



Approximate controllability of impulsive neutral integro-differential equations with deviated arguments and infinite delay in Banach spaces

Venkatesh Usha^{1*} and Mani Mallika Arjunan²

Abstract

In this paper we discussed about the approximate controllability of impulsive neutral integro-differential equations with deviated arguments and infinite delay. In the nonlinear term, we introduce the control parameter. The invertibility of the operator in infinite dimensional spaces is not possible if the generated semigroup is compact. So we remove to assume that concept and there is no need of Lipschitz continuity of nonlinear function. Finally suitable example is also given.

Keywords

Controllability, Impulsive differential equations, Fixed point theorem, Neutral equations, Nonlinear equations.

AMS Subject Classification

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¹Department of Mathematics, PSG College of Arts and Science, Coimbatore - 641 014, Tamil Nadu, India.

²Department of Mathematics, C. B. M. College, Kovaipudur, Coimbatore - 641 042, Tamil Nadu, India.

*Corresponding author: ¹ usha.vsr@gmail.com; ² arjunphd07@yahoo.co.in

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1. Introduction

In this paper we establish the approximate controllability of impulsive neutral integro-differential equations with deviated arguments and infinite delay in a Banach space $(E, \|\cdot\|)$ through the utilization of a fixed point theorem. We discuss

the corresponding model

$$\begin{aligned} \frac{d}{dt} \left[x_t + \mathcal{G} \left(t, x_t, \int_0^t a_1(t, s, x_s) ds \right) \right] &= -A(t)x(t) \\ &+ Bu(t) + \mathcal{F} \left(t, u(t), x(a(x(t), t)) \right) \\ &+ \overline{\mathcal{G}} \left(t, x_t, \int_0^t a_2(t, s, x_s) ds \right), \\ t \in J &= (0, T], t \neq t_i, i = 1, \dots, N, \quad (1.1) \\ x_0(t) &= \phi(t) \in (-\infty, 0], \quad (1.2) \\ \Delta x(t_i) &= I_i(x(t_i)), i = 1, 2, \dots, \delta, \quad (1.3) \end{aligned}$$

where $-A(t) : D(A(t)) \subseteq E \rightarrow E$ is an infinitesimal generator of an analytic semigroup of bounded linear operators on E . B is a bounded linear operator from U to E . The control function $u(\cdot) \in L^2(J, U)$, where U is a Banach space. $D = t_1, t_2, \dots, t_m \subset J = [0, T], 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$. The time history $x_t : (-\infty, 0] \rightarrow E, x_t(\theta) = x(t + \theta)$ lies in the some abstract phase space \mathcal{B}_h defined in [4]. Consider $h_0 : (-\infty, 0] \rightarrow \mathbb{R}$, a continuous function such that $l = \int_{-\infty}^0 h_0(t) dt \leq \infty$. Then $\mathcal{B}_h = \{ \phi : (-\infty, 0] \rightarrow E \text{ is such that for all } b > 0, \|\phi(\theta)\| \text{ is a bounded measurable function on } [-b, 0], \|\phi\|_{[-b, 0]} =$

$\sup_{s \leq \theta \leq 0} \|\phi(\theta)\|$ and $\int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} \|\phi(\theta)\| ds < \infty$. Obviously \mathcal{B}_h is a Banach space with norm $\int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} \|\phi\| ds < \infty, \phi \in \mathcal{B}_h$. The function $\mathcal{F} : J \times E \times E \rightarrow E$ and the functions $\mathcal{G}, \overline{\mathcal{G}} : J \times \mathcal{B}_h \times E \rightarrow E$. The function $a : E \times J \rightarrow E$ and $I_i : E \rightarrow E$ where $i = 1, 2, \dots, \delta$ are appropriate functions. $\phi : [-\infty, 0] \rightarrow E$ is a given Lipschitz continuous function.

The semigroup of bounded linear operators concept is absolutely significant to managing with differential and integro-differential equations in Banach spaces. Lately, for a particular generous type of Impulsive differential equations with deviated arguments in Banach spaces, this concept has been utilized. For more details about semigroup theory and fractional power of operators we refer to [25]. Differential equations involving a variable as well as unknown function and its derivatives, which are taken, generally speaking for differential values of the variable (as distinguished from ordinary differential equations). Such conditions showed up toward the end of the eighteenth century. They were investigated over and again both for their own purpose and regarding taking care of geometric problems. Later on the theory of differential equations with deviating argument is an dominant part of the more extensive theory of functional differential and functional integral equations. These speculations make exceptionally noteworthy branches of nonlinear analysis and final numerous applications in physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors and in other fields of engineering and neutral sciences. One of the fundamental issues considered in the hypothesis of differential equations with deviated arguments is to set up advantageous conditions. The outcome, we will demonstrate in this paper sums up a few ones acquired before in [2, 13, 14].

If any state vector may be steered arbitrarily close to another state vector in a control system, then we call it as approximate controllability of a control system. There are only few papers discussing the controllability of differential equations with deviated arguments on infinite dimensional space. This paper studies the approximate controllability for the nonlinear control system described by integro-differential equation with deviated arguments. The control parameter being inside the nonlinear term. Only Schauder fixed point theorem and a few fundamental hypothesis are used to prove our result. The assumption of the existence of inverse of controllability operator is not required. Since its inverse does not exist in infinite dimensional space if the associated semigroup is compact. Lipschitz continuity of the nonlinear function is also not required.

Controllability theory of nonlinear and nonlinear control problems in infinite and finite dimensional spaces has been studied by many researchers and the details can be found in various papers and monographs see [20, 21]. A control system represented by an abstract neutral differential equation with deviated argument was studied by Muslim et al. [21]. They used semigroup theory of linear operators to study the complete controllability of the given system. For the initial studies

on the control theory of various kind of abstract differential equations, we refer to [5–7, 21, 23].

The approximate controllability of non autonomous non local finite delay differential equations with deviating arguments in a Hilbert space, by establishing sufficient conditions for the existence of mild solutions was studied by Haloi [18] and also Muslim et al. [22]. Without imposing additional assumptions such as analyticity and compactness conditions on the generated semigroup and the nonlinear term, the results were obtained by Sanjukta Das et al.[11]. Pandey et al. [24] used the analytic semigroup theory and fixed point arguments to study the existence and uniqueness of mild solutions of a neutral differential equation with a deviated argument in a Banach space. Lateron Wang et al.[27] proved the approximate controllability of sobolov type fractional evolution system with nonlocal conditions.

The papers of Benchohra et al.[1], Y.K. Chang [4] and Muslim et al.[21] discussed the exact controllability of functional systems with infinite delay. However, in these papers the invertibility of a controllability operator is assumed as a consequence their approach fails in infinite dimensional spaces whenever the generated semigroup is compact.

