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Further results and applications on continuous random variables

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Abstract

New results and new applications of fractional calculus for continuous random variables are presented. New expectation and variance identities of order $\alpha \geq 1$ are established. Under a new fractional normalisation technique, other weighted random variable inequalities are generated and some classical results are deduced as special cases.

Keywords

Integral inequalities, Riemann-Liouville integral, random variable, fractional expectation, fractional variance, fractional moment.

AMS Subject Classification

26D15, 26A33, 60E15.

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1. Introduction

The integral inequalities are very important in physics and applied sciences. For some applications of this theory, we refer the reader [2, 4, 5, 8–11]. Let us now introduce some papers that have motivate the present work. We begin by [1] where, T. Cacoullos and V. Papathanasiou established the following nice covariance identity

$$Cov(h(X),g(X)) = E(z(X)g'(X)), \tag{1.1}$$

where X is a continuous random variable (CRV, for short) with support an interval (a,b) ,$-\infty \leq a < b \leq \infty$, f is probability density function (pdf, for short) of X, g and h are two continuous functions with $\sigma^2 = V(X), |E(z(X)g'(X))| < \infty$ and

$$z(x) := \frac{1}{f(x)}\int_a^x (E[h(X)] - h(t))f(t)dt. \tag{1.2}$$

Under some other conditions, the same authors [2] proved that

$$Var[g(X)] \geq \frac{E^2[z(X)g'(X)]}{E[z(X)h(X)]}, \tag{1.3}$$

They also established the following theorem [3]:

Theorem 1.1. *For every absolutely continuous function h with $h' > 0$, the inequality:*

$$Var[g(X)] \leq E[\frac{z(X)}{h'(X)}(g'(X))^2] \tag{1.4}$$

holds, with equality iff $g = c_1h + c_2$, where

$$zf = \int_a^x (E(h) - h(t))f(t)dt. \tag{1.5}$$

Very recently, based on the notions of [4], Z. Dahmani [5] proved the following theorem:

Theorem 1.2. *Let X be a random variable with a p.d.f. defined on $[a,b]$, such that $\mu = E(X), \sigma^2 = Var(X)$ and $w \in C([a,b]); \int_a^x (b-t)^{\alpha-1}(\mu-t)f(t)dt = (b-x)^{\alpha-1}\sigma^2 w(x)f(x)$. Then, we have*

$$Var_{g(X),\alpha} \geq \frac{\sigma^4(X)}{Var_{X,\alpha}} E^2_{g'(X)w(X)\alpha}. \tag{1.6}$$

Motivated by the papers in [2, 3, 5, 8], in this work, we focus our attention on the application of the Riemann-Liouville fractional integration on random variables. We use some recently introduced concepts on continuous random variables to establish new identities on continuous random variables. We prove new lower bounds for the variances as well as for the expectations. For our results, some classical and excellent results of [1–3], that correspond to the standard integration of order $\alpha = 1$, are generalised for any $\alpha \geq 1$.

2. Preliminaries

Definition 2.1. *[7]The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as*

$$J_a^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \alpha > 0, a < t \leq b. \quad (2.1)$$

For any $\alpha > 0$ and any positive continuous function w defined on $[a,b]$, we recall the definitions [4]:

Definition 2.2. *The fractional $w-$weighted expectation function of order α, for a CRV X with a pdf f defined on $[a,b]$ is given a by:*

$$E_{X,\alpha,w}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau w(\tau) f(\tau) d\tau, a \leq t < b, \quad (2.2)$$

Definition 2.3. *The fractional $w-$weighted expectation function of order α, for $X - E(X)$ is given by*

$$E_{X-E(X),\alpha,w}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X)) \times w(\tau) f(\tau) d\tau. \quad (2.3)$$

Definition 2.4. *The fractional $w-$weighted variance function of order α, for X is defined as*

$$\sigma^2_{X,\alpha,w}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X))^2 w(\tau) f(\tau) d\tau. \quad (2.4)$$

