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Common fixed point theorem for four weakly compatible self maps with (E.A) property of a G-metric space

V. Kiran*

Abstract

In this paper we establish a common fixed point theorem for four weakly compatible self maps with (E.A)property of a G-metric space.

Keywords

G-Metric space, Fixed point, weakly compatible self maps, (E.A)property, Implicit relation.

AMS Subject Classification

54H25,47H10.

Department of Mathematics, Osmania University, Hyderabad, India.

***Corresponding author**: kiranmathou@gmail.com

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Contents

1. Introduction

Several researchers [2,3,4,9,10,11] generalized metric spaces in different ways. Among all these the G-metric space given by Zead Mustafa and Brailey Sims [10,11]is interesting.

G.Jungck [5,6] first introduced compatible mappings and later Jungck and Rhoades [7] introduced the notion of weakly compatible mappings as a generalization of weakly commuting mappings given by Sessa [8]. Recently, Aamri and Moutawakil [1] introduced the concept of (E.A) property.

In present paper we prove a common fixed point theorem for four weakly compatible self maps with (E.A) property in a G-metric space.

2. Preliminaries

Definition 2.1. *[11] Let X be a non empty set and $G : X^3 \to [0,\infty)$ be a function satisfying*
$(G1)$ $G(x,y,z) = 0$ if $x = y = z$
$(G2)$ $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$
$(G4)$ $G(x,y,z) = G(\sigma(x,y,z))$ for all $x,y,z \in X$ where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$ and
$(G5)$ $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$
Then G is called a G-metric on X and the pair (X,G) is called a G- metric space.

Lemma 2.2. *[11] Let (X,G) be a G-metric space then $G(x,y,y) \leq 2G(y,x,x)$ for all $x,y \in X$*

Definition 2.3. *[11] Let (X,G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be G-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number N for which $G(x_n,x_n,x_0) < \varepsilon$ for all $n \geq N$.*

Definition 2.4. *[11] Let (X,G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be G-Cauchy if for each $\varepsilon > 0$ there exists is a natural number N such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$.*

Note that every G-convergent sequence in a G-metric space (X,G) is G-Cauchy.

Definition 2.5. *[11] A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G)*

Definition 2.6. *Let f,g be two self maps mappings of a metric space (X,G). The pair (f,g) is said to be weakly compatible, if $G(fgx,gfx,gfx) = 0$ whenever $G(fx,gx,gx) = 0$.*

That is the mappings f and g are said to be weakly compatible if they commute at their coincident points.

Definition 2.7. *Let f and g be two self maps of a G-metric space (X,G). We say that f and g satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that*
$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ *for some $t \in X$.*

Example 2.8. *[1] Let $X = [0,\infty)$. Define $f,g : X \to X$ by $fx = \frac{x}{4}$ and $gx = \frac{3x}{4}$ for all $x \in X$*
Consider the sequence $x_n = \frac{1}{n}$.
Clearly $\lim_{n\to infty} fx_n = \lim_{n\to infty} gx_n = 0$
Then f and g satisfy (E.A) property

Example 2.9. *Let $X = [2,\infty)$. Define $f,g : X \to X$ by*
$fx = x+1$ and $gx = 2x+1$ for all $x \in X$
Suppose that property (E.A) holds; then there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to infty} fx_n = \lim_{n\to infty} gx_n = t$ for some $t \in x$.
Therefore $\lim_{n\to infty} x_n = t-1$ and $\lim_{n\to infty} x_n = \dfrac{t-1}{2}$
Then $t = 1$, which is a contradiction since $1 \notin X$.
Hence f and g do not satisfy (E.A) property

Clearly weakly compatible and property (E.A) are independent of each other

Example 2.10. *Let $X = [0,\infty)$. Define $f,g : X \to X$ by*
$fx = 3x-2$ and $gx = x^2$ for all $x \in X$
Then f,g satisfy the property (E.A) for the sequence $x_n = \frac{1}{n}+1, n \geq 1$. But f and g are not weakly compatible since they coincide at the points 1 and 2 but they do not commute at 2

Definition 2.11. *A function $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ which is continuous and increasing in each co-ordinate with $\phi(t,t,t,t,t) < t$ for every $t \in \mathbb{R}^+$ is called an Implicit relation*
The set of all implicit relations is denoted by Φ

3. Main Results

Theorem 3.1. *Let f,g,h, and p be selfmaps of a G-metric space(X,G) satisfying the following conditions*

(i) $f(X) \subseteq h(X)$ and $g(X) \subseteq p(X)$

(ii) one of $f(X),g(X),h(X)$ and $p(X)$ is closed subset of X.

