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On the coefficients of some classes of multivalent functions related to complex order

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Abstract
Let $R^b(A,B,p)$, $(b \in C/\{0\})$ denote the class of functions of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ regular in the unit disc $E = \{z : |z| < 1\}$, such that

$$p + \frac{1}{b}\left\{\frac{f'(z)}{z^{p-1}} - p\right\} = \frac{p + Apw(z)}{1 + Bw(z)},\ z \in E$$

where A and B are fixed number $-1 \le B < A \le 1$ and $w(0) = 0, |w(z)| < 1$.
In this paper, coefficient estimates, distortion theorem and maximization theorem for the class $R^b_\lambda(A,B,p)$ are determined, where $R^b_\lambda(A,B,p)$ denote the class of functions $g(z)$ analytic and multivalent in the unit disc E defined by

$$g(z) = (1-\lambda)z^p + \lambda f(z),\ f(z) \in R^b(A,B,p).$$

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1. Introduction

Let V denote the class of functions of the form

$$w(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$.

The present paper is devoted to a unified study of various subclasses of multivalent and univalent functions. For this purpose we mention the class $R^b(A,B,p)$ of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

regular in E and satisfying the condition

$$p + \frac{1}{b}\left\{\frac{f'(z)}{z^{p-1}} - p\right\} = p\frac{(1 + Aw(z))}{1 + Bw(z)},\quad z \in E \tag{1.2}$$

where A and B are fixed numbers such that $-1 \le B < A \le 1$ and b is non-zero complex number or, equivalently (1.2) can be expressed as

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{Abp-B\left\{bp+\frac{f'(z)}{z^{p-1}}-p\right\}}\right|<1, \quad z\in E. \tag{1.3}$$

The class $R^b(A,B,p)$ was introduced by Dixit and Pathak [5]. Further for a given number λ, $0<\lambda\leq 1$, let $R^b_\lambda(A,B,p)$ denote the class of functions $g(z)$ analytic and multivalent in E where

$$g(z)=(1-\lambda)z^p+\lambda f(z),\, f(z)\in R^b(A,B,p). \tag{1.4}$$

In fact by giving specific values to p,b,A,B, and λ in (1.3), we obtain the following important subclasses studied by various authors in earlier works.

1. For $\lambda=1$, we obtain the class of functions studied by Dixit and Pathak [5].
2. For $p=1$, we obtain the class of functions studied by Dixit and Vikas Chandra [7]
3. For $\lambda=1$ and $p=1$, we obtain the class of functions studied by Dixit and Pal [6].
4. For $\lambda=1$, $p=1$ and $b=\cos\alpha\, e^{-i\alpha}$, we obtain the class of functions studied by Dashrath [4].
5. For $\lambda=1, p=1, b=1, A=(1-2\rho)\delta$ and $B=-\delta$ where $0\leq\rho<1$, $0<\delta\leq 1$, we obtain the class of functions studied by Juneja and Mogra [10].
6. For $\lambda=1, p=1, b=1, A=\delta$ and $B=-\delta$, we obtain the class of functions studied by Caplinger and Causey [1] and Padmanabhan [15].
7. For $\lambda=1, b=1$ and $p=1$, we obtain the class of functions studied by Goel and Mehrok [8].

Apart from these, important subclasses can be obtained by giving suitable values to p,b,A,B and λ studied by Mazur ([12], [13]), Nasr [14] and Janowski [9].

In this paper, results concerning coefficient estimates, sufficient condition in terms of coefficients, distortion theorem for the class $R^b_\lambda(A,B,p)$and the maximization of $|a_3-\mu a_2^2|$ over the class $R^b_\lambda(A,B,p)$ have been obtained systematically.

We state below a lemma that is needed in our investigation. The following is due to Keogh and Merkes [11].

Lemma 1.1. *Let* $w(z)=\sum_{k=1}^{\infty}b_kz^k$ *be analytic with* $|w(z)|<1$ *in* E*. If* S *is any complex number then*

$$|b_2-Sb_1^2|\leq \max\ (1,\ |\delta|).$$

Equality may be obtained for functions $w(z)=z^2$ *and* $w(z)=z$.

2. Coefficient estimates

In this section, method of Clunie [2] and Clunie and Keogh [3] will be used to prove the following theorem.

Theorem 2.1. *If* $g(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n\in R^b_\lambda(A,B,p)$, *then* $|a_n|\leq\dfrac{|b|\lambda p(A-B)}{n}$.

The results are sharp.

