Malaya Journal of Matematik, Vol. 7, No. 3, 366-370, 2019
https://doi.org/10.26637/MJM0703/0001

Oscillation of second order nonlinear difference equations with super-linear neutral term

Govindasamy Ayyappan[1]* and Gunasekaran Nithyakala[2]

Abstract
In this paper, the authors establish some new conditions for the oscillation of second order nonlinear difference equation of the form

$$\Delta\left(a_n\Delta\left(y_n+p_ny^{\alpha}_{n-k}\right)\right)+q_ny^{\beta}_{n+1-\ell}=0,$$

where $\alpha>1$ and β are ratio of odd positive integers. Examples are provided to illustrate the importance of the main results.

Keywords
Superlinear neutral term, nonlinear difference equation, oscillation.

AMS Subject Classification
39A11, 39A21

[1] *Department of Mathematics, Periyar University College of Arts and Science, Pappireddipatti -636905, Tamilnadu, India.*
[2] *Department of Mathematics, Periyar University, Salem-636011, Tamilnadu, India.*
***Corresponding author**: [1] ayyapmath@gmail.com; [2]nithya.ramani.82@gmail.com
Article History: Received **6** April **2019**; Accepted **29** May **2019**

Contents

1. Introduction

In this paper, we study the oscillatory behavior of all solutions of nonlinear second order difference equations with a superlinear neutral term of the form

$$\Delta\left(a_n\Delta\left(y_n+p_ny^{\alpha}_{n-k}\right)\right)+q_ny^{\beta}_{n+1-\ell}=0,\quad n\geq n_0 \tag{1.1}$$

where n_0 is a positive integer, subject to the following conditions:

(H_1) $\alpha>1$ and $\beta>0$ are quotient of odd positive integers;

(H_2) $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ for $n\geq n_0$ are positive real sequences such that $\lim_{n\to\infty}p_n=0$, and $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$;

(H_3) $k\in\mathbb{N}$ and $\ell\in\mathbb{N}_0$.

Let $\eta=\max\{k,\ \ell-1\}$. By a solution of equation (1.1), we mean a real sequence $\{y_n\}$ defined for $n\geq n_0-\eta$, that satisfies equation (1.1) for all $n\geq n_0$. A nontrivial solution of equation (1.1). A nontrivial solution of equation (1.1) is said to be oscillatory if its terms are neither eventually positive nor eventually negative, and otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been a lot of interest in studying the oscillatory and asymptotic behavior of solutions of various classes of second order nonlinear difference equations, see [1, 2, 10, 12] and the references contained therein. In [4, 6, 15–18], the authors considered equation of the form (1.1) and obtained criteria for the oscillation of (1.1) under the conditions either

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty \tag{1.3}$$

for the case $0<\alpha\leq 1$. In [7, 9], the authors investigated the oscillatory behavior of equation (1.1) when $\alpha>1$, under the assumptions (1.2) and (1.3), respectively.

In view of the above observations our aim in this paper is to present some new sufficient conditions which ensure that all solutions of equation (1.1) when the condition (1.3) holds and our results are different from that of [7, 9]. For related results concerning second order differential equations with sublinear or superlinear neutral terms, we refer the reader to [3, 8, 13, 14].

2. Main Results

In the following for convenience we denote

$$z_n = y_n + p_n y_{n-k}^{\alpha},$$
$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s},$$

and

$$P_n = 1 - M^{\alpha-1} p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} \geq 0$$

for every constant $M > 0$ and $n \geq n_1 \geq n_0$. For convenience, for some $0 < \gamma \leq 1$ and $n \geq n_1$, we get

$$Q_n = q_n A_{n+1}^{(1+\gamma)(\beta-1)} P_{n+1-\ell}^{\beta}.$$

We begin with a new oscillation result for equation (1.1) when $\beta \geq 1$.

Theorem 2.1. *Let assumptions* $(H_1)-(H_3)$ *and* $\beta \geq 1$ *be hold. If*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[Q_s A_{s+1} - \frac{1}{4a_s A_{s+1}} \right] = \infty, \tag{2.1}$$

then equation (1.1) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1.1), say $y_n > 0$, $y_{n+1-\ell} > 0$ and $y_{n-k} > 0$ for $n \geq n_2$ for some $n_2 \geq n_1$. It is easy to see that $z_n > 0$ for $n \geq n_2$, and equation (1.1) becomes

$$\Delta(a_n \Delta z_n) + q_n y_{n+1-\ell}^{\beta} = 0. \tag{2.2}$$

Hence $\Delta(a_n \Delta z_n) \leq 0$ for all $n \geq n_2$, which implies that $a_n \Delta z_n$ is decreasing for all $n \geq n_2$. Thus (I) $\Delta z_n > 0$ or (II) $\Delta z_n < 0$ for all $n \geq n_2$.