(iii)
$$G(fx,gy,gy) \leq \phi\Big(G(px,hy,hy),G(fx,px,px),G(hy,gy,gy), G(px,gy,gy),G(fx,hy,hy)\Big)$$
for every $x,y \in X$ and $\phi \in \Phi$

(iv) The pairs $(f,p)$and (g,h) are weakly compatible

(v) The pairs (f,p) or (g,h) satisfies the property $(E.A)$

Then f,g,h and p have a unique common fixed point in X

Proof. We first prove the existence of a common fixed point in one of the two cases of the condition (v) and the other case follows similarly with appropriate changes. Here we prove in case (g,h) satisfies the property $(E.A)$. Then there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = z$ for some $z \in X$
Since $g(X) \subseteq p(X)$, there exists $\{y_n\}$ in X such that
$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} py_n = z$
We now prove $\lim_{n\to\infty} fy_n = z$
Let $\lim_{n\to\infty} fy_n = l$ and if $l \neq z$ then $G(l,z,z) \neq 0$
From (iii)of the Theorem 3.1,we have

$$G(fy_n,gx_n,gx_n) \leq \phi\Big(G(py_n,hx_n,hx_n),G(fy_n,py_n,py_n),G(hx_n,gx_n,gx_n), G(py_n,gx_n,gx_n),G(fy_n,hx_n,hx_n)\Big)$$

on letting $n \to \infty$ we get

$$\begin{aligned} G(l,z,z) &\leq \phi\Big(G(z,z,z),G(l,z,z),G(z,z,z),G(z,z,z),G(l,z,z)\Big) \\ &= \phi\Big(0,G(l,z,z),0,0,G(l,z,z)\Big) \\ &\leq \phi\Big(G(l,z,z),G(l,z,z),G(l,z,z),G(l,z,z),G(l,z,z)\Big) \\ &< G(l,z,z) \end{aligned}$$

which is a contradiction since $l = z$
Hence we have $\lim_{n\to\infty} fy_n = z$.
Suppose $p(X)$ is closed subset of X then there exists u in X such that $pu = z$
Therefore we have
$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} py_n = z = pu$
Now from (iii)of the Theorem 3.1,we have

$$G(fu,gx_n,gx_n) \leq \phi\Big(G(pu,hx_n,hx_n),G(fu,pu,pu),G(hx_n,gx_n,gx_n), G(pu,gx_n,gx_n),G(fu,hx_n,hx_n)\Big)$$

on letting $n \to \infty$ we get

$$\begin{aligned} G(fu,z,z) &\leq \phi\Big(G(z,z,z),G(fu,z,z),G(z,z,z), G(z,z,z),G(fu,z,z)\Big) \\ &= \phi\Big(0,G(fu,z,z),0,0,G(fu,z,z)\Big) \\ &\leq \phi\Big(G(fu,z,z),G(fu,z,z),G(fu,z,z), G(fu,z,z), G(fu,z,z)\Big) \\ &< G(fu,z,z) \end{aligned}$$

which is a contradiction hence $fu = z$
Therefore $fu = pu = z$
Since $f(X) \subseteq h(X)$, then there exists a point v in X such that $fu = hv = z$.
Now we claim that gv=z.
If $gv \neq z$ then $G(z, gv, gv) > 0$
From (iii)of the Theorem 3.1,we have

$$G(fu, gv, gv) \leq \phi\Big(G(pu, hv, hv), G(fu, pu, pu), G(hv, gv, gv), G(pu, gv, gv), G(fu, hv, hv)\Big).$$

$$\begin{aligned} G(z, gv, gv) &\leq \phi\Big(G(z,z,z), G(z,z,z), G(z, gv, gv), G(z, gv, gv), G(z,z,z)\Big) \\ &= \phi\Big(0, 0, G(z, gv, gv), G(z, gv, gv), 0)\Big) \\ &< \phi\Big(G(z, gv, gv), G(z, gv, gv), G(z, gv, gv), G(z, gv, gv), G(z, gv, gv)\Big) \\ &< G(z, gv, gv) \end{aligned}$$

which is a contradiction,hence $gv = z$,thus $hv = gv = z$.
Since the pair (f, p) is weakly compatible then $fpu = pfu$ implies $fz = pz$.
We now show that$fz = z$.
If $fz \neq z$ then $G(fz, z, z) > 0$.
From(iii) of the Theorem 3.1, we have

$$G(fz, gv, gv) \leq \phi\Big(G(pz, hv, hv), G(fz, pz, pz), G(hv, gv, gv), G(pz, gv, gv), G(fz, hv, hv)\Big)$$