Equations with a deviating arguments describe many processes with an after effect; such equations appear, for example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment. The presence of a deviation-delay in the systems studied often turns out to be the cause of phenomena substantially affecting the motion of the process. For example, in automatic regulators, the delay is the interval of time, always present, which the system needs to react to the input impulse. The plentiful applications of differential equations with deviating arguments has motivated the rapid development of the theory of differential equations with deviating arguments and their generalization in the recent years see [13, 14, 16, 19, 26]. Extension of the theory of differential equations with deviating argument as well as stimuli of developments within various fields of science and technology contribute to the need for further development. This theory in recent years has attracted the attention of vast number of researchers, interested in both in the theory and its applications. For more details, we refer [2, 3, 8, 9, 16, 17]

By the motivation of above mentioned literature we have proved the existence of mild solutions for an impulsive neutral integro-differential equation with infinite delay and with deviated argument in a Banach space $(E, \|\cdot\|)$ through the utilization of the Schauder fixed point theorem. In section 2, we gave some definitions, preliminaries, some lemmas and theorems. In section 3 we have presented the existence results for the structure (1.1) – (1.3) under Banach Contraction principle and Schauder fixed point theorem. Finally we have provided the example based on the proof.



2. Preliminaries

In this section for our convenience we state some basic definitions, preliminaries, lemmas and assumptions that will be used to prove our main results. We briefly outlined the facts concerning analytic, fractional powers of operators. For further details, we follow the monographs [12, 25]. Let E and U be Banach spaces. The state function $x(\cdot)$ takes values in Banach space E . The control function $u(\cdot) \in L^2(J, U)$, where U is a Banach space.

Here we define few notations that are used in the following sections. $M = \sup \|S(t) : 0 \leq t \leq T\|$, $M_b = \|B\|$, $\|\alpha\| = \int_0^b |\alpha_i(s)| ds$ and let $\lim_{r \rightarrow \infty} \frac{\pi_i(r)}{r} = \eta$. We define the space \mathcal{B}'_h where

$$\mathcal{B}'_h = \left\{ x : (-\infty, T] \rightarrow E : x_k \in C(J_k, E) \text{ and there exists } x(t_k^-) \right.$$

$$\left. \text{and } x(t_k^+), \text{ with } x(t_k) = x(t_k^-), x_0 = \phi \in \mathcal{B}_h, k = 0, 1, \dots, m \right\}$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \dots, m$. Set $\|\cdot\|_l$ be a seminorm in \mathcal{B}'_h defined by

$$\|x\|_l = \|x_0\|_{\mathcal{B}'_h} + \sup\{|x(s)| : 0 \leq s \leq t\}, x \in \mathcal{B}'_h$$

We define the following operators, $\Gamma_0^T = \int_0^T S(T-s) BB^* S^*(T-s) ds$ and $R(\lambda, \Gamma_0^T) = (\lambda I + \Gamma_0^T)^{-1}$.

Definition 2.1. The system is said to be controllable on the interval J if for every $x_0, x_T \in E$, there exists a control $u \in L^2(J, U)$ such that the mild solution of (1.1) to (1.3) satisfies $x(0) = x_0, x(T) = x_T$.

Lemma 2.2 ([10]). Assume $x \in \mathcal{B}'_h$ be a function and for $t \in J$, $x_t \in (-\infty, 0]$, then

$$l\|x(t)\| \leq \|x_t\|_{(-\infty, 0]} \leq \|\phi\|_{(-\infty, 0]} + l \sup_{s \in [0, t]} \|x(s)\|.$$

In order to study the existence results for the problem (1.1) to (1.3) we need to list the following hypotheses:

(H1) A generates a strongly continuous semigroup $S(t)$ in the Banach space E and there exists a constant M such that $\|S(t)\| \leq M$.

(H2) The nonlinear map $\mathcal{F} : [0, T] \times E \times U \rightarrow E$ and there exists a positive constant $\alpha(\cdot) \in L_1(J, R^+)$ and a non-decreasing function $f_i \in L_1(C \times U, R^+)$ and $i = 1, 2, \dots, m$ such that

$$\|\mathcal{F}(t, x, u)\| \leq \sum_{i=1}^m \alpha_i(t) f_i(x, u), \forall (t, x, u) \in J \times E \times U$$

(H4) The function $\mathcal{G} : [0, T] \times B \times E \rightarrow E$ is completely continuous and there exists a positive constant M_G such that

$$\|\mathcal{G}(t, \varphi_1, \varphi_2)\| \leq M_G [1 + \|\varphi_1\|_{\mathcal{B}'_h} + \|\varphi_2\|_E],$$

(H5) There exists positive constant M_G , the function $\overline{\mathcal{G}} : [0, T] \times B \times E \rightarrow E$ is completely continuous and uniformly bounded such that

$$\|\overline{\mathcal{G}}(t, z, w)\| \leq M_G [1 + \|z\|_{\mathcal{B}'_h} + \|w\|_E],$$

(H6) There exist positive constants M_{a_i} where $i = 1, 2$ and the map $a_i : J \times J \times B \rightarrow E$ where $\{(t, s) \in J \times J : t \geq s\}$ are continuous such that

$$\|a_i(t, s, x)\| \leq M_{a_i} [1 + \|x\|_{\mathcal{B}'_h}],$$

(H7) The functions $I_k : E_\alpha(t_0) \rightarrow D(A)$ where $k = 1, 2, \dots, m$ are continuous functions and there exist positive constants M_I to ensure that

$$\|I_k(x(t_k)) - I_k(y(t_k))\| \leq M_I \|x(t_k) - y(t_k)\|, \\ k = 1, 2, \dots, m, \forall x(t_k), y(t_k) \in J_k = J\{t_1, t_2, \dots, t_m\}$$

(H8) The functions I_k are continuous where $k = 1, 2, \dots, m$ and there exist a positive constant d_k such that

$$\|I_k(z)\| \leq \beta_k, k = 1, 2, \dots, m, \forall z \in E$$

$$\text{and } \sum_{k=1}^m \beta_k = \beta$$

(H9) $\lambda R(\lambda, \Gamma_0^T) \rightarrow 0$ as $\lambda \rightarrow 0^+$

(H10) There exists a Banach space $(U, \|\cdot\|_U)$ continuously included in E such that $AS(t) \in L(U, E)$ for all $t \in J$ and $S(\cdot)x \in C(J; E)$, for every $x \in U$. There exists a constant \tilde{M}_G such that $\|AS(t)\|_{L(U, E)} \leq \tilde{M}_G$ for all $t \in J$.

Definition 2.3. A piecewise continuous function $u(\cdot) : (-\infty, T] \rightarrow E$ is called a mild solution for the problem (1.1) – (1.3) if $u(t) = \phi(t)$, $t \in (-\infty, 0]$, $\Delta u(t_i) = I_i(u(t_i))$, $i = 1, 2, \dots, \delta$ and $u(\cdot)$ satisfies the integral equation

$$x(t) = \left\{ \begin{array}{l} S(t, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ + \mathcal{G}\left(t, x_t, \int_0^t a_1(t, s, x_s) ds\right) \\ + \int_0^t S(t, \tau) \mathcal{A}(\tau) \\ \quad \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right. \\ \left. + Bu(\tau) \right] d\tau + \sum_{0 < t_i < t} S(t, t_i) I_i(x(t_i)) \end{array} \right.$$

for each $t \in [0, T]$.



3. Approximate Controllability results

In this section we present the existence results for the structure (1.1) – (1.3) under Banach contraction principle and Schauder fixed point theorem.