Proof. Since $g(z)\in R^b_\lambda(A,B,p)$, then from the definition of the classes $R^b_\lambda(A,B,p)$ and $R^b(A,B,p)$, it follows that there exists a function $w(z)$ satisfying

$$\frac{g'(z)}{z^{p-1}}=p(1-\lambda)+\lambda\left[\frac{p+\{bp(A-B)+pB\}w(z)}{1+Bw(z)}\right]$$

which gives

$$w(z)\left[\lambda bp(A-B)+pB-\frac{Bg'(z)}{z^{p-1}}\right]=\frac{g'(z)}{z^{p-1}}-p$$

where $w(0)=0$, $|w(z)|<1$ for $|z|<1$, that is

$$w(z)\left[\lambda bp(A-B)-B\lambda\sum_{n=p+1}^{\infty}na_nz^{n-p}\right]$$

$$=\lambda\sum_{n=p+1}^{\infty}na_nz^{n-p} \tag{2.1}$$

Equating corresponding coefficients on both sides of (2.1) we observe that the coefficient a_n on the right hand side of (2.1) depends only on $a_{p+1},a_{p+2},......,a_{n-p}$ on the left side of (2.1). Hence it follows from (2.1) that

$$w(z)\left[\lambda bp(A-B)-B\lambda\sum_{k=p+1}^{n-1}ka_kz^{k-p}\right]$$

$$=\sum_{k=p+1}^{n}ka_kz^{k-p}+\sum_{k=n+1}^{\infty}c_kz^{k-p}$$

c'_{ks} being some complex number. Since $|w(z)|<1$, we have by means of Parseval's identity

$$|\lambda|^2|b|^2p^2(A-B)^2+B^2\lambda^2\sum_{k=p+1}^{n-1}k^2|a_k|^2r^{2k-2p}$$
$$\geq\sum_{k=p+1}^{n}k^2|a_k|^2r^{2k-2p}+\sum_{k=n+1}^{\infty}|c_k|^2r^{2k-2p}$$

if we take limit as r approaches 1, then

$$|b|^2p^2\lambda^2(A-B)^2+B^2\lambda^2\sum_{k=p+1}^{n-1}k^2|a_k|^2\geq\sum_{k=p+1}^{n}k^2|a_k|^2$$

or

$$|b|^2p^2\lambda^2(A-B)^2+B^2\sum_{k=p+1}^{n-1}k^2|a_k|^2 \geq \sum_{k=p+1}^{n}k^2|a_k|^2 = \sum_{k=p+1}^{n}k^2|a_k|^2+n^2|a_n|^2$$

or

$$(1-B^2)\sum_{k=p+1}^{n-1}k^2|a_k|^2+n^2|a_n|^2 \leq |b|^2\lambda^2p^2(A-B)^2$$

or

$$|a_n| \leq \frac{|b|p\lambda(A-B)}{n}, \quad n=p+1,p+2,...$$

The sharpness of the result follows for the functions

$$f(z)=(1-\lambda)z^p+\lambda\int_0^z\left[pt^{p-1}+\left\{\frac{(A-B)bpt^{n-1}}{1-Bt^{n-1}}\right\}\right]dt$$

for $n \geq p+1$ and $z \in E$. □

3. A sufficient condition for a function to be in $R_\lambda^b(A,B,p)$

Theorem 3.1. *Let* $g(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be analytic in E.*

If for some $A,B(-1 \leq B < A \leq 1)$ *and*

$$\sum_{n=p+1}^{\infty}n|a_n|[1+|B|] \leq (A-B)\lambda p|b|. \quad (3.1)$$

Then $g(z) \in R_\lambda^b(A,B,p)$.

Proof. We prove this theorem by the technique of Clunie and Keogh [3]. Suppose that (3.1) holds and that

$$g(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n, \text{then for} |z|<1,$$

$$\left|\frac{g'(z)}{z^{p-1}}-p\right|-\left[\lambda bp(A-B)-B\left\{\frac{g'(z)}{z^{p-1}}-p\right\}\right]$$
$$=\left|\sum_{n=p+1}^{\infty}na_nz^{n-p}\right|-\left|\lambda bp(A-B)-B\lambda\sum_{n=p+1}^{\infty}na_nz^{n-p}\right|$$
$$\leq \sum_{n=p+1}^{\infty}n|a_n|r^{n-p}-|b|\lambda p(A-B)+|B|\lambda\sum_{n=p+1}^{\infty}na_nr^{n-p}$$
$$\leq \sum_{n=p+1}^{\infty}n|a_n|-|b|\lambda p(A-B)+|B|\lambda\sum_{n=p+1}^{\infty}n|a_n|$$
$$=\sum_{n=p+1}^{\infty}n|a_n|(1+|B|)-|b|\lambda p(A-B) \leq 0$$

Hence it follows that

$$\left|\frac{\frac{g'(z)}{z^{p-1}}-p}{\lambda bp(A-B)-B\left(\frac{g'(z)}{z^{p-1}}-p\right)}\right|<1, \quad z \in E,$$

therefore $g \in R_\lambda^b(A,B,p)$.

