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On $(1,2)^{\star}$-$\breve{g}$-closed sets in bitopological spaces

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Abstract
The aim of this paper, we introduce a new class of sets called $(1,2)^{\star}$-$\hat{\hat{g}}_1$-closed sets, $(1,2)^{\star}$-$\mathscr{G}$-closed sets and $(1,2)^{\star}$-$\breve{g}$-closed sets in bitopological spaces and we obtain several characterizations of this class and investigate the relationships with other some closed sets in bitopological spaces.

Keywords
$(1,2)^{\star}$-$\hat{\hat{g}}$-closed sets, $(1,2)^{\star}$-$\hat{\hat{g}}_1$-closed sets, $(1,2)^{\star}$-$\mathscr{G}$-closed sets and $(1,2)^{\star}$-$\breve{g}$-closed sets.

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Contents

1. Introduction

N. Levine [8] was introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. J. Kelly [6] was introduced the notion of bitopological spaces. Recently many Researchers[rep. [12], [13]] introduced by various types of generalized closed sets and studied their fundamental properties are investigated in bitopological spaces. M.Garg [4] was introduced a new class of sets namely $\hat{\hat{g}}$-closed sets in topological spaces and K. M. Dharmalingam *et. al.*, [2] extend the $\hat{\hat{g}}$-closed sets in bitopological spaces.

In this paper, we introduce a new class of sets called $(1,2)^{\star}$-$\hat{\hat{g}}_1$-closed sets, $(1,2)^{\star}$-$\mathscr{G}$-closed sets and $(1,2)^{\star}$-$\breve{g}$-closed sets in bitopological spaces and we obtain several characterizations of this class and investigate the relationships with other some closed sets in bitopological spaces.

2. Preliminaries

Definition 2.1. *Let A be a subset of a bitopological space X.*

Then A is called $\tau_{1,2}$-open [7] if $A = P \cup Q$, for some $P \in \tau_1$ and $Q \in \tau_2$.

The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Definition 2.2. *[7] Let A be a subset of a bitopological space X. Then*

1. *the $\tau_{1,2}$-interior of A, denoted by $\tau_{1,2}$-int(A), is defined by $\cup\{U : U \subseteq A$ and U is $\tau_{1,2}$-open$\}$.*
2. *the $\tau_{1,2}$-closure of A, denoted by $\tau_{1,2}$-cl(A), is defined by $\cap\{U : A \subseteq U$ and U is $\tau_{1,2}$-closed$\}$.*

Remark 2.3. *[7] Notice that $\tau_{1,2}$-open subsets of X need not necessarily form a topology.*

Definition 2.4. *Let A be a subset of a bitopological space X. Then A is called*

1. *$(1,2)^{\star}$-semi-open set [7] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int(A)).*
2. *$(1,2)^{\star}$-preopen set [7] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*
3. *$(1,2)^{\star}$-α-open set [7] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A))).*
4. *$(1,2)^{\star}$-β-open set [9] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A))).*

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.5. *Let A be a subset of a bitopological space X. Then A is called*

1. *$(1,2)^{\star}$-g-closed set [11] if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open.*

2. $(1,2)^{\star}$-*sg-closed set* [1] *if* $(1,2)^{\star}$-*scl(A)* $\subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $(1,2)^{\star}$-*semi-open.*
3. $(1,2)^{\star}$-*gs-closed set* [1] *if* $(1,2)^{\star}$-*scl(A)* $\subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $\tau_{1,2}$-*open.*
4. $(1,2)^{\star}$-αg-*closed set* [3] *if* $(1,2)^{\star}$-$\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $\tau_{1,2}$-*open.*
5. $(1,2)^{\star}$-$g\beta$-*closed set* [3] *if* $(1,2)^{\star}$-$\beta cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $\tau_{1,2}$-*open.*
6. $(1,2)^{\star}$-$\hat{\hat{g}}$-*closed set* [2] *if* $\tau_{1,2}$-$cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* $(1,2)^{\star}$-*sg-open. The complements of the above closed sets are their respective open sets.*
7. $(1,2)^{\star}$-*locally closed* [10] *if* $A = M \cap N$ *where* M *is* $\tau_{1,2}$-*open and* N *is* $\tau_{1,2}$-*closed.*

