■

4. Distance degree sequences of subdivision and total graphs

In this section we consider one more derived graph viz., the subdivision graph. Note that a subdivision graph is obtained by introducing a degree two vertex on an edge of a graph. Here we give distance degree sequence of a subdivision graph with each edge being subdivided r- times. We start with subdivision of a tree.

Let $H(T,r)$ be a subdivision of a tree T with each edge subdivided by r vertices.

Theorem 4.1. *The distance matrix of $H(T,r)$ the $r-$ subdivided tree is given by*

$$D(H(T,r)) = \begin{bmatrix} [A]_{p\times p} & 0 \\ 0 & [B]_{r(p-1)\times r(p-1)} \end{bmatrix}_{((r+1)p-r)\times((r+1)p-r)} \tag{4.1}$$

Proof: Here we give the distance degree sequences of each vertex the tree and the subdivided vertices.

Note that for the tree vertices the set of eccentric vertices remain the same. On subdivision of edges r- times we see that the distance between the tree vertices gets multiplied by $r+1$. So the $ecc_{H(T,r)}(u) = (r+1)ecc_T(u)$, for each $u \in V(T)$.

So in $H(T,r)$ we see that the $deg(u)$ remains the same and the tree vertices that were neighbors of u(*in* T) are at distance $r+1$ in $H(T,r)$ with the distance between u and the r new vertices being $\{1,2,3,4,\ldots,r\}$. Hence the number of vertices at each distance can be achieved upto $(r+1)ecc_T(u)$. Now for the new vertices introduced at each edge we will write down the distances. Or in otherwords we write down the distance matrix of $H(T,r)$ derived from that of T. Given the $D(T)$, $D(H(T,r))$ will have $p + r(p-1) = (r+1)p - r$ vertices, out of which we will retain the first p entries as the vertices of T. Hence in $D(H(T,r))$ each entry of the submatrix of order

$p \times p$ is just the multiple of $(r+1)$ of the entries of $D(T)$, as they correspond to the tree vertices. And we denote this block matrix as A. The next block matrix of order $r(p-1) \times r(p-1)$ has all new vertices introduced as subdivision vertices. We denote this by B.
Picking the $(i,j)^{th}$ entry in $D(T)$, if it is one, then in $D(H(T,r))$ the $(i,u)^{th}$ and $(i,v)^{th}$ entry will be one the same number of times. It is clear that no non-tree vertex is an eccentric vertex in $H(T,r)$, and hence the eccentricity of any vertex of $H(T,r)$ will be at most $(r+1)ecc_T(x)$. So the pendant vertices of T are the peripheral vertices of $H(T,r)$ also. For any vertex, say x, of the tree T, the distances from x to all vertices will be as follows:

$dds_{H(T,r)}(x) = (1,(j)^{r+1},(r+1+j)^{r+1},\ldots,((r+1)(i-1)+j)^{r+1},\ldots,((r+1)(ecc_T(x)-1)+j)^{r+1})$ for all $1 \le j \le r+1$ and $1 \le i \le ecc_T(x)$.

Below we enlist the distance degree sequences of the subdivision vertices $H(T,r)$.

Let the subdivision vertices between two tree vertices say v_i and v_j be
$\{u_{i,j,1},u_{i,j,2},u_{i,j,3},\ldots,u_{i,j,r}\}$.

Let $e = v_iv_j$ be the edge then we have from Theorem 3.2, $dds_{LT}(e) = (d_0(e),d_1(e),d_2(e),\ldots,d_{ecc(e)}(e))$, where $ecc(e)$ is the eccentricity of e in $L(T)$ that is $min(ecc_T(u),ecc_T(v))$. Here we consider different cases:

Case 1: Let the number of subdivisions r be odd.
Then the distance degree sequence of $u_{i,j,k}$ is given by
$dds(u_{i,j,k}) =$
$\{1,2^k,(d_1(u))^{r-2k+1},(d_1(e))^{2k},(d_2(u))^{r-2k+1},(d_2(e))^{2k},\ldots,$
$(d_{ecc(e)}(u))^{r-2k+1},(d_{ecc(e)}(e))^{2k}\}$ for $1 \le k \le \frac{r-1}{2}$.
$dds(u_{i,j,\frac{r+1}{2}}) =$
$\{1,2^{\frac{r+1}{2}},(d_1(e))^{r+1},(d_2(e))^{2k},(d_2(u))^{r+1},(d_3(e))^{r+1},\ldots,$
$(d_{ecc(e)}(e))^{r+1}\}$.
$dds(u_{i,j,k}) =$
$\{1,2^{r-k+1},(d_1(u))^{2k-r+1},(d_1(e))^{2(r-k+1)},(d_2(u))^{2k-r+1},$
$(d_2(e))^{2(r-k+1)},\ldots,(d_{ecc(e)}(u))^{2k-r+1},(d_{ecc(e)}(e))^{2(r-k+1)}\}$
for $\frac{r+3}{2} \le k \le r$.
Case 2: Let the number of subdivisions r be even.
Now the distance degree sequences are given as follows:
$dds(u_{i,j,k}) =$
$\{1,2^k,(d_1(u))^{r-2k+1},(d_1(e))^{2k},(d_2(u))^{r-2k+1},(d_2(e))^{2k},\ldots,$
$(d_{ecc(e)}(u))^{r-2k+1},(d_{ecc(e)}(e))^{2k}\}$ for $1 \le k \le \frac{r}{2}$.
$dds(u_{i,j,k}) =$
$\{1,2^{r-k+1},(d_1(u))^{2k-r+1},(d_1(e))^{2(r-k+1)},(d_2(u))^{2k-r+1},$
$(d_2(e))^{2(r-k+1)},\ldots,(d_{ecc(e)}(u))^{2k-r+1},(d_{ecc(e)}(e))^{2(r-k+1)}\}$
for $\frac{r+2}{2} \le k \le r$.
By putting all the distances of the tree vertices and the subdivision vertices together, we get the distance matrix of the $H(T,r)$ as a block matrix given in the statement of the theorem. ■

Remark 4.2. *The total graph $T(G)$ of a graph G can be obtained by subdividing the graph G once at all edges and then taking its second power. Referring to Theorem 4.1 and then applying the Lemma 2.1, we can get the distance degree sequence of total graph of a graph.*

Corollary 4.3. *If two trees A and B have the same distance degree sequence then the total graphs of A and B also have the same distance degree sequence.*

Proof: We have $T(A)$ as the second power graph of $H(A,2)$ and $T(B)$ as the second power graph of $H(B,2)$. Then by Theorem 4.1, Lemma 2.1 we can observe that the $DDS(T(A)) = DDS(T(B))$. ■

5. Distance degree sequence of Mycieleski graph

Here we consider Mycieleski graph of a graph G and give the distance degree sequence of it. For ready reference we give the definition of the Mycieleski of a graph here.
Mycielski graph: Let G be a graph with n vertices. Let these n vertices be labeled as $\{u_1,u_2,\ldots,u_n\}$. The Mycielski graph $\mu(G)$ contains G itself as a subgraph, together with $n+1$ additional vertices: a vertex v_i corresponding to each vertex u_i of G, and an extra vertex w. Each vertex v_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge u_iu_j of G, the Mycielski graph includes two edges, v_iu_j and u_iv_j. Thus, if G has n vertices and m edges, then $\mu(G)$ has $2n+1$ vertices and $3m+n$ edges.

Theorem 5.1. *Let G be any connected graph with $V(G) = \{u_1,u_2,\ldots,u_n\}$. Then the Mycieleski of G, $\mu(G)$, has its distance degree sequence as follows:*
$dds_{(\mu(G))}(u_i) = (d_0,2d_1,2d_2+2,2d_3+d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)},d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)}))$;
$dds_{\mu(G)}(v_i) = (d_0,d_1+1,n+d_2,n-1-d_1-d_2)$;
$dds_{\mu(G)}(w) = (d_0,n,n)$.