Now we consider Case (I). Since $\{z_n\}$ is a positive increasing sequence, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (y_n + p_n y_{n-k}^{\alpha}) = \lim_{n\to\infty} y_n$$

since $\lim_{n\to\infty} p_n = 0$. Further there exists $n_3 \geq n_2$ and a constant $d > 0$ such that $z_n > d$ and $y_n > d$ for all $n \geq n_3$. Using the last inequality in (1.1), we obtain

$$\Delta(a_n \Delta z_n) + d^{\beta} q_n \leq 0, \quad n \geq n_3. \tag{2.3}$$

Summing (2.3) from n_3 to $n-1$, we have

$$a_n \Delta z_n - a_{n_3} \Delta z_{n_3} + d^{\beta} \sum_{s=n_3}^{n-1} q_s \leq 0, \quad n \geq n_3. \tag{2.4}$$

But (2.1) implies that

$$\sum_{n=n_3}^{\infty} q_n = \infty,$$

which together with (2.4) yields

$$\lim_{n\to\infty} a_n \Delta z_n = -\infty.$$

This contradicts the eventual positivity of $a_n \Delta z_n$.

Next, we consider Case (II). Define the sequence $\{w_n\}$ by

$$w_n = \frac{a_n \Delta z_n}{z_n}, \quad n \geq n_2. \tag{2.5}$$

Then $w_n < 0$ for $n \geq n_2$, and the decreasing nature of $a_n \Delta z_n$ implies that

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s} \quad \text{for} \quad s \geq n \geq n_2. \tag{2.6}$$

Summing (2.6) from n to $j-1 \geq n$, we obtain

$$z_j - z_n \leq a_n \Delta z_n \left(\sum_{s=n}^{j-1} \frac{1}{a_s} \right),$$

which by letting $j \to \infty$, leads to

$$\frac{a_n \Delta z_n}{z_n} A_n \geq -1 \quad \text{for} \quad n \geq n_2, \tag{2.7}$$

or

$$w_n A_n \geq -1 \quad \text{for} \quad n \geq n_2. \tag{2.8}$$

Now from (2.7) that

$$\Delta\left(\frac{z_n}{A_n}\right) = \frac{A_n \Delta z_n - z_n \Delta A_n}{A_n A_{n+1}} = \frac{A_n a_n \Delta z_n + z_n}{a_n A_n A_{n+1}} \geq 0$$

for $n \geq n_2$, and thus

$$\frac{z_n}{A_n} \geq \frac{z_{n-k}}{A_{n-k}} \quad \text{for} \quad n \geq n_2 + k. \tag{2.9}$$

From the definition of z_n, we have

$$y_n = z_n - p_n y_{n-k}^{\alpha} \geq z_n - p_n z_{n-k}^{\alpha}, \quad n \geq n_2 + k$$

and using (2.9), we obtain

$$y_n \geq z_n - p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} z_n^{\alpha} = \left(1 - p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} z_n^{\alpha-1}\right) z_n. \tag{2.10}$$

Since z_n is positive and decreasing, we get

$$z_n \leq z_{n_2} = M > 0 \quad \text{for} \quad n \geq n_2. \tag{2.11}$$

Using (2.11) in (2.10), we obtain

$$y_n \geq \left(1 - M^{\alpha-1} p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}}\right) z_n = P_n z_n \quad \text{for} \quad n \geq n_2. \quad (2.12)$$

Since $\left\{\frac{z_n}{A_n}\right\}$ is positive and increasing, we get

$$\frac{z_n}{A_n} \geq \frac{z_{n_2}}{A_{n_2}} = d > 0 \quad \text{for} \quad n \geq n_2. \quad (2.13)$$