3. On $(1,2)^{\star}$-$\breve{g}$-closed sets

Definition 3.1. *A subset H of a bitopological space* (X,τ_1,τ_2) *is called a*

1. $(1,2)^{\star}$-$\hat{\hat{g}}_1$-*closed set if* $\tau_{1,2}$-$cl(H) \subseteq G$ *whenever* $H \subseteq G$ *and* G *is* $(1,2)^{\star}$-$\hat{\hat{g}}$-*open.*
2. $(1,2)^{\star}$-$\mathscr{G}$-*closed set if* $(1,2)^{\star}$-$scl(H) \subseteq G$ *whenever* $H \subseteq G$ *and* G *is* $(1,2)^{\star}$-$\hat{\hat{g}}_1$-*open.*
3. $(1,2)^{\star}$-$\breve{g}$-*closed set if* $\tau_{1,2}$-$cl(H) \subseteq G$ *whenever* $H \subseteq G$ *and* G *is* $(1,2)^{\star}$-$\mathscr{G}$-*open.*
4. $(1,2)^{\star}$-$\breve{g}_{\alpha}$-*closed set if* $(1,2)^{\star}$-$\alpha cl(H) \subseteq G$ *whenever* $H \subseteq G$ *and* G *is* $(1,2)^{\star}$-$\mathscr{G}$-*open.*
5. *The* $(1,2)^{\star}$-$\mathscr{G}$-*Kernel of the set H , denoted by* $(1,2)^{\star}$-$\mathscr{G}$-*ker(H), is the intersection of all* $(1,2)^{\star}$-$\mathscr{G}$-*open supersets of H.*

The complements of the above (1), (2) and (3) used closed sets are their respective open sets.

Remark 3.2. *In a bitopological space* (X,τ_1,τ_2), *for a subset H the following relations are true.*

1. *if H is* $\tau_{1,2}$-*closed set, then H is* $(1,2)^{\star}$-$\hat{\hat{g}}_1$-*closed.*
2. *if H is* $\tau_{1,2}$-*closed set, then H is* $(1,2)^{\star}$-$\mathscr{G}$-*closed.*
3. *if H is* $(1,2)^{\star}$-*semi-open set, then H is* $(1,2)^{\star}$-$\mathscr{G}$-*open.*

Converse part of the above is not true as shown in the following Example.

Example 3.3. *Let* $X = \{p,q,r,s\}$, $\tau_1 = \{\phi,\{p\},\{r,s\},\{p,r,s\}, X\}$ *and* $\tau_2 = \{\phi, X\}$ *then* $\tau_{1,2} = \{\phi,\{p\},\{r,s\},\{p,r,s\},X\}$. *In the bitopological space* (X,τ_1,τ_2), *then*

1. *the subset* $\{r,s\}$ *is* $(1,2)^{\star}$-$\hat{\hat{g}}_1$-*closed but not* $\tau_{1,2}$-*closed.*
2. *the subset* $\{p,q,r\}$ *is* $(1,2)^{\star}$-$\mathscr{G}$-*closed but not* $\tau_{1,2}$-*closed.*
3. *the subset* $\{s\}$ *is* $(1,2)^{\star}$-$\mathscr{G}$-*open but not* $(1,2)^{\star}$-*semi-open.*

Proposition 3.4. *For a subset H of a bitopological space* (X,τ_1,τ_2), *the following implications are true.*

1. *H is* $\tau_{1,2}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-$\breve{g}$-*closed.*
2. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-$\breve{g}_{\alpha}$-*closed.*
3. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-*sg-closed.*
4. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-*g-closed.*
5. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-αg-*closed.*
6. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-*gs-closed.*
7. *H is* $(1,2)^{\star}$-$\breve{g}$-*closed set* $\Rightarrow$ *H is* $(1,2)^{\star}$-*gsp-closed.*

Proof.