Definition 3.1. We define the mild solutions of (1.1) as $x(\cdot) \in \mathcal{PC} \times C(J, U)$, $x_0 = \phi$ and which satisfies the following integral equation

$$x(t) = \begin{cases} S(t, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ -\mathcal{G}\left(t, x_t, \int_0^t a_1(t, s, x_s) ds\right) + \int_0^t S(t, \tau) \\ \quad \mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right. \\ \left. + Bv(\tau) \right] d\tau + \sum_{0 < t_i < t} S(t, t_i)I_i(x(t_i)) \end{cases} \quad (3.1)$$

For $\lambda > 0$ we define an operator $\Psi_\lambda(x, u) = (\tilde{x}, v)$ on $\mathcal{PC} \times C(J, U)$ where

$\mathcal{PC}([-\infty, T], E_\alpha(t_0)) = \mathcal{PC}^\alpha = \{x : [-\infty, T] \rightarrow E_\alpha(t_0) : x \text{ is continuous at } t \neq t_i \text{ and left continuous at } t = t_i \text{ and } x(t_i^+) \text{ exists, for all } i = 1, 2, \dots, \delta\}$,

with norm $\|x\|_{\mathcal{PC}^\alpha} = \sup_{t \in [-\infty, T]} \|x(t)\|_\alpha$ is also a Banach

space and $E_\alpha(t_0)$ be a Banach subspace for some $0 < \alpha < 1$ and $t_0 \in [0, T]$.

$$v(t) = B^*S^*(T, t)\mathcal{R}(\lambda, \Gamma_0^T)p(x, u) \quad (3.2)$$

$$z(t) = \begin{cases} S(t, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ -\mathcal{G}\left(t, x_t, \int_0^t a_1(t, s, x_s) ds\right) + \int_0^t S(t, \tau) \\ \quad \mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right. \\ \left. + Bv(\tau) \right] d\tau + \sum_{0 < t_i < t} S(t, t_i)I_i(x(t_i)) \end{cases} \quad (3.3)$$

where

$$p(x(\cdot)) = x_T - S(T, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ + \mathcal{G}\left(T, x_T, \int_0^T a_1(T, s, x_s) ds\right) - \int_0^T S(T, \tau) \\ \quad \mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ - \int_0^T S(T, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau$$

$$- \int_0^T S(T, \tau)\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \\ - \sum_{0 < t_i < t} S(T, t_i)I_i(x(t_i)) \quad (3.4)$$

The system (1.1) is approximately controllable if for all $\lambda > 0$ there exists a fixed point of the operator Ψ^λ which is the mild solution of the system (1.1). For any arbitrary $h_1 \in E$, the control

$$u(t) = B^*S^*(T, t)\mathcal{R}(\lambda, \Gamma_0^T)p(x, u) \quad (3.5) \\ p(x, u) = h_1 - S(T, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ + \mathcal{G}\left(T, x_T, \int_0^T a_1(T, s, x_s) ds\right) - \int_0^T S(T, \tau) \\ \quad \times \mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \delta, x_\delta) d\delta\right) d\tau \\ - \int_0^T S(T, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ - \int_0^T S(T, \tau)\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \quad (3.6)$$

transfers initial state x_0 to

$$z(t) = h_1 - \lambda(\lambda I + \Gamma_0^T)^{-1} \left[h_1 - S(T, 0)[\phi(0) \right. \\ \left. + \mathcal{G}(0, \phi(0), 0)] + \mathcal{G}\left(T, x(T), \int_0^T a_1(T, s, x_s) ds\right) \right. \\ \left. - \int_0^T S(T, \tau)\mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \delta, x_\delta) d\delta\right) d\tau \right. \\ \left. - \int_0^T S(T, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \right. \\ \left. - \int_0^T S(T, \tau)\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \right] \quad (3.7)$$

Proof. By substituting (3.2) and (3.3) in

$$z(t) = S(t, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ - \mathcal{G}\left(t, x_t, \int_0^t a_1(t, s, x_s) ds\right) \\ + \int_0^t S(t, \tau)\mathcal{A}(\tau)\mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau)\overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right. \\ \left. + Bv(\tau) \right] d\tau \quad (3.8)$$



and at $t = T$ the value of $z(t)$,

$$\begin{aligned}
 z(T) &= S(T, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\
 &\quad - \mathcal{G}\left(T, x(T), \int_0^T a_1(T, s, x_s) ds\right) \\
 &\quad + \int_0^T S(t, \tau) \mathcal{A}(\tau) \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad + \int_0^T S(T, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad + \int_0^T S(T, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right] \\
 &\quad + BB^* S^*(T-t) \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} p(x, u) \Big] d\tau \\
 &= S(T, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\
 &\quad - \mathcal{G}\left(T, x(T), \int_0^T a_1(T, s, x_s) ds\right) \\
 &\quad + \int_0^T S(t, \tau) \mathcal{A}(\tau) \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad + \int_0^T S(T, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad + \int_0^T S(T, \tau) \mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \\
 &\quad + \Gamma_0^T \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} p(x, u) \\
 &= h_1 - p(x, u) + \Gamma_0^T \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} p(x, u) \\
 &= h_1 - p(x, u) + \left[I - \lambda \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} \right] p(x, u) \\
 &= h_1 - p(x, u) + p(x, u) - \lambda \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} p(x, u) \\
 z(T) &= h_1 - \lambda \mathcal{R}(\lambda, \Gamma_0^T)^{-1} p(x, u)
 \end{aligned}
 \tag{3.9}$$

The control $u(t) = B^* S^*(T, t) \mathcal{R}(\lambda I + \Gamma_0^T)^{-1} p(x, u)$ transfers the initial state x_0 to

$$\begin{aligned}
 z_T &= h_1 - \lambda \mathcal{R}(\lambda, \Gamma_0^T)^{-1} \left[h_1 - S(T, 0)[\phi(0) \right. \\
 &\quad + \mathcal{G}(0, \phi(0), 0)] + \mathcal{G}\left(T, x_T, \int_0^T a_1(T, s, x_s) ds\right) \\
 &\quad - \int_0^T S(T, \tau) \mathcal{A}(\tau) \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \delta, x_\delta) d\delta\right) d\tau \\
 &\quad - \int_0^T S(T, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad \left. - \int_0^T S(T, \tau) \mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \right]
 \end{aligned}$$

Hence proved. \square

Theorem 3.2. Assume that the hypotheses (H1)-(H9) hold, then for all $0 < \alpha \leq 1$ the system (1.1) has a mild solution on

J and for $\phi(t) \in \mathcal{D}(A)$ for each $t \in [-\infty, 0]$,

$$\begin{aligned}
 &\left[1 + \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \right] \left\{ \left[\mathcal{M}_G l + \mathcal{M} \mathcal{M}_0 \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \right. \\
 &\left. \left[1 + \mathcal{M}_{a_1} T \right] + \mathcal{M} \mathcal{M}_G T \mathcal{M}_G l \left[1 + \mathcal{M}_{a_2} T \right] \right. \\
 &\left. + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \eta \right\} < 1
 \end{aligned}$$

Proof. We define

$$\begin{aligned}
 W_{r_0} &= \{ (x(\cdot), u(\cdot)) \in \mathcal{P}\mathcal{C}([-\infty, T], E(t_0)) \times C(J \times U) \\
 &\quad : \|x_t\| + \|v(t)\| \leq r_0 \},
 \end{aligned}
 \tag{3.10}$$

where r_0 is a positive constant. W_{r_0} is a closed convex subset of a Banach space $\mathcal{P}\mathcal{C}([-\infty, T], E(t_0)) \times C(J \times U)$.