Definition 2.5. *The fractional $w-$weighted expectation of order α for X is defined as*

$$E_{X,\alpha,w} = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau w(\tau) f(\tau) d\tau. \quad (2.5)$$

Definition 2.6. *The fractional $w-$weighted variance of order α for X is given by:*

$$\sigma^2_{X,\alpha,w} = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} (\tau - E(X))^2 w(\tau) f(\tau) d\tau. \quad (2.6)$$

3. Main Results

We begin this section by proving the following theorem. We have:

Theorem 3.1. *Let X be a CRV with support an interval $[a,b]$, $-\infty < a < b < \infty$. Suppose that X has a pdf f and let w be a positive continuous function on $[a,b]$. Then, the following equality holds for any $\alpha \geq 1$:*

$$E_{zg',\alpha,\omega} = E_{gh,\alpha,w} - E_{h,\alpha,w} E_{g,\alpha,w}, \quad (3.1)$$

where g is an absolutely continuous function with $\left|E_{zg',\alpha,\omega}\right| < \infty, J_a^{\alpha} w f(b) = 1$, h is a given function and z satisfies:

$$z(t) = \frac{1}{(b-t)^{\alpha-1} w(t) f(t)} \times \int_a^t (b-u)^{\alpha-1} w(u) f(u) (E_{h,\alpha,w} - h(u)) du. \quad (3.2)$$

Proof.

We have

$$\begin{aligned} E_{zg',\alpha,\omega} &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} g'(t) \frac{1}{(b-t)^{\alpha-1} w(t) f(t)} \\ &\times \int_a^t (b-u)^{\alpha-1} w(u) f(u) (E_{h,\alpha,w} - h(u)) du \\ &\times w(t) f(t) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b g'(t) \int_a^t (b-u)^{\alpha-1} w(u) f(u) \\ &\times (E_{h,\alpha,w} - h(u)) du dt. \end{aligned}$$

Therefore,

$$E_{zg',\alpha,\omega} = \frac{1}{\Gamma(\alpha)}\left\{g(t)\int_a^t (b-u)^{\alpha-1}w(u)f(u)\right.$$

$$\left.\times(E_{h,\alpha,w}-h(u))du|_{t=a}^{t=b}\right\} - \frac{1}{\Gamma(\alpha)}\left\{\int_a^b g(t)\right.$$

$$\left.\times(b-t)^{\alpha-1}w(t)f(t)(E_{h,\alpha,w}-h(t))dt\right\}$$

$$= \frac{1}{\Gamma(\alpha)}\left\{g(b)\int_a^b (b-u)^{\alpha-1}w(u)f(u)\right.$$

$$\left.\times(E_{h,\alpha,w}-h(u))du\right\} - \frac{1}{\Gamma(\alpha)}\left\{\int_a^b g(t)\right.$$

$$\left.\times(b-t)^{\alpha-1}w(t)f(t)(E_{h,\alpha,w}-h(t))dt\right\}$$

$$= g(b)\left\{\frac{E_{h,\alpha,w}}{\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}w(u)f(u)du\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}w(u)f(u)h(u)du\right\}$$

$$-\left\{\frac{E_{h,\alpha,w}}{\Gamma(\alpha)}\int_a^b g(t)(b-t)^{\alpha-1}w(t)f(t)dt\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_a^b g(t)(b-t)^{\alpha-1}w(t)f(t)h(t)dt\right\}$$

$$= g(b)\left\{E_{h,\alpha,w}J_a^\alpha wf(b)-E_{h,\alpha,w}\right\}$$

$$-E_{h,\alpha,w}E_{g,\alpha,w}+E_{gh,\alpha,w}.$$

Thanks to the hypothesis $J_a^\alpha wf(b)=1$, we obtain

$$E_{zg',\alpha,\omega} = E_{gh,\alpha,w}-E_{h,\alpha,w}E_{g,\alpha,w}.$$

The proof is thus achieved.