1. If H is any $\tau_{1,2}$-closed set in X and G is any $(1,2)^{\star}$-$\mathscr{G}$-open set containing H, then $H = \tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-$\breve{g}$-closed.
2. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set in X and G is any $(1,2)^{\star}$-$\mathscr{G}$-open set containing H, then $(1,2)^{\star}$-$\alpha cl(H) \subseteq \tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-$\breve{g}_{\alpha}$-closed.
3. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set in X and G is any $(1,2)^{\star}$-semi-open set containing H, since every $(1,2)^{\star}$-semi-open set is $(1,2)^{\star}$-$\mathscr{G}$-open and H is $(1,2)^{\star}$-$\breve{g}$-closed, we have $(1,2)^{\star}$-$scl(H) \subseteq \tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-sg-closed.
4. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set and G is any $(1,2)^{\star}$-open set containing H, since every $(1,2)^{\star}$-open set is $(1,2)^{\star}$-$\mathscr{G}$-open, we have $\tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-g-closed.
5. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set in X and G is any $(1,2)^{\star}$-open set containing H, since every $(1,2)^{\star}$-open set is $(1,2)^{\star}$-$\mathscr{G}$-open, we have $(1,2)^{\star}$-$\alpha cl(H) \subseteq \tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-αg-closed.
6. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set in X and G is any $(1,2)^{\star}$-open set containing H, since every $(1,2)^{\star}$-open set is $(1,2)^{\star}$-$\mathscr{G}$-open, we have $(1,2)^{\star}$-$scl(H) \subseteq \tau_{1,2}$-$cl(H) \subseteq G$. Hence H is $(1,2)^{\star}$-gs-closed.
7. If H is a $(1,2)^{\star}$-$\breve{g}$-closed set in X and G is any $(1,2)^{\star}$-open set containing H, every $(1,2)^{\star}$-open set is $(1,2)^{\star}$-$\mathscr{G}$-open, we have $(1,2)^{\star}$-$\beta cl(H) \subseteq \tau_{1,2}$-$cl(H) \subseteq G$. Thus H is $(1,2)^{\star}$-gsp-closed.

Remark 3.5. *These relations are shown in the diagram.*

$$\begin{array}{ccccc}
(1,2)^{\star}\text{-}\alpha\text{-closed} & \leftarrow & \tau_{1,2}\text{-closed} & \rightarrow & (1,2)^{\star}\text{-semi-closed} \\
\downarrow & & \downarrow & & \downarrow \\
(1,2)^{\star}\text{-}\breve{g}_{\alpha}\text{-closed} & \leftarrow & (1,2)^{\star}\text{-}\breve{g}\text{-closed} & \rightarrow & (1,2)^{\star}\text{-}sg\text{-closed} \\
\downarrow & & \downarrow & & \downarrow \\
(1,2)^{\star}\text{-}\alpha g\text{-closed} & \leftarrow & (1,2)^{\star}\text{-}g\text{-closed} & \rightarrow & (1,2)^{\star}\text{-}g\beta\text{-closed}
\end{array}$$

Remark 3.6. *In a bitopological spaces* (X,τ_1,τ_2), *then*

1. *the family of* $(1,2)^{\star}$-$\breve{g}$-*closed sets and the family of* $(1,2)^{\star}$-α-*closed sets are independent of each other.*

2. *the family of $(1,2)^{\star}$-$\breve{g}$-closed sets and the family of $(1,2)^{\star}$-semi-closed sets are independent of each other.*

As shown in the following Examples.

Example 3.7. *Let $X=\{p,q,r\}$, $\tau_1=\{\phi,\{p,q\},X\}$ and $\tau_2=\{\phi,X\}$ then $\tau_{1,2}=\{\phi,\{p,q\},X\}$. In the bitopological space (X,τ_1,τ_2), then the subset $\{r\}$ is $(1,2)^{\star}$-α-closed and $(1,2)^{\star}$-semi-closed but not $(1,2)^{\star}$-$\breve{g}$-closed.*

Example 3.8. *Let $X=\{x,y,z\}$, $\tau_1=\{\phi,\{y,z\},X\}$ and $\tau_2=\{\phi,X\}$ then $\tau_{1,2}=\{\phi,\{y,z\},X\}$. In the bitopological space (X,τ_1,τ_2), then the subset $\{y\}$ is $(1,2)^{\star}$-$\breve{g}$-closed but not $(1,2)^{\star}$-α-closed and $(1,2)^{\star}$-semi-closed.*

Lemma 3.9. *A subset H is $(1,2)^{\star}$-$\breve{g}$-closed $\iff \tau_{1,2}$-$cl(A)\subseteq (1,2)^{\star}$-$\mathscr{G}$-$ker(H)$.*

Proof. $\Rightarrow$ Assuming that H is $(1,2)^{\star}$-$\breve{g}$-closed. Then $\tau_{1,2}$-$cl(H)\subseteq U$ whenever $H\subseteq U$ and U is $(1,2)^{\star}$-$\mathscr{G}$-open. Let $x\in\tau_{1,2}$-$cl(H)$. If $x\notin(1,2)^{\star}$-$\mathscr{G}$-$ker(H)$, then there is $(1,2)^{\star}$-$\mathscr{G}$-open set U containing H such that $x\notin U$. Since U is $(1,2)^{\star}$-$\mathscr{G}$-open set containing H, we have $x\notin\tau_{1,2}$-$cl(H)$ and this is a contradiction.