Step 1:

For $0 < \lambda \leq 1$, there is a positive constant $r_0 = r_0(\lambda)$ such that $\Psi^\lambda : W_{r_0} \rightarrow W_{r_0}$.

$$\Psi^\lambda(x, u)(t) = (\tilde{x}, v)$$

If $(x, v) \in W_{r_0}$, we have $\|v(t)\| =$

$$\begin{aligned}
 &B^* S^*(T, t) \mathcal{R}(\lambda, \Gamma_0^T) \times \left[x_T - S(T, 0)[\phi(0) \right. \\
 &\quad + \mathcal{G}(0, \phi(0), 0)] + \mathcal{G}\left(T, x_T, \int_0^T a_1(T, s, x_s) ds\right) \\
 &\quad - \int_0^T S(T, \tau) \mathcal{A}(\tau) \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \delta, x_\delta) d\delta\right) d\tau \\
 &\quad - \int_0^T S(T, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\
 &\quad - \int_0^T S(T, \tau) \mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) d\tau \\
 &\quad \left. - \sum_{0 < t_i < t} S(T - t_i) I_i(x(t_i)) \right]
 \end{aligned}$$

$$z(t) = \begin{cases} \phi, & t \in (-\infty, 0]; \\ S(t, 0)[\phi(0) + \mathcal{G}(0, \phi(0), 0)] \\ - \mathcal{G}\left(t, x_t, \int_0^t a_1(t, s, x_s) ds\right) \\ + \int_0^t S(t, \tau) \mathcal{A}(\tau) \\ \mathcal{G}\left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \overline{\mathcal{G}}\left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta\right) d\tau \\ + \int_0^t S(t, \tau) \left[\mathcal{F}\left(\tau, u(\tau), x(a(x(\tau), \tau))\right) \right] \\ + Bv(\tau) \Big] d\tau + \sum_{0 < t_i < t} S(t, t_i) I_i(x(t_i)) \end{cases}$$

(3.11)



We express the function $\Phi(\cdot)$ defined by

$$\Phi(t) = \begin{cases} S(t, 0)\phi(0), & \text{if } t \in [0, T], \\ \phi(t), & \text{if } t \in (-\infty, 0]. \end{cases}$$

We can decompose $x(t) = \Phi(t) + \tilde{x}(t)$, $t \in (-\infty, T]$

where $\tilde{x} = 0$, $t \leq 0$ and for $t > 0$

$\tilde{x}(t) =$

$$\begin{aligned} & S(t, 0)[\mathcal{G}(0, \phi(0), 0)] \\ & - \mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right) \\ & + \int_0^t \mathcal{A}(\tau)S(t, \tau) \times \\ & \mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau \\ & + \int_0^t S(t, \tau) \times \\ & \overline{\mathcal{G}}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau \\ & + \int_0^t S(t, \tau) \times \\ & \left[\mathcal{F}\left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau))\right)\right. \\ & \left. + Bu(\tau)\right] d\tau \\ & + \sum_{0 < t_i < t} S(t, t_i)I_i(\tilde{x}(t_i) + \Phi(t_i)), \quad t \in [0, T]. \end{aligned} \quad (3.12)$$

We can define

$$\mathcal{W}_{r_0}^\alpha = \{(\tilde{x}(\cdot), u(\cdot)) \in \mathcal{PC}([-\infty, T], E(t_0)) \times C(J \times U) : \|\tilde{x}\| + \|u(t)\| \leq r_0 \text{ and } \tilde{x}(0) = 0\}$$

For further proof, we need the following estimation.

$$\begin{aligned} & \|\tilde{x}_t + \Phi_t\|_{\mathcal{B}_h} \\ & \leq \|\tilde{x}_t\|_{\mathcal{B}_h} + \|\Phi\|_{\mathcal{B}_h} \\ & \leq l \sup_{s \in [0, t]} \|\tilde{x}(s)\| + \|\tilde{x}_0\|_{\mathcal{B}_h} + l \sup_{s \in [0, t]} \|\Phi(s)\| + \|\phi\|_{\mathcal{B}_h} \\ & \leq lr_0 + lM\|\phi(0)\| + \|\phi\|_{\mathcal{B}_h} \\ & \leq lr_0 + q \end{aligned}$$

where $q = lM\|\phi(0)\| + \|\phi\|_{\mathcal{B}_h}$.

Let $\pi_i(r_0) = \sup \{f_i(x(a(x(s), s)), v) : \|x\| + \|v(t)\| \leq r_0, \forall(x, v) \in \mathcal{PC}((-\infty, T], E(t_0)) \times C(J \times U)\}$ and $(x, v) \in W_{r_0}$

then

$\|v(t)\| =$

$$\begin{aligned} & \left\|B^*S^*(T, t)\mathcal{R}(\lambda, \Gamma_0^T) \times \left[x_T - S(t, 0)[\mathcal{G}(0, \phi(0), 0)]\right.\right. \\ & \left.+\mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right) - \int_0^t \mathcal{A}(\tau)\right. \\ & \left.\times S(t, \tau)\mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau\right. \end{aligned}$$

$$\begin{aligned} & \left. - \int_0^t S(t, \tau)\overline{\mathcal{G}}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau\right. \\ & \left. - \int_0^t S(t, \tau)\left[\mathcal{F}\left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau))\right)\right.\right. \\ & \left. - \sum_{0 < t_i < t} S(t, t_i)I_i(\tilde{x}(t_i) + \Phi(t_i))\right]\Bigg\|, \quad t \in [0, T]. \\ & \leq \|B^*\| \|S^*(T, t)\| \|\mathcal{R}(\lambda, \Gamma_0^T)\| \\ & \quad \times \left[\|x_T - S(t, 0)[\mathcal{G}(0, \phi(0), 0)]\| + \left\|\int_0^t \mathcal{A}(\tau)S(t, \tau)\right.\right. \\ & \quad \left.\left.\times \mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau\right\| \right. \\ & \quad \left. + \left\|\int_0^t S(t, \tau)\overline{\mathcal{G}}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau\right\| \right. \\ & \quad \left. + \left\|\int_0^t S(t, \tau)\right.\right. \\ & \quad \left.\left.\times \left[\mathcal{F}\left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau))\right)\right] d\tau\right\| \right. \\ & \quad \left. - \left\|\sum_{0 < t_i < t} S(t, t_i)I_i(\tilde{x}(t_i) + \Phi(t_i))\right\|\right], \quad t \in [0, T]. \end{aligned} \quad (3.14)$$