Remark 3.2. *If we take $\alpha=1, w(x)=1$ in (3.1), we obtain (1.1).*

Lemma 3.3. *Let X be a CRV with support an interval $[a,b]$ $,-\infty\le a<b\le\infty$, and a pdf $f$and suppose that $w:[a,b]\to\mathbb{R}^+$ is a continuous function. Then, for any $\alpha\ge 1$, we have*

$$\sigma^2_{g,\alpha,w} = E_{zg',\alpha,\omega}, \tag{3.3}$$

where g is an absolutely continuous function such that $\left|E_{zg',\alpha,\omega}\right|<\infty, g(b)=E(g)$ and

$$z(t) = \frac{1}{(b-t)^{\alpha-1}w(t)f(t)}$$

$$\times\int_a^t (b-u)^{\alpha-1}(E(g)-g(u))w(u)f(u)du.$$

Proof:
We have

$$E_{zg',\alpha,\omega} = \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}g'(t)\frac{1}{(b-t)^{\alpha-1}w(t)f(t)}$$

$$\times\int_a^t (b-u)^{\alpha-1}(E(g)-g(u))$$

$$\times w(u)f(u)du w(t)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b g'(t)\int_a^t (b-u)^{\alpha-1}(E(g)-g(u))$$

$$\times w(u)f(u)dudt.$$

Hence, we get

$$E_{zg',\alpha,\omega} = \frac{1}{\Gamma(\alpha)}g(t)\int_a^t (b-u)^{\alpha-1}(E(g)-g(u))$$

$$\times w(u)f(u)du|_a^b - \frac{1}{\Gamma(\alpha)}\int_a^b g(t)$$

$$\times(b-t)^{\alpha-1}(E(g)-g(t))w(t)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}g(b)\int_a^b (b-u)^{\alpha-1}(E(g)-g(u))$$

$$\times w(u)f(u)du - \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}$$

$$\times(g(t)(E(g)-g^2(t))w(t)f(t)dt.$$

We use the fact that $g(b)=E(g)$, then it yields that

$$E_{zg',\alpha,\omega} = \frac{1}{\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}(E^2(g)-E(g)g(u))$$

$$\times w(u)f(u)du - \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}$$

$$\times(g(t)E(g)-g^2(t))w(t)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}((g^2(t)+E^2(g)$$

$$-2g(t)E(g))w(t)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(g(t)-E(g))^2w(t)f(t)dt$$

$$= \sigma^2_{g,\alpha,w}.$$

Remark 3.4. *If we take $g(x)=x$ in Lemma 3.3, we obtain $E(X)=b$, and*

$$\sigma^2_{X,\alpha,w} = E_{z,\alpha,\omega}, \tag{3.4}$$

with,

$$z(t) = \frac{1}{(b-t)^{\alpha-1}w(t)f(t)}\int_a^t (b-u)^{\alpha-1}(E[X]-u) \times w(u)f(u)du = \frac{1}{(b-t)^{\alpha-1}w(t)f(t)}\int_a^t (b-u)^{\alpha}w(u)f(u)du. \tag{3.5}$$

Now, taking three continuous functions U,V,Q defined on $[a,b]$, we prove the following result:

Theorem 3.5. *Let U,V and Q be three continuous functions on $[a,b]$ and $J_a^\alpha Q(b)=1$, then*

$$J_a^\alpha UVQ(b) - J_a^\alpha UQ(b)J_a^\alpha VQ(b) = J_a^\alpha\left[(U-J_a^\alpha UQ(b)) \times (V-J_a^\alpha VQ(b))\,Q(b)\right] \tag{3.6}$$

is valid, for any $\alpha \geq 1$.