Further, $\{A_n\}$ is positive and converging to zero, there exists $n_3 \geq n_2 + k$ such that

$$0 < A_n^{\gamma} \leq d \quad \text{for} \quad n \geq n_3. \quad (2.14)$$

Hence by (2.13) and (2.14)

$$z_n \geq A_n^{1+\gamma}, \quad n \geq n_3. \quad (2.15)$$

By (2.12), from (2.2), we obtain

$$\begin{aligned} \Delta(a_n \Delta z_n) &= -q_n y_{n+1-\ell}^{\beta} \\ &\leq -q_n P_{n+1-\ell}^{\beta} z_{n+1-\ell}^{\beta} \\ &\leq -q_n P_{n+1-\ell}^{\beta} z_{n+1}^{\beta}, \quad n \geq n_3, \end{aligned} \quad (2.16)$$

where, we also used the decreasing nature of z_n in the last estimate. Now (2.16), in view of (2.15) leads to

$$\begin{aligned} \Delta(a_n \Delta z_n) &\leq -q_n A_{n+1}^{(1+\gamma)(\beta-1)} P_{n+1-\ell}^{\beta} z_{n+1} \\ &= -Q_n z_{n+1} \quad \text{for} \quad n \geq n_3. \end{aligned} \quad (2.17)$$

From (2.5) one obtains

$$\begin{aligned} \Delta w_n &= \frac{\Delta(a_n \Delta z_n)}{z_{n+1}} - \frac{z_n}{a_n z_{n+1}} w_n^2 \\ &\leq \frac{\Delta(a_n \Delta z_n)}{z_{n+1}} - \frac{w_n^2}{a_n}, \end{aligned} \quad (2.18)$$

where we have used again the decreasing nature of $\{z_n\}$. Combining (2.18) and (2.17), we have

$$\Delta w_n \leq -Q_n - \frac{w_n^2}{a_n}, \quad n \geq n_3. \quad (2.19)$$

Using (2.19), we get

$$\begin{aligned} \Delta(A_n w_n) &= w_n \Delta A_n + A_{n+1} \Delta w_n \\ &= -\frac{w_n}{a_n} + A_{n+1} \Delta w_n \\ &\leq -\frac{w_n}{a_n} - A_{n+1} Q_n - \frac{A_{n+1} w_n^2}{a_n} \\ &\leq -A_{n+1} Q_n + \frac{1}{4 a_n A_{n+1}}, \end{aligned}$$

and summing this resulting inequality from n_3 to n and using (2.8) yields

$$\begin{aligned} \sum_{s=n_3}^{n} \left[Q_s A_{s+1} - \frac{1}{4 a_s A_{s+1}} \right] &\leq A_{n_3} w_{n_3} - A_{n+1} w_{n+1} \\ &\leq 1 + A_{n_3} w_{n_3} < \infty \end{aligned}$$

for $n \geq n_3$, contradicting (2.1). This completes the proof. □

When $\beta = 1$, we have the following immediate corollary from Theorem 2.1.

Corollary 2.2. *Let assumptions $(H_1) - (H_3)$ and $\beta = 1$ be hold. If*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[q_s P_{s+1-\ell} A_{s+1} - \frac{1}{4 a_s A_{s+1}} \right] = \infty, \quad (2.20)$$

then equation (1.1) is oscillatory.

Next, we present an oscillation criteria for the case $0 < \beta < 1$.

Theorem 2.3. *Let assumptions $(H_1) - (H_3)$ and $0 < \beta < 1$ be hold. If*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[L q_s P_{s+1-\ell}^{\beta} A_{s+1} - \frac{1}{4 a_s A_{s+1}} \right] = \infty, \quad (2.21)$$

for some constant $L > 0$, then equation (1.1) is oscillatory.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1.1), say $y_n > 0$, $y_{n+1-\ell} > 0$ and $y_{n-k} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Proceeding as in the proof of Theorem 2.1, we obtain two cases (I) $\Delta z_n > 0$ of (II) $\Delta z_n < 0$ for all $n \geq n_1$. The proof of Case (I) is similar to that of Theorem 2.1. For Case (II), we see that $\{z_n\}$ is positive and decreasing with $\lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n$. Then we have either $\lim_{n\to\infty} z_n = d_1 > 0$ or $\lim_{n\to\infty} z_n = 0$. The first case implies that $\lim_{n\to\infty} y_n = d_1$. Thus, there exist $d_2 > 0$ and $n_2 \in \mathbb{N}$ such that $y_n \geq d_2$ for all $n \geq n_2$. Then using this estimate in equation (1.1) and arguing as in Case (I), we obtain a contradiction. For the other case, there exists $n_3 \in \mathbb{N}$ such that