By using the notations defined and the Hypotheses (H1) – (H9), we have

$$\begin{aligned} A_1 &= \|B^*\| \|S^*(T, t)\| \|\mathcal{R}(\lambda, \Gamma_0^T)\| \\ &\leq \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \\ A_2 &= \|S(t, 0)[\phi + \mathcal{G}(0, \phi(0), 0)]\| \\ &\leq \mathcal{M}\|\phi\|_{\mathcal{B}_h} + \mathcal{M}\mathcal{M}_G[1 + \|\phi\|_{\mathcal{B}_h}] \\ A_3 &= \left\|\mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right)\right\| \\ &\leq \mathcal{M}_G[1 + (lr_0 + q) + \mathcal{M}_{a_1}T(1 + (lr_0 + q))] \\ &\leq \mathcal{M}_G + \mathcal{M}_Glr_0 + \mathcal{M}_Gq + \mathcal{M}_G\mathcal{M}_{a_1}T \\ &\quad + \mathcal{M}_G\mathcal{M}_{a_1}Tlr_0 + \mathcal{M}_G\mathcal{M}_{a_1}Tq \\ &\leq \mathcal{M}_G + \mathcal{M}_Gq + \mathcal{M}_G\mathcal{M}_{a_1}T + \mathcal{M}_G\mathcal{M}_{a_1}Tq \\ &\quad + \mathcal{M}_Gr_0[l + \mathcal{M}_{a_1}T] \\ &\leq p_1 + \mathcal{M}_Glr_0[1 + \mathcal{M}_{a_1}T] \end{aligned}$$

where $p_1 = \mathcal{M}_G + \mathcal{M}_Gq + \mathcal{M}_G\mathcal{M}_{a_1}T + \mathcal{M}_G\mathcal{M}_{a_1}Tq$.

$$\begin{aligned} A_4 &= \left\|\int_0^t \mathcal{A}(\tau)S(t, \tau) \times \right. \\ & \quad \left.\mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta\right) d\tau\right\| \\ &\leq \widetilde{\mathcal{M}}_G T \left[\mathcal{M}_G + \mathcal{M}_Gq + \mathcal{M}_G\mathcal{M}_{a_1}T\right. \\ & \quad \left.+ \mathcal{M}_G\mathcal{M}_{a_1}Tq + \mathcal{M}_Gr_0[l + \mathcal{M}_{a_1}T]\right] \end{aligned}$$



$$\begin{aligned} &\leq p_2 + \widetilde{\mathcal{M}}_G T \cdot \mathcal{M}_G l r_0 [1 + \mathcal{M}_{a_1} T] \\ \text{where } p_2 &= \widetilde{\mathcal{M}}_G T \left[\mathcal{M}_G + \mathcal{M}_G q + \mathcal{M}_G \mathcal{M}_{a_1} T \right. \\ &\quad \left. + \mathcal{M}_G \mathcal{M}_{a_1} T q \right] \\ A_5 &= \left\| \int_0^t S(t, \tau) \times \right. \\ &\quad \left. \overline{\mathcal{G}} \left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta \right) d\tau \right\| \\ &\leq \mathcal{M} \mathcal{M}_{\overline{G}} T \left[\mathcal{M}_{\overline{G}} + \mathcal{M}_{\overline{G}} q + \mathcal{M}_{\overline{G}} \mathcal{M}_{a_2} T \right. \\ &\quad \left. + \mathcal{M}_{\overline{G}} \mathcal{M}_{a_2} T q + \mathcal{M}_{\overline{G}} r_0 [l + \mathcal{M}_{a_2} T l] \right] \\ &\leq p_3 + \mathcal{M} \mathcal{M}_{\overline{G}} T \cdot \mathcal{M}_{\overline{G}} l r_0 [1 + \mathcal{M}_{a_2} T] \\ \text{where } p_3 &= \mathcal{M} \mathcal{M}_{\overline{G}} T \left[\mathcal{M}_{\overline{G}} + \mathcal{M}_{\overline{G}} q + \mathcal{M}_{\overline{G}} \mathcal{M}_{a_2} T \right. \\ &\quad \left. + \mathcal{M}_{\overline{G}} \mathcal{M}_{a_2} T q \right] \\ A_6 &= \left\| \int_0^t S(t, \tau) \right. \\ &\quad \times \left[\mathcal{F} \left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau)) \right) \right] d\tau \left\| \\ &\leq \mathcal{M} T \sum_{i=1}^m \alpha_i(\tau) f_i[u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau))] \\ &\leq \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \pi_i(r_0) \\ A_7 &= \left\| \sum_{0 < t_i < t} S(t, t_i) I_i(\tilde{x}(t_i) + \Phi(t_i)) \right\| \leq M \beta_k \end{aligned}$$

By substituting all these estimations in the above equation, we get

$$\begin{aligned} \|v(t)\| &\leq \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \left\{ \|x_T\| + \mathcal{M} \|\phi\|_B + \mathcal{M} \mathcal{M}_G [1 + \|\phi\|_B] \right. \\ &\quad + p_1 + \mathcal{M}_G l r_0 [1 + \mathcal{M}_{a_1} T] \\ &\quad + p_2 + \widetilde{\mathcal{M}}_G T \cdot \mathcal{M}_G l r_0 [1 + \mathcal{M}_{a_1} T] \\ &\quad + p_3 + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l r_0 [1 + \mathcal{M}_{a_2} T] \\ &\quad \left. + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \pi_i(r_0) + M \sum_{k=1}^m \beta_k \right\} \\ &\leq \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \left[\|x_T\| + \mathcal{M} \|\phi\|_B + \mathcal{M} \mathcal{M}_G [1 + \|\phi\|_B] \right. \\ &\quad \left. + p + M \sum_{k=1}^m \beta_k \right] \\ &\quad + \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \left\{ r_0 \left[\mathcal{M}_G l + \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \left[1 + \mathcal{M}_{a_1} T \right] \right. \\ &\quad \left. + r_0 \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l [1 + \mathcal{M}_{a_2} T] \right\} \end{aligned}$$

$$\begin{aligned} &+ \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \pi_i(r_0) \left. \right\} \\ &= Q_1 \end{aligned}$$

where $p = \min\{p_1, p_2, p_3\}$.