The following functions shows that the result is sharp.

$$g(z)=z^p+\frac{(A-B)\lambda bpz^n}{(1+|B|)n} \text{ for } n \geq p+1 \text{ and } z \in E.$$

□

4. Distortion Theorem

Theorem 4.1. *If* $g(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \in R_\lambda^b(A,B,p)$, *then*

$$Re\frac{g'(z)}{z^{p-1}} \geq (1-\lambda)p+$$

$$\lambda\frac{[p-ABr^2pRe(b)-B^2r^2Re(1-b)p-(A-B)p|b|r]}{1-B^2r^2} \quad (4.1)$$

and

$$Re\frac{g'(z)}{z^{p-1}} \leq (1-\lambda)p+$$

$$\lambda\frac{[p-ABr^2pRe(b)-B^2r^2Re(1-b)p+(A-B)|b|]}{1-B^2r^2} \quad (4.2)$$

Proof. Since $f \in R_\lambda^b(A,B,p)$. Therefore by Theorem 3 of Dixit and Pathak [5], we have

$$Re\frac{f'(z)}{z^{p-1}} \geq$$

$$\frac{p-ABr^2pRe(b)-B^2r^2Re(1-b)p-(A-B)p|b|r}{1-B^2r^2}$$

and

$$Re\frac{f'(z)}{z^{p-1}} \leq$$

$$\frac{p-ABr^2pRe(b)-B^2r^2Re(1-b)p+(A-B)p|b|r}{1-B^2r^2}$$

using (1.4)

$$g'(z)=(1-\lambda)pz^{p-1}+\lambda f'(z)$$

$$\frac{g'(z)}{z^{p-1}}=(1-\lambda)p+\frac{\lambda f'(z)}{z^{p-1}}$$

$$Re\frac{g'(z)}{z^{p-1}} \geq (1-\lambda)p+$$

$$\lambda\frac{[p-ABr^2pRe(b)-B^2r^2Re(1-b)p-(A-B)p|b|r]}{1-B^2r^2}.$$

□

Theorem 4.2. *If* $g(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n\in R_\lambda^b(A,B,p)$ *and* μ *is any complex number then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\le\frac{\lambda p|b|(A-B)}{p+2}$$

$$\max\left\{1,\frac{|B(p+1)^2+(p+2)\mu pb\lambda(A-B)|}{(p+1)^2}\right\}. \quad (4.3)$$

The result is sharp.

Proof. Since $g\in R_\lambda^b(A,B,p)$, we have

$$\frac{g'(z)}{z^{p-1}}=p(1-\lambda)+\lambda\left[\frac{p+\{bp(A-B)+pB\}w(z)}{1+Bw(z)}\right]$$

where $w(z)=\sum_{k=1}^{\infty}b_kz^k$ is analytic in E and satisfying the conditions $w(0)=0$, $|w(z)|<1$ for $z\in E$.

$$w(z)=\frac{\sum_{n=p+1}^{\infty}na_nz^{n-p}}{\lambda bp(A-B)}$$
$$\left[1+\frac{B}{\lambda bp(A-B)}\sum_{n=p+1}^{\infty}na_nz^{n-p}+...\right]$$
$$=\frac{1}{\lambda bp(A-B)}$$
$$\left[\sum_{n=p+1}^{\infty}na_nz^{n-p}+\frac{B}{\lambda bp(A-B)}\left(\sum_{n=p+1}^{\infty}na_nz^{n-p}\right)^2+...\right]$$
$$=\frac{1}{\lambda bp(A-B)}$$
$$\left[(p+1)a_{p+1}z+(p+2)a_{p+2}z^2+...+\frac{B}{\lambda bp(A-B)}(p+1)^2(a_{p+1})^2z^2+...\right]$$

and then comparing the coefficient of z and z^2 on both sides, we have

$$b_1=\frac{(p+1)a_{p+1}}{\lambda bp(A-B)},$$

$$b_2=\frac{1}{\lambda bp(A-B)}\left[(p+2)a_{p+2}+\frac{B(p+1)^2(a_{p+1})^2}{\lambda bp(A-B)}\right]$$

Thus

$$a_{p+1}=\frac{\lambda bp(A-B)b_1}{p+1}$$

and

$$(p+2)a_{p+2}=\lambda bp(A-B)b_2-\frac{B(p+1)^2(a_{p+1})^2}{\lambda bp(A-B)}$$

or

$$a_{p+2}=\frac{\lambda bp(A-B)b_2}{(p+2)}-\frac{B(p+1)^2(a_{p+1})^2}{(p+2)\lambda bp(A-B)}.$$