$$\begin{aligned} G(fz, z, z) &\leq \phi\Big(G(fz, z, z), G(fz, fz, fz), G(z,z,z), G(fz, z, z), G(fz, z, z)\Big) \\ &= \phi\Big(G(fz, z, z), 0, 0, G(fz, z, z), G(fz, z, z)\Big) \\ &\leq \phi\Big(G(fz, z, z), G(fz, z, z), G(fz, z, z), G(fz, z, z), G(fz, z, z)\Big) \\ &< G(fz, z, z) \end{aligned}$$

which is a contradiction,hence $fz = z$,
thus $fz = pz = z$.
Proving that z is common fixed point of f and p.
Since the pair (g, h) is weakly compatible then $ghv = hgv$ implies $gz = hz$.

We now prove that $gz = z$
If $gz \neq z$ then $G(z, gz, gz) > 0$
From(iii) of the Theorem 3.1, we have

$$G(fz, gz, gz) \leq \phi\Big(G(pz, hz, hz), G(fz, pz, pz), G(hz, gz, gz), G(pz, gz, gz), G(fz, hz, hz)\Big)$$

$$\begin{aligned} G(z, gz, gz) &\leq \phi\Big(G(z, gz, gz), G(z,z,z), G(gz, gz, gz), G(z, gz, gz), G(z, gz, gz)\Big) \\ &= \phi\Big(G(z, gz, gz), 0, 0, G(z, gz, gz), G(z, gz, gz)\Big) \\ &\leq \phi\Big(G(z, gz, gz), G(z, gz, gz), G(z, gz, gz), G(z, gz, gz), G(z, gz, gz)\Big) \\ &< G(z, gz, gz) \end{aligned}$$

which is a contradiction,hence $gz = z$,
thus $gz = hz = z$.
Proving that z is common fixed point of g and h.
Showing that z is a common fixed point of f, g, h and p.
The proof is similar in the other cases of the condition (ii) with appropriate changes.

Uniqueness: Let w be the another common fixed point of f, g, h and p.
If $z \neq w$ then $G(z, w, w) > 0$
From (iii)of the Theorem 3.1,we have

$$\begin{aligned} G(z, w, w) &= G(fz, gw, gw) \\ &\leq \phi\Big(G(pz, hw, hw), G(fz, pz, pz), G(hw, gw, gw), G(pz, gw, gw), G(fz, hw, hw)\Big) \\ &\leq \phi\Big(G(z, w, w), G(z,z,z), G(w, w, w), G(z, w, w), G(z, w, w)\Big) \\ &= \phi\Big(G(z, w, w), 0, 0, G(z, w, w), G(z, w, w)\Big) \\ &\leq \phi\Big(G(z, gz, gz), G(z, gz, gz), G(z, gz, gz), G(z, gz, gz), G(z, gz, gz)\Big) \\ &< G(z, gz, gz) \end{aligned}$$

which is a contradiction,hence $w = z$.
Therefore z is the the unique common fixed point of f, g, h and p. □

Corollary 3.2. *Let f, g and p be self maps of a G-metric space(X, G)*
satisfying the following conditions

(i) $f(X) \subseteq p(X)$ and $g(X) \subseteq p(X)$

(ii) one of $f(X), g(X)$ and $p(X)$ is closed subset of X.

(iii) $G(fx,gy,gy) \leq \phi\Big(G(px,py,py),G(fx,px,px),G(py,gy,gy), G(px,gy,gy),G(fx,py,py)\Big)$

for every $x,y \in X$ *and* $\phi \in \Phi$

(iv) The pairs (f,p)*and* (g,p) *are weakly compatible*

(v) The pairs (f,p) *or* (g,p) *satisfies the property* $(E.A)$

Then f,g *and* p *have a unique common fixed point in* X

Proof. On taking $h=p$ in the Theorem 3.1, the corollary follows. □

Corollary 3.3. *Let* f *and* p *be self maps of a G-metric space* (X,G) *satisfying the following conditions*

(i) $f(X) \subseteq p(X)$

(ii) one of $f(X)$ *and* $p(X)$ *is closed subset of* X.

(iii) $G(fx,fy,fy) \leq \phi\Big(G(px,py,py),G(fx,px,px),G(py,fy,fy), G(px,fy,fy),G(fx,py,py)\Big)$

for every $x,y \in X$ *and* $\phi \in \Phi$

(iv) The pair (f,p) *is weakly compatible*

(v) The pair (f,p) *satisfies the property* $(E.A)$

Then f *and* p *have a unique common fixed point in* X

Proof. On taking $h=p$ and $f=g$ in the Theorem 3.1, the corollary follows □

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