We estimate the following by using

$$\begin{aligned} \|z(t)\| &\leq \left\| S(t, 0) [\phi(0) + \mathcal{G}(0, \phi(0), 0)] \right. \\ &\quad - \mathcal{G} \left(t, x_t, \int_0^t a_1(t, s, x_s) ds \right) \\ &\quad + \int_0^t S(t, \tau) \mathcal{A}(\tau) \mathcal{G} \left(\tau, x_\tau, \int_0^\tau a_1(\tau, \zeta, x_\zeta) d\zeta \right) d\tau \\ &\quad + \int_0^t S(t, \tau) \overline{\mathcal{G}} \left(\tau, x_\tau, \int_0^\tau a_2(\tau, \zeta, x_\zeta) d\zeta \right) d\tau \\ &\quad + \int_0^t S(t, \tau) \left[\mathcal{F} \left(\tau, u(\tau), x(a(x(\tau), \tau)) \right) \right. \\ &\quad \left. + Bv(\tau) \right] d\tau + \sum_{0 < t_i < t} S(t, t_i) I_i(x(t_i)) \left\| \right. \\ &\leq \mathcal{M} \mathcal{M}_G [1 + \|\phi\|_{\mathcal{B}_h}] + p + M \sum_{k=1}^m \beta_k \\ &\quad + r_0 \left[\mathcal{M}_G l + \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \left[1 + \mathcal{M}_{a_1} T \right] \\ &\quad + r_0 \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l [1 + \mathcal{M}_{a_2} T] \\ &\quad + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \pi_i(r_0) \\ &= Q_2 \end{aligned}$$

By adding Q_1 and Q_2 , dividing by r_0 and letting $r_0 \rightarrow \infty$ we get

$$\begin{aligned} \lim_{r_0 \rightarrow \infty} \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \left\{ \left[\mathcal{M}_G l + \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \left[1 + \mathcal{M}_{a_1} T \right] \right. \\ &\quad + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l [1 + \mathcal{M}_{a_2} T] \\ &\quad \left. + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \frac{\pi_i(r_0)}{r_0} \right\} \\ &\quad + \left[\mathcal{M}_G l + \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \left[1 + \mathcal{M}_{a_1} T \right] \\ &\quad + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l [1 + \mathcal{M}_{a_2} T] \\ &\quad + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \frac{\pi_i(r_0)}{r_0} \\ &\leq \left[1 + \frac{1}{\lambda} \mathcal{M}_b \mathcal{M} \right] \left\{ \left[\mathcal{M}_G l + \widetilde{\mathcal{M}}_G T \mathcal{M}_G l \right] \right. \\ &\quad \times \left[1 + \mathcal{M}_{a_1} T \right] + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l [1 + \mathcal{M}_{a_2} T] \\ &\quad \left. + \mathcal{M} T \sum_{i=1}^m \|\alpha_i\| \eta \right\} < 1 \end{aligned} \tag{3.15}$$

Therefore by the theorem (3.2) we get

$$\|\Psi^\lambda(t)\| = \|v(t)\| + \|z(t)\| \leq r_0.$$



From the above inequality shows Ψ^λ maps W_{r_0} into itself.

Step 2:

As per infinite-dimensional version of Arzela-Ascoli theorem and from step 1, we need the following proofs.

(i) For all $t \in J$, we define a set

$$\mathcal{V}(t) = \{\Psi^\lambda(x, u)(t) : (x, u) \in W_{r_0}\}.$$

We need to prove is relatively compact.

(ii) For an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\Psi^\lambda(x, u)(t_1) - \Psi^\lambda(x, u)(t_2)\| < \varepsilon$$

if $(x, u) \in W_{r_0}, |t_1 - t_2| \leq \delta$, for all $t_1, t_2 \in J$. In the case where $t = 0$, the set $V(t)$ is trivial, since $V(0) = \phi(0)$. Therefore assume t be a fixed real number, such that $0 < t \leq T$. For a given real number γ such that $0 < \gamma < t$. Then define an operator Ψ_γ^λ such that

$$\Psi_\gamma^\lambda(x, u)(t) = \left[S(\tau)\Psi^\lambda(x, u)(t - \gamma), B^*S^*(T, \tau) \mathcal{R}(\lambda, \Gamma_0^T)p(x, u) \right].$$

$$\Psi_1^\lambda(x, u)(t - \rho) = S(t - \rho)[x_0 + \mathcal{G}(0, \phi(0), 0)]$$

The set

$$V_\gamma(t) = \left\{ \Psi_1^\lambda(x, u)(t) + \mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right) \right\}$$

is relatively compact since $S(t)$ is compact.

In another words there exists a finite set $\{y_i, 1 \leq i \leq n\}$ in $\mathcal{P}\mathcal{C} \times U$ such that

$$V_\gamma(t) \subset \cup_{i=1}^m B(y_i, \varepsilon/2)$$

where $B(y_i, \varepsilon/2)$ is an open ball in $\mathcal{P}\mathcal{C} \times U$ with centre at y_i and radius $\varepsilon/2$. Also, we have

$$\begin{aligned} & \|(\Psi_1^\lambda(x, u))(t) - (\Psi_\gamma^\lambda(x, u))(t)\| \\ & \leq \left\| \int_{t-\rho}^t S(t, \tau) B B^* S^*(T, s) (\lambda I + \Gamma_0^T)^{-1} p(x, u) ds \right. \\ & \quad \left. + \int_{t-\rho}^t S(t, \tau) \mathcal{F}\left(\tau, u_\tau, (\tilde{x} + \Phi)(a(\tilde{x} + \Phi(\tau), \tau))\right) d\tau \right\| \\ & \leq \frac{1}{\lambda} M^2 M_b^2 P_\rho + M \sum_{i=1}^m \int_0^t \alpha_i(\tau) \pi_i(r_0) d\tau \\ & \leq \varepsilon/2 \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} P_\rho &= \|x_T\| + M\|\phi(0) + \mathcal{G}(0, \phi(0), 0)\| \\ &+ p_1 + \mathcal{M}_G l r_0 \left[1 + \mathcal{M}_{a_1} T \right] \\ &+ p_2 + \mathcal{M} \widetilde{\mathcal{M}}_G T \mathcal{M}_G l r_0 \left[1 + \mathcal{M}_{a_1} T \right] \\ &+ p_3 + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l r_0 \left[1 + \mathcal{M}_{a_2} T \right] \\ &+ \mathcal{M} \sum_{i=1}^m \int_0^t \|\alpha_i(\tau)\| \pi_i(r_0) d\tau + \sum_{k=1}^m S(T, t_k) I_k(x(t_k)) \end{aligned}$$

$$\begin{aligned} V_\gamma(t) &= \left\{ \Psi_1^\lambda(x, u)(t) \right. \\ &\quad \left. + \mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right) \right. \\ &\quad \left. : (x, u) \in W_{r_0} \right\} \subset \cup_{i=1}^m B(y_i, \varepsilon) \end{aligned}$$

Hence the set $V(t)$ is also relatively compact.

Step 3:

Further we need to prove the set $\{\Psi^\lambda(x, u)(\cdot) | (x, u) \in W_{r_0}\}$ is equicontinuous on $[0, T]$.