Hence

$$a_{p+2}-\mu(a_{p+1})^2=\frac{\lambda bp(A-B)b_2}{(p+2)}-$$
$$\frac{B(p+1)^2(a_{p+1})^2}{(p+2)\lambda bp(A-B)}-\mu(a_{p+1})^2$$
$$a_{p+2}-\mu(a_{p+1})^2=\frac{\lambda bp(A-B)b_2}{(p+2)}$$
$$-\left[\frac{B(p+1)^2}{(p+2)\lambda bp(A-B)}+\mu\right](a_{p+1})^2$$

$$=\frac{\lambda bp(A-B)b_2}{(p+2)}$$

-

$$\left[\frac{B(p+1)^2+\mu(p+2)\lambda bp(A-B)}{(p+2)\lambda bp(A-B)}\right]\frac{\lambda^2b^2p^2(A-B)^2b_1^2}{(p+1)^2}$$
$$=\frac{\lambda bp(A-B)}{(p+2)}\left[b_2-\{B(p+1)^2+\mu(p+2)\lambda bp(A-B)\}\frac{b_1^2}{(p+1)^2}\right]$$
$$=\frac{\lambda bp(A-B)}{(p+2)}$$
$$\left[b_2-\left\{B+\frac{(p+2)\mu\lambda bp(A-B)}{(p+1)^2}\right\}b_1^2\right].$$

Using Lemma 1.1, we obtain

$$\left|a_{p+2}-\mu\left(a_{p+1}\right)^2\right|\le\frac{\lambda p|b|(A-B)}{(p+2)}$$

$$\max\left\{1,\frac{|(p+1)^2B+(p+2)\mu\lambda bp(A-B)|}{(p+1)^2}\right\}.$$

Which is (4.3) of Theorem 4.2, when

$$\left|\frac{B(p+1)^2+(p+2)\mu\lambda bp(A-B)}{(p+1)^2}\right|>1.$$

We choose the function

$$g(z)=\frac{Bp+(A-B)\lambda bp}{Bp}z^p-\frac{(A-B)\lambda bp}{B^2}$$

$$\left[\frac{Bz^p}{p} - \frac{B^2 z^{p+1}}{p+1} + \frac{B^3 z^{p+2}}{p+2} - ......\right]$$

and when

$$\left|\frac{B(p+1)^2 + (p+2)\mu\lambda bp(A-B)}{(p+1)^2}\right| < 1,$$

we have

$$g(z) = \frac{Bp + (A-B)\lambda bp}{Bp} z^p - \frac{(A-B)\lambda bp}{B^2}$$

$$\left[\frac{z^p}{p} + \frac{Bz^{p+1}}{p+1} + \frac{B^2 z^{p+2}}{p+2} + ...\right].$$

□

References

[1] T. R. Caplinger and W. M. Causey, A class of univalent functions, *Proc. Amer. Math. Soc.,* 39 (1973), 357-361.

[2] J. Clunie, On meromorphic schlicht functions, *J. London Math. Soc.,* 34 (1959), 215-229.

[3] J. Clunie and F. R. Keogh, On starlike and convex schlicht functions, *J. London Math. Soc.,* 35 (1960), 229-233.

[4] Dashrath, *On some classes related to spiral-like univalent and multivalent functions*, Ph.D. Thesis, Kanpur University, Kanpur, (U.P.), India, 1983.

[5] K. K. Dixit and A. L. Pathak, On a class of multivalent functions, *Ganita,* 60 (1) (2009), 15-21.

[6] K. K. Dixit and S. K. Pal, On a class of univalent functions related to complex order, *Indian J. Pure Appl. Math.,* 26 (9) (1995), 889-896.

[7] K. K. Dixit and V. Chandra, On the coefficients of some classes of analytic functions related to complex order, *Bull. Cal. Math. Soc.,* 101 (4) (2009), 351-358.

[8] R. M. Goel and B. S. Mehrok, A subclass of univalent functions, *J. Austral Math. Soc.,* 35 (1983), 1-17.

[9] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.,* 17 (1969), 633-637.

[10] O. P. Juneja and M. L. Mogra, A class of univalent functions, *Bull. Sci. Math.,* 103 (4) (1979), 435-447.

[11] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.,* 20 (1) (1969), 8-12.

[12] T. R. Mazur, On some extremal problems in the class of functions with bounded rotation, *Scientific Bulletin of Lodz Technical Univ. Math.,* 9 (1977), 77-88.

[13] T.R. Mazur, On the coefficients of convex combination of certain Holomorphic functions, *Complex Analysis*, App. 83. Sofia (1985), 161-165.

[14] M. A. Nasr, On the radius of convexity of convex combination of certain analytic functions, *Glasnik mathematicki*, 10 (2) (1975), 257-262.

[15] K. S. Padmanabhan, On a certain class of functions whose derivatives have a positive real part in the unit disc, *Ann. Polon. Math.,* 23 (1970), 73-81.

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