$\Leftarrow$ let $\tau_{1,2}$-$cl(H)\subseteq(1,2)^{\star}$-$\mathscr{G}$-$ker(H)$. If U is any $(1,2)^{\star}$-$\mathscr{G}$-open set containing H, then $\tau_{1,2}$-$cl(H)\subseteq(1,2)^{\star}$-$\mathscr{G}$-$ker(H)\subseteq U$. Therefore, H is $(1,2)^{\star}$-$\breve{g}$-closed.

Proposition 3.10. *In a bitopological space (X,τ_1,τ_2), if P and Q are $(1,2)^{\star}$-$\breve{g}$-closed sets then $P\cup Q$ is $(1,2)^{\star}$-$\breve{g}$-closed.*

Proof. If $P\cup Q\subseteq G$ and G is $(1,2)^{\star}$-$\mathscr{G}$-open, then $P\subseteq G$ and $Q\subseteq G$. Since P and Q are $(1,2)^{\star}$-$\breve{g}$-closed, $G\supseteq\tau_{1,2}$-$cl(P)$ and $G\supseteq\tau_{1,2}$-$cl(Q)$ and hence $G\supseteq\tau_{1,2}$-$cl(P)\cup\tau_{1,2}$-$cl(Q)=\tau_{1,2}$-$cl(P\cup Q)$. Thus $P\cup Q$ is $(1,2)^{\star}$-$\breve{g}$-closed.

Proposition 3.11. *In a bitopological space (X,τ_1,τ_2), if a set H is $(1,2)^{\star}$-$\breve{g}$-closed, then $\tau_{1,2}$-$cl(H)-H$ contains no nonempty $(1,2)^{\star}$-$\mathscr{G}$-closed.*

Proof. Assuming that H is $(1,2)^{\star}$-$\breve{g}$-closed. Let K be a $(1,2)^{\star}$-$\mathscr{G}$-closed subset of $\tau_{1,2}$-$cl(H)-H$. Then $H\subseteq K^c$. Since H is $(1,2)^{\star}$-$\breve{g}$-closed, $\tau_{1,2}$-$cl(H)\subseteq K^c$.

Consequently, $K\subseteq(\tau_{1,2}$-$cl(H))^c$. We already have $K\subseteq\tau_{1,2}$-$cl(H)$. Thus $K\subseteq\tau_{1,2}$-$cl(H)\cap(\tau_{1,2}$-$cl(H))^c$ and $K=\phi$.

Remark 3.12. *The converse of Proposition 3.11 is not true as shown in the following Example.*

Example 3.13. *Let $X=\{i,j,k,l\}$, $\tau_1=\{\phi,X\}$ and $\tau_2=\{\phi,\{i,l\},X\}$ then $\tau_{1,2}=\{\phi,\{i,l\},X\}$. In the bitopological space (X,τ_1,τ_2), then $(1,2)^{\star}$-$\breve{g}$-closed is $\{\phi,\{j,k\},\{i,j,k\},\{j,k,l\},X\}$ and $(1,2)^{\star}$-$\mathscr{G}$-closed is $\{\phi,\{i\},\{l\},\{i,j\},\{i,k\},\{i,l\},\{j,k\},\{i,j,k\},\{i,j,l\},\{i,k,l\},X\}$. If $H=\{j\}$ then $\tau_{1,2}$-$cl(H)-H=\{k\}$ does not contain any nonempty $(1,2)^{\star}$-$\mathscr{G}$-closed. But $(1,2)^{\star}$-$\hat{\hat{g}}_1$-closed is not $(1,2)^{\star}$-$\breve{g}$-closed.*

Theorem 3.14. *In a bitopological space (X,τ_1,τ_2), if a set H is $(1,2)^{\star}$-$\breve{g}$-closed, then $\tau_{1,2}$-$cl(H)-H$ contains no nonempty $\tau_{1,2}$-closed set.*

Proof. Suppose that H is $(1,2)^{\star}$-$\breve{g}$-closed. Let K be a $\tau_{1,2}$-closed subset of $\tau_{1,2}$-$cl(H)-H$. Then $H\subseteq K^c$. Since H is $(1,2)^{\star}$-$\breve{g}$-closed, we have $\tau_{1,2}$-$cl(H)\subseteq K^c$.