When $0 < t_1 + \theta < t_2 + \theta \leq T$, we have

$$\begin{aligned} & \|v(t_1) - v(t_2)\| \\ & \leq \|B^*S^*(T, t_1) - B^*S^*(T, t_2)\| \\ & \quad \times \frac{1}{\lambda} \left[\|x_T\| + M\|\phi(0) + \mathcal{G}(0, \phi(0), 0)\| \right. \\ & \quad \left. + p_1 + \mathcal{M}_G l r_0 \left[1 + \mathcal{M}_{a_1} T \right] \right. \\ & \quad \left. + p_2 + \mathcal{M} \widetilde{\mathcal{M}}_G T \mathcal{M}_G l r_0 \left[1 + \mathcal{M}_{a_1} T \right] \right. \\ & \quad \left. + p_3 + \mathcal{M} \mathcal{M}_{\overline{G}} T \mathcal{M}_{\overline{G}} l r_0 \left[1 + \mathcal{M}_{a_2} T \right] \right. \\ & \quad \left. + \mathcal{M} T \sum_{i=1}^m \|\alpha_i(\tau)\| \pi_i(r_0) \right. \\ & \quad \left. + M \sum_{k=1}^m d_k \right] \\ & \quad + \int_0^{t-\rho} S(t - \rho, \tau) \\ & \quad \times \mathcal{G}\left(t, \tilde{x}_t + \Phi_t, \int_0^t a_1(t, s, \tilde{x}_s + \Phi_s) ds\right) d\tau \end{aligned} \tag{3.17}$$

And we have

$$\begin{aligned} & \|z(t_1) - z(t_2)\| \\ & = \|S(t_1 + \theta) - S(t_2 + \theta)\| \\ & \quad \times \left\| \mathcal{G}\left(t_1, \tilde{x}_{t_1} + \Phi_{t_1}, \int_0^{t_1} a_1(t_1, s, \tilde{x}_s + \Phi_s) ds\right) \right. \\ & \quad \left. - \mathcal{G}\left(t_2, \tilde{x}_{t_2} + \Phi_{t_2}, \int_0^{t_2} a_1(t_2, s, \tilde{x}_s + \Phi_s) ds\right) \right\| \\ & \quad + \int_{t_1+\theta}^{t_2+\theta} \|S(t_2 + \theta, \tau)\| \|\mathcal{A}(\tau)\| \\ & \quad \times \left\| \mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \varsigma, \tilde{x}_\varsigma + \Phi_\varsigma) d\varsigma\right) \right\| d\tau \\ & \quad + \int_0^{t_1+\theta} \|S(t_1 + \theta, \tau) - S(t_2 + \theta, \tau)\| \|\mathcal{A}(\tau)\| \\ & \quad \times \left\| \mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_1(\tau, \varsigma, \tilde{x}_\varsigma + \Phi_\varsigma) d\varsigma\right) \right\| d\tau \\ & \quad + \int_{t_1+\theta}^{t_2+\theta} \|S(t_2 + \theta, \tau)\| \\ & \quad \times \left\| \mathcal{G}\left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \varsigma, \tilde{x}_\varsigma + \Phi_\varsigma) d\varsigma\right) \right\| d\tau \end{aligned}$$



$$\begin{aligned}
 & + \int_0^{t_1+\theta} \|S(t_1+\theta, \tau) - S(t_2+\theta, \tau)\| \\
 & \quad \times \left\| \overline{\mathcal{G}} \left(\tau, \tilde{x}_\tau + \Phi_\tau, \int_0^\tau a_2(\tau, \zeta, \tilde{x}_\zeta + \Phi_\zeta) d\zeta \right) \right\| d\tau \\
 & + \int_{t_1+\theta}^{t_2+\theta} \|S(t_2+\theta, \tau)\| \\
 & \quad \times \left\| \mathcal{F} \left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau)) \right) \right\| d\tau \\
 & + \int_0^{t_1+\theta} \|S(t_1+\theta, \tau) - S(t_2+\theta, \tau)\| \\
 & \quad \times \left\| \mathcal{F} \left(\tau, u(\tau), (\tilde{x} + \Phi)(a(\tilde{x}(\tau) + \Phi(\tau), \tau)) \right) \right\| d\tau \\
 & + \int_{t_1+\theta}^{t_2+\theta} \|S(t_2+\theta, \tau)\| \|Bu(\tau)\| d\tau \\
 & + \int_0^{t_1+\theta} \|S(t_1+\theta, \tau) - S(t_2+\theta, \tau)\| \|Bu(\tau)\| d\tau \\
 & + \sum_{t_1 < t_i < t_2} S(t_2+\theta, t_i) \|I_i(\tilde{x}(t_i) + \Phi(t_i))\| \\
 & + \sum_{0 < t_i < t_1} [S(t_2+\theta, t_i) - S(t_1+\theta, t_i)] \|I_i(\tilde{x}(t_i) + \Phi(t_i))\|
 \end{aligned} \tag{3.18}$$

Since $S(\cdot)$ is compact and by Lebesgue Dominated Convergence theorem all the terms tends to 0 as $t_1 \rightarrow t_2$ which gives the equicontinuity and boundedness of $\psi_1^\lambda(W_{r_0})$. This will imply that $\psi_1^\lambda(W_{r_0})$ is equicontinuous and bounded. V is also equicontinuous.

Therefore by Arzela-Ascoli theorem in $\mathcal{P}\mathcal{C}$, the set $\psi_1^\lambda(W_{r_0})$ is relatively compact and hence by the Schauder fixed point theorem ψ^λ has a fixed point. \square

Theorem 3.3. Assume that the hypotheses of the previous theorem (3.2) are satisfied then the system (1.1) is approximately controllable on $[0, T]$.

Proof. Assume (x^λ, u^λ) be a fixed point of ψ^λ in W_{r_0} and any fixed point of ψ^λ is a mild solution of (1.1) on $[0, T]$ under the control

$$u^\lambda = B^* S^*(T, t) R(\lambda, \Gamma_0^T) p(x^\lambda, u^\lambda)$$

and satisfies $x^\lambda(T) = h - \lambda R(\lambda, \Gamma_0^T) p(x, u)$.

By using the property that $\mathcal{F}, \mathcal{G}, \overline{\mathcal{G}}$ are uniformly bounded then there exist constants K_1 and K_2 which are positive such that

$$\begin{aligned}
 & \left\{ \left\| \mathcal{F}(s, u^\lambda(s), x^\lambda(a(x^\lambda(s), s))) \right\| \right. \\
 & + \left\| \mathcal{G} \left(t, x_t^\lambda, \int_0^T a_1(t, s, x_s^\lambda) ds \right) \right\| \\
 & + \left\| \overline{\mathcal{G}} \left(t, x_t^\lambda, \int_0^T a_2(t, s, x_s^\lambda) ds \right) \right\| \left. \right\} \leq K_1
 \end{aligned}$$

and $\left\| \mathcal{G} \left(t, x_t^\lambda, \int_0^T a_1(t, s, x_s^\lambda) ds \right) \right\| \leq K_2$. Consequently the subsequence,

$$\begin{aligned}
 & \left\{ \mathcal{F}(s, u^\lambda(s), x^\lambda(a(x^\lambda(s), s))) \right. \\
 & + \mathcal{G} \left(t, x_t^\lambda, \int_0^T a_1(t, s, x_s^\lambda) ds \right. \\
 & \left. \left. + \overline{\mathcal{G}} \left(t, x_t^\lambda, \int_0^T a_2(t, s, x_s^\lambda) ds \right) \right\}
 \end{aligned}$$

is converges weakly to

$$\begin{aligned}
 & \left\{ \mathcal{F}(s, u(s), x(a(x(s), s))) + \mathcal{G} \left(t, x_t, \int_0^T a_1(t, s, x_s) ds \right. \right. \\
 & \left. \left. + \overline{\mathcal{G}} \left(t, x_t, \int_0^T a_2(t, s, x_s) ds \right) \right\}
 \end{aligned}$$

The function $a(x^\lambda(s), s) \rightarrow a(x(s), s)$ and $x^\lambda(s) \rightarrow x(s)$ when a is continuous.