Proof
We have

$$\begin{aligned} & J_a^\alpha\left[(U-J_a^\alpha UQ(b))(V-J_a^\alpha VQ(b))\right](b) \\ = & \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(U(t)-J_a^\alpha UQ(b)) \times (V(t)-J_a^\alpha VQ(b))\,Q(t)dt \\ = & \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}U(t)V(t)Q(t)dt - \frac{J_a^\alpha VQ(b)}{\Gamma(\alpha)} \times \int_a^b (b-t)^{\alpha-1}U(t)Q(t)dt - \frac{J_a^\alpha UQ(b)}{\Gamma(\alpha)} \\ & \times \int_a^b (b-t)^{\alpha-1}V(t)Q(t)dt + \frac{J_a^\alpha UQ(b)J_a^\alpha VQ(b)}{\Gamma(\alpha)} \times \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}Q(t)dt \\ = & J_a^\alpha UVQ(b) - 2J_a^\alpha UQ(b)J_a^\alpha VQ(b) + J_a^\alpha UQ(b)J_a^\alpha VQ(b)J_a^\alpha Q(b). \end{aligned}$$

Since $J_a^\alpha Q(b)=1$, then we ca write

$$\begin{aligned} & J_a^\alpha\left[(U-J_a^\alpha UQ(b))(V-J_a^\alpha VQ(b))\,Q\right](b) \\ = & J_a^\alpha UVQ(b) - J_a^\alpha UQ(b)J_a^\alpha VQ(b). \end{aligned}$$

Taking into account the above theorem, we prove the following theorem:

Theorem 3.6. *Let X be a continuous random variable with support an interval $[a,b]$, $-\infty < a < b < \infty$, having a pdf f and let $w:[a,b]\to\mathbb{R}^+$ be a continuous function. Then, for any $\alpha \geq 1$, we have*

$$\frac{E^2_{zg',\alpha,w}}{E_{zh',\alpha,w}} \leq E_{[g-E_{g,\alpha,w}]^2,\alpha,w}, 0<\alpha, \tag{3.7}$$

where g is an absolutely continuous function with $\left|E_{zg',\alpha,\omega}\right| < \infty$ and $J_a^\alpha wf(b)=1$, h is a given function and z is given by

$$z(t) = \frac{1}{(b-t)^{\alpha-1}w(t)f(t)}\int_a^t (b-u)^{\alpha-1}w(u)f(u) \times (E_{h,\alpha,w}-h(u))du.$$

Proof
Thanks to (3.1), we observe that

$$\begin{aligned} E_{zg',\alpha,\omega} &= E_{gh,\alpha,w} - E_{h,\alpha,w}E_{g,\alpha,w} \\ &= J_a^\alpha ghwf(b) - J_a^\alpha hwf(b)J_a^\alpha gwf(b). \end{aligned}$$

In (3.6), we take $U=g, V=h$ and $Q=wf$. Then, we observe that

$$\begin{aligned} E_{zg',\alpha,\omega} &= J_a^\alpha\left[(g-J_a^\alpha gwf(b))(h-J_a^\alpha hwf(b))\,wf\right](b) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(g(t)-J_a^\alpha gwf(b)) \times (h(t)-J_a^\alpha hwf(b))\,wf(t)dt. \end{aligned}$$

Thanks to Cauchy-Shwarz inequality, we obtain

$$\begin{aligned} & E^2_{zg',\alpha,\omega} \\ \leq & \left(\frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(g(t)-J_a^\alpha gwf(b))^2 wf(t)dt\right) \times \left(\frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(h(t)-J_a^\alpha hwf(b))\,wf(t)dt\right) \\ = & J_a^\alpha\left[(g-J_a^\alpha gwf(b))^2 wf\right](b)J_a^\alpha\left[(h-J_a^\alpha hwf(b))^2 wf\right](b) \\ = & E_{(g-J_a^\alpha gwf(b))^2,\alpha,w}E_{(h-J_a^\alpha hwf(b))^2,\alpha,w} \\ = & E_{\left(g-E_{g,\alpha,w}\right)^2,\alpha,w}E_{\left(h-E_{h,\alpha,w}\right)^2,\alpha,w}. \end{aligned} \tag{3.8}$$

Again, taking $g=h$ in (3.1), we get

$$\begin{aligned} E_{zh',\alpha,\omega} &= E_{h^2,\alpha,w} - \left(E_{h,\alpha,w}\right)^2 \\ &= E_{h^2,\alpha,w} - 2\left(E_{h,\alpha,w}\right)^2 + \left(E_{h,\alpha,w}\right)^2. \end{aligned}$$