$$0 < z_n < k \quad (2.22)$$

for all $n \geq n_3$ and $K = L^{\frac{1}{\beta-1}} > 0$. Now proceeding as in the proof of Theorem 2.1 (Case (II)), we obtain (2.16) and then using (2.22) yields

$$\begin{aligned} \Delta(a_n \Delta z_n) &\leq -q_n P_{n+1-\ell}^{\beta} z_{n+1}^{\beta} \\ &= -q_n P_{n+1-\ell}^{\beta} \frac{z_{n+1}}{z_{n+1}^{1-\beta}} \\ &\leq -q_n P_{n+1-\ell}^{\beta} \frac{z_{n+1}}{K^{1-\beta}} \\ &= -L q_n P_{n+1-\ell}^{\beta} z_{n+1}, \quad n \geq n_3. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. This completes the proof. □

3. Example

In this section, we present some examples to show the importance of the main results. First we give an example for the case $\beta > 1$.

Example 3.1. *Consider the second order neutral type difference equation*

$$\Delta\left(n(n+1)\Delta\left(y_n+\frac{y_{n-k}^{\alpha}}{n^2}\right)\right)+(n+1)^5y_{n+1-\ell}^3=0,\ \ n\geq 1. \tag{3.1}$$

Here $\alpha>1$ is a quotient of odd integers, $\beta=3$, the delays $k\geq 1$ and $\ell\geq 0$, and $a_n=n(n+1)$, $p_n=\frac{1}{n^2}$ and $q_n=(n+1)^5$. We set $\gamma=1$. It is easy to see that (H_2) holds. Further $A_n=\frac{1}{n}$ and $P_n=1-\frac{M^{\alpha-1}}{n^{2-\alpha}(n-k)^{\alpha}}$. Also

$$Q_nA_{n+1}-\frac{1}{4a_nA_{n+1}}=(n+1)\left(1-\frac{M^{\alpha-1}}{(n+1-\ell)^{2-\alpha}(n+1-\ell-k)^{\alpha}}\right)^3-\frac{1}{4n}.$$

Moreover

$$\lim_{n\to\infty}\left(Q_nA_{n+1}-\frac{1}{4a_nA_{n+1}}\right)=\infty,$$

and therefore condition (2.1) of Theorem 2.1 is satisfied. Hence the equation (3.1) is oscillatory.

Next we give an example for the case $\beta=1$.

Example 3.2. *Consider the second order neutral type difference equation*

$$\Delta\left(n(n+1)\Delta\left(y_n+\frac{y_{n-k}^{\frac{5}{3}}}{n^{\frac{5}{3}}}\right)\right)+\frac{n+1}{n}y_{n+1-\ell}=0,\ \ n\geq 1. \tag{3.2}$$

Here $\alpha=\frac{5}{3}$, $\beta=1$, the delays $k\geq 1$ and $\ell\geq 0$, and $a_n=n(n+1)$, $p_n=\frac{1}{n^{\frac{5}{3}}}$ and $q_n=\frac{n+1}{n}$. We set $\gamma=\frac{1}{2}$. It is easy to see that (H_2) holds. Also $A_n=\frac{1}{n}$ and $P_n=1-\frac{M^{\frac{2}{3}}}{(n-k)^{\frac{5}{3}}}$. Further

$$q_nP_{n+1-\ell}A_{n+1}-\frac{1}{4a_nA_{n+1}}=\frac{3}{4n}-\frac{M^{\frac{2}{3}}}{n(n+1-\ell-k)^{\frac{5}{3}}}.$$

Therefore condition (2.20) of Corollary 2.2 is satisfied and hence equation (3.2) is oscillatory.

Finally, we provide an example for the case $0<\beta<1$.

Example 3.3. *Consider the second order neutral type difference equation*

$$\Delta\left(2^n\Delta\left(y_n+\frac{y_{n-k}^{\alpha}}{2^n}\right)\right)+n2^ny_{n+1-\ell}^{\frac{1}{3}}=0,\ \ n\geq 1. \tag{3.3}$$

Here $\alpha>1$ and $\beta=\frac{1}{3}$ and the delays $k\geq 1$ and $\ell\geq 0$, and $a_n=2^n$, $p_n=\frac{1}{2^n}$ and $q_n=n2^n$. We set $\gamma=1$. It is easy to see that condition (H_2) holds. Also $A_n=\frac{1}{2^{n-1}}$, and $P_n=1-\frac{M^{\alpha-1}}{2^{n-\alpha k}}$. Further

$$Lq_nP_{n+1-\ell}^{\beta}A_{n+1}-\frac{1}{4a_nA_{n+1}}=Ln\left(1-\frac{M^{\alpha-1}}{2^{n+1-\ell-\alpha k}}\right)-\frac{1}{4},$$

and this tends to infinity for any constant $L>0$. Therefore condition (2.21) of Theorem 2.3 is satisfied and hence equation (3.3) is oscillatory.