Proof: Let G be a connected graph with $V(G) = \{u_1,u_2,\ldots,u_n\}$. Then by the structure of the Mycielski graph $\mu(G)$ contains a vertex v_i corresponding to each vertex u_i of G, and an extra vertex w. The edge set consists of one copy of edges of G and each vertex v_i is connected by an edge to w, and for each edge u_iu_j of G, two new edges, v_iu_j and u_iv_j. Here we consider different cases to prove the result.

From the construction of Mycielski graph $\mu(G)$ we have three kinds vertices u_is as in G, v_is and w. For each kind we give the distance degree sequence as follows:
1) For every vertex u_i, note that $ecc(u_i) = 4$.
We now write for each entry:
The number of vertices at distance one from u_i is $2(d_1)$ by the structure of $\mu(G)$.
The number of vertices at distance two are; the vertex w and the d_2 vertices u_js together with d_2+1 vertices v_i, v_js. Hence, $2d_2+2$ vertices are at distance two from u_i in $\mu(G)$.
The number of vertices at distance 3 will be the number of vertices at distance 3 in G of the form u_js, the same number contributed in the form v_js, from the copy and remaining

vertices at distance $\{4,5,6,\ldots,eccu_i\}$ of vertices in the copied version of the form v_js (will be at distance 3 through w). Hence, $2d_3+d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)}$ number of vertices are at distance 3 in $\mu(G)$.
All the remaining vertices at distance $\{4,5,6,\ldots,eccu_i\}$ in original graph are at distance 4 in $\mu(G)$. Hence, $d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)}$ vertices are at distance 4.
That is, the $dds_{(\mu(G))}(u_i) = (d_0, 2d_1, 2d_2+2, 2d_3+d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)}, d_4+d_5+d_6+\ldots+d_{ecc_G(u_i)}))$.
2) For every vertex v_i, note that $ecc(v_i)=3$.
We now write for each entry:
It is easy to see that the number of vertices at distance 1 is d_1+1.
The number of vertices at distance 2 is $n+d_2$ as d_2+1 of G of the form u_js are at distance 2 and from the construction of Mycielski graph $n-1$ vertices of the copy of the form v_js, through the vertex w.
Rest of the vertices are at distance 3, hence $n-1-d_1-d_2$.
3) For the vertex w the $dds_{(\mu(G))}(w) = (d_0,n,n)$, trivially from the construction of Mycielski graph.
■

Remark 5.2. *Applying the above theorem iteratively we can get distance degree sequences of any iterated Mycielskian of a graph G.*

Corollary 5.3. *If G and H are two graphs having the same distance degree sequence then $\mu(G)$ and $\mu(H)$ also have the same distance degree sequence.*

6. Bipartite Graphs

Theorem 6.1. *There exists a bipartite DDR graph G with each vertex having $dds = (1,k,k,\ldots,k,1)$ where k occurs n times with n odd only if $k=2$.*

Proof: Let G be a bipartite DDR graph with the DDS as given in the hypothesis with n odd, $n \geq 3$. From the sequence it is clear that whenever G is bipartite then the alternate neighbors belong to one partite set and the rest to the other. This implies that one partite set say, V_1 has $(i+1)k$ vertices and the second V_2 has $(ik+2)$ vertices. Each vertex of V_1 has degree k implies degree sum of $V_1 = k(i+1)k = k^2i+k^2$. All neighbors of V_1 are in V_2 and there are $ik+2$ number of vertices in V_2 with each vertex having degree k. Hence, $k(ik+2) = k^2i+k^2$ implying $ik^2+2k = ik^2+k^2$. Therefore, $k=2$. ■

Theorem 6.2. *A DDR graph G with its $DDS(G) = (1,k,k,\ldots,k,1)^p$ is bipartite if and only if G is an even cycle.*

Proof: From Theorem 6.1 the above result follows trivially. ■

7. Conclusion

In this paper we have tried giving the distance degree sequences of some of the derived graphs of graphs. But finding the distance degree sequences of non-tree graphs seems difficult at this point of time. Also the following problem is unclear and hence we pose it as an open one.

Problem 1: Let G and H be two non-tree graphs having the same distance degree sequence then is it possible for $L(G)$ and $L(H)$ to have the same distance degree sequence.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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