Thanks to the hypothesis $J_a^\alpha wf(b)=1$, it yields that

$$\begin{aligned} E_{zh',\alpha,\omega} &= E_{h^2,\alpha,w} - 2\left(E_{h,\alpha,w}\right)^2 + \left(E_{h,\alpha,w}\right)^2 J_a^\alpha wf(b) \\ &= J_a^\alpha\left(h^2wf\right)(b) - 2E_{h,\alpha,w}J_a^\alpha(hwf)(b) + \left(E_{h,\alpha,w}\right)^2 J_a^\alpha wf(b) \\ &= J_a^\alpha\left(\left[h^2 - 2E_{h,\alpha,w}h + \left(E_{h,\alpha,w}\right)^2\right]wf\right)(b) \\ &= J_a^\alpha\left(\left[h-E_{h,\alpha,w}\right]^2 wf\right)(b) \\ &= E_{\left(h-E_{h,\alpha,w}\right)^2,\alpha,w}. \end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), we deduce that

$$\frac{E^2_{zg',\alpha,w}}{E_{zh',\alpha,w}} \leq E_{[g-E_{g,\alpha,w}]^2,\alpha,w}.$$

The proof is thus achieved.

Remark 3.7. *If we take $\alpha = 1$ and $w(x) = 1$ in (3.7), we obtain (1.3).*

Lemma 3.8. *Let X be a CRV with support an interval $[a,b]$, $-\infty < a < b < \infty$, having a pdf f and let $w : [a,b] \to \mathbb{R}^+$ be a continuous function. Then, for any $\alpha \geq 1$, we have*

$$\begin{aligned} & E_{(g-E_{g,\alpha,w})^2,\alpha,w} \\ = \; & \frac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}(g(x)-g(y))^2 \\ & \times f(x)f(y)w(x)w(y)dxdy, \end{aligned} \tag{3.10}$$

where g is an absolutely continuous function and $J_a^{\alpha} wf(b) = 1$.

Proof.
We have

$$\begin{aligned} & \tfrac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}(g(x)-g(y))^2 \\ & \times f(x)f(y)w(x)w(y)dxdy = \tfrac{1}{[\Gamma(\alpha)]^2}\Big[\int_a^b (b-y)^{\alpha-1} \\ & \times f(y)w(y)\int_a^y (b-x)^{\alpha-1}g^2(x)f(x)w(x)dxdy \\ & + \int_a^b (b-y)^{\alpha-1}g^2(y)f(y)w(y)\int_a^y (b-x)^{\alpha-1} \\ & \times f(x)w(x)dxdy - 2\int_a^b (b-y)^{\alpha-1}g(y)f(y) \\ & \times w(y)\int_a^y (b-x)^{\alpha-1}g(x)f(x)w(x)dxdy\Big]. \end{aligned} \tag{3.11}$$

Then, we can write

$$\begin{aligned} & \int_a^b (b-y)^{\alpha-1}f(y)w(y)\int_a^y (b-x)^{\alpha-1}g^2(x) \\ & \times f(x)w(x)dxdy = \int_a^b (b-y)^{\alpha-1}f(y)w(y)dy \\ & \int_a^b (b-x)^{\alpha-1}g^2(x)f(x)w(x)dx - \int_a^b (b-y)^{\alpha-1} \\ & \times g^2(y)f(y)w(y)\int_a^y (b-x)^{\alpha-1}f(x)w(x)dxdy \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} & \int_a^b (b-y)^{\alpha-1}g(y)f(y)w(y)\int_a^y (b-x)^{\alpha-1}g(x) \\ & \times f(x)w(x)dxdy = \left(\int_a^b (b-y)^{\alpha-1}g(y)f(y)w(y)dy\right)^2 \\ & - \int_a^b (b-y)^{\alpha-1}g(y)f(y)w(y)\int_a^y (b-x)^{\alpha-1} \\ & \times g(x)f(x)w(x)dxdy. \end{aligned} \tag{3.13}$$