Consequently, $K\subseteq(\tau_{1,2}$-$cl(H))^c$. Hence, $K\subseteq\tau_{1,2}$-$cl(H)\cap(\tau_{1,2}$-$cl(H))^c=\phi$. Therefore $K=\phi$.

Proposition 3.15. *In a bitopological space (X,τ_1,τ_2), if H is $(1,2)^{\star}$-$\breve{g}$-closed set and $H\subseteq K\subseteq\tau_{1,2}$-$cl(H)$, then K is $(1,2)^{\star}$-$\breve{g}$-closed set.*

Proof. Let $K\subseteq U$ where U is $(1,2)^{\star}$-$\mathscr{G}$-open set. Since $H\subseteq K$, $H\subseteq U$. Since H is $(1,2)^{\star}$-$\breve{g}$-closed set, $\tau_{1,2}$-$cl(H)\subseteq U$. Since $K\subseteq\tau_{1,2}$-$cl(H)$, $\tau_{1,2}$-$cl(K)\subseteq\tau_{1,2}$-$cl(H)\subseteq U$. Therefore K is $(1,2)^{\star}$-$\breve{g}$-closed set.

Proposition 3.16. *Let $H\subseteq Y\subseteq X$ and suppose that H is $(1,2)^{\star}$-$\breve{g}$-closed. Then H is $(1,2)^{\star}$-$\breve{g}$-closed relative to Y.*

Proof. Let $H\subseteq Y\cap G$, where G is $(1,2)^{\star}$-$\mathscr{G}$-open. Then $H\subseteq G$ and hence $\tau_{1,2}$-$cl(H)\subseteq G$. This implies that $Y\cap\tau_{1,2}$-$cl(H)\subseteq Y\cap G$. Thus H is $(1,2)^{\star}$-$\breve{g}$-closed relative to Y.

Proposition 3.17. *In a bitopological space (X,τ_1,τ_2), if H is a $(1,2)^{\star}$-$\mathscr{G}$-open and $(1,2)^{\star}$-$\breve{g}$-closed, then H is $\tau_{1,2}$-closed.*

Proof. Since H is $(1,2)^{\star}$-$\mathscr{G}$-open and $(1,2)^{\star}$-$\breve{g}$-closed, $\tau_{1,2}$-$cl(H)\subseteq H$ and hence H is $\tau_{1,2}$-closed.

Theorem 3.18. *Let H be a $(1,2)^{\star}$-locally closed set. Then H is $\tau_{1,2}$-closed $\iff$ H is $(1,2)^{\star}$-$\breve{g}$-closed.*

Proof. It is fact that every $\tau_{1,2}$-closed set is $(1,2)^{\star}$-$\breve{g}$-closed.

Conversely, $H\cup(X-\tau_{1,2}$-$cl(H))$ is $\tau_{1,2}$-open, since H is $(1,2)^{\star}$-locally closed. Now $H\cup(X-\tau_{1,2}$-$cl(H))$ is $(1,2)^{\star}$-$\mathscr{G}$-open set such that $H\subseteq H\cup(X-\tau_{1,2}$-$cl(H))$. Since H is $(1,2)^{\star}$-$\breve{g}$-closed, then $\tau_{1,2}$-$cl(H)\subseteq H\cup(X-\tau_{1,2}$-$cl(H))$. Thus, we have $\tau_{1,2}$-$cl(H)\subseteq H$ and hence H is a $\tau_{1,2}$-closed.

Proposition 3.19. *In a bitopological space (X,τ_1,τ_2), for each $x\in X$, either $\{x\}$ is $(1,2)^{\star}$-$\mathscr{G}$-closed or $\{x\}^c$ is $(1,2)^{\star}$-$\breve{g}$-closed.*

Proof. Suppose that $\{x\}$ is not $(1,2)^{\star}$-$\mathscr{G}$-closed. Then $\{x\}^c$ is not $(1,2)^{\star}$-$\mathscr{G}$-open and the only $(1,2)^{\star}$-$\mathscr{G}$-open set containing $\{x\}^c$ is the space X itself. Therefore $\tau_{1,2}$-$cl(\{x\}^c)\subseteq X$ and so $\{x\}^c$ is $(1,2)^{\star}$-$\breve{g}$-closed.