4. Conclusion

In this paper, we present some new oscillation criteria for the equation (1.1). The obtained results simplifies the known results in the literature in the sense that we need only one condition instead of two conditions required in [7, 9, 11] for the oscillation of all solutions of equation (1.1). Further the results presented in this paper extend and generalize the results known [4–6, 15–18] for the case $0<\alpha<1$ to the case $\alpha>1$.

References

[1] R.P.Agarwal, M.Bohner, S.R.Grace and D.O'Regan, Discrete Oscillation Theory, *Hindawi Publ. Corp., New York,* 2005.

[2] R.P.Agarwal, S.R.Grace and D.O'Regan, Oscillation Theory for Difference and Functional Equations, *Kluwer Acad. Publ., Dordrecht,* 2000.

[3] R.P.Agarwal, M. Bohner, T.Li and C.Zhang, Oscillation of second order differential equations with a sublinear neutral term, *Carpathian J. Math.,* 30(2014), 1-6.

[4] G.Ayyappan and G.Nithyakala, Some oscillation results for even order delay difference equations with a sublinear neutral term, *Abstr. Appl. Anal.,* 2018, Article ID 2590158, 6 Pages.

[5] M.Bohner, H.A.El-Morshedy, S.R.Grace and I.Sager, Oscillation of second order nonlinear difference equations with sublinear neutral term, *Math. Moravica,* 23(2019), 1-10.

[6] C.Dharuman, J.R.Graef, E.Thandapani and K.S.Vidhya, Oscillation of second order difference equations with sublinear neutral term, *J. Math. Appl.,* 40(2017), 59-67.

[7] C.Dharuman and E.Thandapani, Oscillation of solutions of nonlinear difference equations with a super-linear neutral term, *Nonauton. Dyn. Syst.,* 5(2018), 52-58.

[8] S.R.Grace and J.R.Graef, Oscillatory behavior of second order nonliear differential equations with a sublinear neutral term, *Math. Model. Anal.,* 23 (2018), 217-226.

[9] B.Kamaraj and R.Vasuki, Oscillation of second order difference equation with a superlinear neutral term, *J. Advances in Math. Comput. Sci.,* 23(2017), 1-10.

[10] X.Liu, Oscillation of solutions of neutral difference equations with a nonlinear term, *J. Comput. Math. Appl.,* 52(2006), 439-448.

[11] S.Mehar Banu and S.Nalani, Oscillation of second order difference equation with several super linear neutral terms, *Adv. Diff. Equ.,* 2018, 2018:345.

[12] S.Selvarangam, E.Thandapani and S.Pinelas, Oscillation theorem for second order nonlinear neutral difference equations, *J. Inequal. Appl.,* 2014, 2014:417.

[13] S.Tamilvannan, E.Thandapani and J.Dzurina, Oscillation of second order nonlinear differential equation with sublinear neutral term, *Diff. Equ. Appl.,* 9(2017), 29-35.

[14] S.Tamilvannan, E.Thandapani and S.R.Grace, Oscillation theorem for second order nonlinear differential equation with a nonlinear neutral term, *Inter. J. Dynam. Syst. Diff. Equ.,* 7(2017), 316-327.

[15] E. Thandapani, Z. Liu, R. Arul and P.S. Raja, Oscillation and asymptotic behavior of second order difference equtaions with nonlinear neutral term, *Appl. Math. E-Notes,* 4(2004), 59-67.

[16] E.Thandapani, S.Pandian and R.K.Balasubramanian, Oscillation of solutions of nonlinear neutral difference equations with nonlinear neutral term, *Far East J. Appl. Math.,* 15(2004), 47-62.

[17] M.K.Yildiz and H.Ogunmez, Oscillation results of higher order nonlinear neutral delay difference equations with a nonlinear neutral term, *Hacet. J. Math. Stat.,* 43(2014), 809-814.

[18] Z.Zhang, J.Chen and C.Zhang, Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term, *Comput. Math. Appl.,* 41(2001), 1487-1494.

⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