By (3.11),(3.12) and (3.13), we obtain

$$\begin{aligned} & \frac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}(g(x)-g(y))^2 \\ & \times f(x)f(y)w(x)w(y)dxdy \\ = \; & \frac{1}{[\Gamma(\alpha)]^2}\Bigg[\int_a^b (b-y)^{\alpha-1}f(y)w(y)dy\int_a^b (b-x)^{\alpha-1}g^2(x) \\ & \times f(x)w(x)dx - \left(\int_a^b (b-y)^{\alpha-1}g(y)f(y)w(y)dy\right)^2\Bigg] \\ = \; & J_a^{\alpha} fw(b) J_a^{\alpha} g^2 fw(b) - (J_a^{\alpha} gfw(b))^2 \\ = \; & J_a^{\alpha} g^2 fw(b) - (J_a^{\alpha} gfw(b))^2 \\ = \; & E_{g^2,\alpha,w} - E^2_{g,\alpha,w} \\ = \; & E_{g^2,\alpha,w} - 2E^2_{g,\alpha,w} + E^2_{g,\alpha,w} \\ = \; & E_{g^2,\alpha,w} - 2E^2_{g,\alpha,w} + J_a^{\alpha} fw(b) E^2_{g,\alpha,w} \\ = \; & E_{(g-E_{g,\alpha,w})^2,\alpha,w}. \end{aligned}$$

Finally, based on Lemma 3.7, we present to the reader the following results:

Theorem 3.9. *Let X be a CRV with support an interval $[a,b]$, $-\infty < a < b < \infty$, having a pdf f and let $w : [a,b] \to \mathbb{R}^+$ be a continuous function. Then, for any $\alpha \geq 1$, the following inequality holds*

$$E_{[g-E_{g,\alpha,w}]^2,\alpha,w} \leq E_{\frac{z(g')^2}{h'},\alpha,w}, \tag{3.14}$$

where g is an absolutely continuous function, with $\left|E_{zg',\alpha,\omega}\right| < \infty$ and $J_a^{\alpha} wf(b) = 1$, h is continuous function on $[a,b]$, with $0 < h'$ and

$$\begin{aligned} z(t) \; = \; & \frac{1}{(b-t)^{\alpha-1}w(t)f(t)} \\ & \times \int_a^t (b-u)^{\alpha-1}w(u)f(u)(E_{h,\alpha,w} - h(u))du. \end{aligned} \tag{3.15}$$

Proof: By Lemma 3.7, we have

$$\begin{aligned} & E_{[g-E_{g,\alpha,w}]^2,\alpha,w} \\ = \; & \frac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}[g(y)-g(x)]^2 \\ & \times f(x)w(x)f(y)w(y)dxdy \\ = \; & \frac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}\left(\int_x^y g'(t)dt\right)^2 \\ & \times f(x)w(x)f(y)w(y)dxdy. \end{aligned} \tag{3.16}$$

Thanks to the condition on h' and by Cauchy-Schwarz inequality, we get

$$\left(\int_x^y g'(t)dt\right)^2 = \left(\int_x^y \sqrt{h'(t)}\left(\frac{g'(t)}{\sqrt{h'(t)}}\right)dt\right)^2 \leq (h(y)-h(x))\int_x^y \frac{[g'(t)]^2}{h'(t)}dt. \quad (3.17)$$

By (3.16) and (3.17), we can write

$$E_{[g-E_{g,\alpha,w}]^2,\alpha,w} \leq \frac{1}{[\Gamma(\alpha)]^2}\int_a^b\int_a^y (b-x)^{\alpha-1}(b-y)^{\alpha-1}(h(y)-h(x)) \times \int_x^y \frac{[g'(t)]^2}{h'(t)}dt f(x)w(x)f(y)w(y)dxdy.$$