$$\begin{aligned}
 & \|x^\lambda(T) - x_T\| \\
 & \leq \|\lambda(\lambda I + \Gamma_0^T)^{-1} [x_T - S(T)[\phi(0) + g(0, \phi(0), 0)]\| \\
 & + \left\| \lambda(\lambda I + \Gamma_0^T)^{-1} \mathcal{G} \left(T, x_T^\lambda, \int_0^T a_1(T, s, x_s^\lambda) ds \right) \right\| \\
 & + \int_0^T S(T, s) \left\| \lambda(\lambda I + \Gamma_0^T)^{-1} \mathcal{A}(t) \right. \\
 & \quad \left. \left[\mathcal{G} \left(t, x_t^\lambda, \int_0^T a_1(t, s, x_s^\lambda) ds \right) \right. \right. \\
 & \quad \left. \left. - \mathcal{G} \left(t, x_t, \int_0^T a_1(t, s, x_s) ds \right) \right] \right\| dt \\
 & + \int_0^T S(T, s) \left\| \lambda(\lambda I + \Gamma_0^T)^{-1} \mathcal{A}(t) \right. \\
 & \quad \left. \times \mathcal{G} \left(t, x_t, \int_0^T a_1(t, s, x_s) ds \right) \right\| dt \\
 & + \int_0^T S(T, s) \left\| \lambda(\lambda I + \Gamma_0^T)^{-1} \right. \\
 & \quad \left. \times \left[\overline{\mathcal{G}} \left(t, x_t^\lambda, \int_0^T a_2(t, s, x_s^\lambda) ds \right) \right. \right. \\
 & \quad \left. \left. - \overline{\mathcal{G}} \left(t, x_t, \int_0^T a_2(t, s, x_s) ds \right) \right] \right\| dt \\
 & + \int_0^T S(T, s) \left\| \lambda(\lambda I + \Gamma_0^T)^{-1} \right. \\
 & \quad \left. \times \overline{\mathcal{G}} \left(t, x_t, \int_0^T a_2(t, s, x_s) ds \right) \right\| dt \\
 & + \int_0^T S(T, s) \|\lambda(\lambda I + \Gamma_0^T)^{-1} [\mathcal{F}(s, x_s^\lambda) - \mathcal{F}(s)]\| ds \\
 & + \int_0^T S(T, s) \|\lambda(\lambda I + \Gamma_0^T)^{-1} [\mathcal{F}(s)]\| ds \\
 & + \sum_{0 < t_i < T} S(T, t_i) [I_i(x^\lambda(t_i)) - I_i(x(t_i))] \rightarrow 0 \tag{3.19}
 \end{aligned}$$

as $\lambda^+ \rightarrow 0$. Hence the approximate controllability of (1.1). \square



4. Application

To epitomize our hypothetical results, now, we consider the following INIDE with infinite delay of the structure

$$\begin{aligned} & \frac{d}{dt} \left[z(t,x) + \left[\int_{-\infty}^t a_1(t,x,s-t) Q_1(z(s,x)) ds \right. \right. \\ & \quad \left. \left. + \int_0^t \int_{-\infty}^s K_1(s,\tau) Q_2(z(\tau,x)) d\tau ds \right] \right] \\ & = \frac{\partial^2}{\partial x^2} \left[z(t,x) + Bu(t,x) \right] \\ & + \left[\int_{-\infty}^t a_2(t,x,s-t) Q_3(z(s,x)) ds \right. \\ & \quad \left. + \int_0^t \int_{-\infty}^s K_2(s,\tau) Q_4(z(\tau,x)) d\tau ds \right] + \mu_1(x, z(t,x)) \\ & + \mu_2(t,x, z(t+\theta,x)), x \in (0, \pi), t > 0 \quad (4.1) \\ & \quad z(t,0) = z(t,\pi) = 0, t > 0 \quad (4.2) \\ & \quad z(t,x) = \phi(t,x), t \in [-\infty, 0], x \in [0, \pi] \quad (4.3) \\ & \Delta u(t_i, \cdot) = u(t_i^+) - u(t_i^-, \cdot) = I_i(u(t_i, \cdot)), i = 1, 2, \dots, \delta \quad (4.4) \end{aligned}$$

where $\mu_1(x, u(t,x)) = \int_0^x \mathcal{V}(x,y) u(y, \tilde{u}(t)(b_1|u(t,y)|)) dy$ for all $(t,x) \in [0, \infty) \times [0, \pi]$. The function μ_2 is measurable in x , locally Hölder continuous in t and $\theta \in [-\infty, 0]$ locally Lipschitz continuous in u and uniformly in x . ϕ is Lipschitz continuous on $[-\infty, 0]$ with Lipschitz constant $\mathcal{K}_\phi > 0$ and it follows the conditions $\phi(0,0) = 0$ and $\phi(0,1) = 0$. In μ_1, \tilde{u} is locally Hölder continuous in t and it satisfies the condition $\tilde{u}(0) = 0$. Here $\mathcal{V}(\cdot, \cdot) \in C'([0, \pi] \times [0, \pi], \mathbb{R})$. We defined μ_2 as

$$\mu_2[t,x, u(t+\theta,x)] = \int_{-\infty}^0 \int_0^\pi \mathcal{V}_0(t) P(s,y,x) u(t+s,y) dy ds$$

In such a way that the function A which is measurable and

$$\sup_{r \in [-\infty, \infty]} \int_0^\pi \int_0^\pi A^2(t,y,w) dy dw < \infty$$

Let $E = L^2([0, \pi])$. We characterize $A(T)u = u(t,x)$ and we defined $A : E \rightarrow E$ by $Ax = \frac{d^2z}{dx^2}$ where the domain of A is

$$D(A) = \left\{ z(\cdot) \in E : z \in E, z, z' \text{ are absolutely continuous, } z'' \in E, z'(0) = z'(1) = 0 \right\}.$$

We have

$$Az = \sum_{n=1}^\infty (-n^2 \pi^2) \langle z, e_n \rangle e_n, z \in D(A),$$

where $e^n(\theta) = \sqrt{2} \cos(n\pi\theta), 0 < x < 1, n = 1, 2, \dots$. Then

$$\begin{aligned} S(t)z &= \sum_{n=1}^\infty 2e^{-n^2 \pi^2 t} \cos(n\pi x) \int_0^1 \cos(n\pi x) \\ &\quad \times \int_0^1 \cos(n\pi x) z(\psi) d\psi + \int_0^1 z(\psi) d\psi, z \in E \end{aligned}$$

Further the functions $a_i, i = 1, 2, Q_i, i = 1, 2, 3, 4, K_i, i = 1, 2$ and $\mu_i, i = 1, 2$ are continuous and $\mu_i < c_1$ where $i = 1, 2$, and similarly we have $a_i < c_2, i = 1, 2, Q_i < c_3, i = 1, 2, 3, 4, K_i < c_4, i = 1, 2$ are positive constants. Finally there exists constants d_k such that $\|I_k(x)\| \leq d_k$. Therefore the problem 4.1 can be expressed as (1.1). Hence the (4.1) is approximately controllable.

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