On the other hand, since

$$\begin{cases} x\leq t\leq y \\ a\leq x\leq y \\ a\leq y\leq b \end{cases} \Leftrightarrow \begin{cases} a\leq x\leq t \\ t\leq y\leq b \\ a\leq t\leq b \end{cases},$$

then, by changing the order of integration, it yields that

$$E_{[g-E_{g,\alpha,w}]^2,\alpha,w} \leq \frac{1}{[\Gamma(\alpha)]^2}\int_a^b \frac{[g'(t)]^2}{h'(t)}\left[\int_t^b\int_a^t (b-x)^{\alpha-1}(b-y)^{\alpha-1} \times (h(y)-h(x))f(x)w(x)f(y)w(y)dxdy\right]dt.$$

(3.19)

Also, we have

$$\begin{aligned}
&\int_t^b\int_a^t (b-x)^{\alpha-1}(b-y)^{\alpha-1}(h(y)-h(x))f(x)w(x) \\
&\times f(y)w(y)dxdy = \int_a^b\int_a^t (b-x)^{\alpha-1}(b-y)^{\alpha-1} \\
&\times (h(y)-h(x))f(x)w(x)f(y)w(y)dxdy \\
&-\int_a^t\int_a^t (b-x)^{\alpha-1}(b-y)^{\alpha-1}(h(y)-h(x))f(x)w(x) \\
&\times f(y)w(y)dxdy \\
=& \int_a^b (b-y)^{\alpha-1}h(y)f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}f(x)w(x)dx \\
&-\int_a^b (b-y)^{\alpha-1}f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}h(x)f(x)w(x)dx \\
&-\int_a^t (b-y)^{\alpha-1}h(y)f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}f(x)w(x)dx \\
&+\int_a^t (b-y)^{\alpha-1}f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}h(x)f(x)w(x)dx \\
=& \int_a^b (b-y)^{\alpha-1}h(y)f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}f(x)w(x)dx \\
&-\int_a^b (b-y)^{\alpha-1}f(y)w(y)dy\int_a^t (b-x)^{\alpha-1}h(x)f(x)w(x)dx.
\end{aligned} \quad (3.20)$$

Multiplying (3.20) by $\frac{1}{\Gamma(\alpha)}$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_t^b\int_a^t (b-x)^{\alpha-1}(b-y)^{\alpha-1}(h(y)-h(x))f(x)w(x) \\
&\times f(y)w(y)dxdy = J_a^{\alpha}(hfw)(b)\int_a^t (b-x)^{\alpha-1}f(x) \\
&\times w(x)dx - J_a^{\alpha}(fw)(b)\int_a^t (b-x)^{\alpha-1}h(x)f(x)w(x)dx \\
=& J_a^{\alpha}(hfw)(b)\int_a^t (b-x)^{\alpha-1}f(x)w(x)dx \\
&-\int_a^t (b-x)^{\alpha-1}h(x)f(x)w(x)dx \\
=& \int_a^t (b-x)^{\alpha-1}\left[J_a^{\alpha}(hfw)(b)-h(x)\right]f(x)w(x)dx \\
=& \int_a^t (b-x)^{\alpha-1}\left[E_{h,\alpha,w}-h(x)\right]f(x)w(x)dx. \quad (3.21)
\end{aligned}$$

By (3.18) and (3.21), we get

$$\begin{aligned}
&E_{[g-E_{g,\alpha,w}]^2,\alpha,w} \\
\leq& \frac{1}{\Gamma(\alpha)}\int_a^b \frac{[g'(t)]^2}{h'(t)}\left[\int_a^t (b-x)^{\alpha-1}\left[E_{h,\alpha,w}-h(x)\right]f(x)w(x)dx\right]dt \\
=& \frac{1}{\Gamma(\alpha)}\int_a^b \frac{[g'(t)]^2}{h'(t)}\left[(b-t)^{\alpha-1}w(t)f(t)z(t)\right]dt \\
=& J_a^{\alpha}\left(\frac{z(g')^2}{h'}wf\right)(b).
\end{aligned}$$

Theorem 3.9 is thus achieved.

Remark 3.10. *If we take* $\alpha = 1$, *and* $w(x) = 1$ *in (3.14), we obtain (1.4).*

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