Remark 3.20. *In a bitopological space (X,τ_1,τ_2), let K be a $\tau_{1,2}$-closed set. Then the following properties hold: If H is $(1,2)^{\star}$-$\mathscr{G}$-closed in (X,τ_1,τ_2), then $H\cap F$ is $(1,2)^{\star}$-$\mathscr{G}$-closed.*

Corollary 3.21. *If H is $(1,2)^{\star}$-$\breve{g}$-closed set and F is a $\tau_{1,2}$-closed set, then $H\cap F$ is $(1,2)^{\star}$-$\breve{g}$-closed set.*

Proof. Let U be a $(1,2)^{\star}$-$\mathscr{G}$-open set such that $H\cap F\subseteq U$. By Remark 3.20, it shows that $H\subseteq U\cup(X\backslash F)$ and $U\cup(X\backslash F)$ is $(1,2)^{\star}$-$\mathscr{G}$-open. Since H is $(1,2)^{\star}$-$\breve{g}$-closed, we have $\tau_{1,2}$-$cl(H)\subseteq U\cup(X\backslash F)$ and so $\tau_{1,2}$-$cl(H\cap F)\subseteq\tau_{1,2}$-$cl(H)\cap\tau_{1,2}$-$cl(F)=\tau_{1,2}$-$cl(H)\cap F\subseteq(U\cup(X\backslash F))\cap F=U\cap F\subseteq U$. Therefore $H\cap F$ is $(1,2)^{\star}$-$\breve{g}$-closed.

4. On $(1,2)^{\star}$-$\breve{g}$-closure and $(1,2)^{\star}$-$\breve{g}$-interior

Definition 4.1. *In a bitopological spaces (X,τ_1,τ_2), for every set $H \subseteq X$, we define*

1. *$(1,2)^{\star}$-$\breve{g}$-closure of H to be the intersection of all $(1,2)^{\star}$-$\breve{g}$-closed sets containing H.*
   i.e., $(1,2)^{\star}$-$\breve{g}$-$cl(H) = \cap\{K : H \subseteq K \in (1,2)^{\star}$-$\breve{g}$-$closed\}$.
2. *$(1,2)^{\star}$-$\breve{g}$-$int(H)$ is defined as the union of all $(1,2)^{\star}$-$\breve{g}$-open sets contained in H.*
   i.e., $(1,2)^{\star}$-$\breve{g}$-$int(H) = \cup\{K : K \subseteq H$ and K is $(1,2)^{\star}$-$\breve{g}$-$open\}$.

Lemma 4.2. *In a bitopological spaces (X,τ_1,τ_2), for any $H \subseteq X$, $H \subseteq (1,2)^{\star}$-$\breve{g}$-$cl(H) \subseteq \tau_{1,2}$-$cl(H)$.*

Proof. It follows from Proposition 3.4(1).

Lemma 4.3. *For any $H \subseteq X$, $(1,2)^{\star}$-sg-$cl(H) \subseteq (1,2)^{\star}$-$\breve{g}$-$cl(H)$, where $(1,2)^{\star}$-sg-$cl(H)$ is given by $(1,2)^{\star}$-sg-$cl(H) = \cap\{K : H \subseteq K \in (1,2)^{\star}$-$sg$-$closed\}$.*

Proof. It follows from Proposition 3.4(3).

Proposition 4.4. *For any bitopological space (X,τ_1,τ_2), the following relations are hold:*

1. *If H is $\tau_{1,2}$-open set, then H is $(1,2)^{\star}$-$\breve{g}$-open.*
2. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-$\breve{g}_{\alpha}$-open.*
3. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-g-open.*
4. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-αg-open.*
5. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-gs-open.*
6. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-gsp-open.*
7. *If H is $(1,2)^{\star}$-$\breve{g}$-open set, then H is $(1,2)^{\star}$-sg-open.*

Proof. Obvious.

Proposition 4.5. *If H and K are $(1,2)^{\star}$-$\breve{g}$-open sets, then $H \cap K$ is $(1,2)^{\star}$-$\breve{g}$-open.*

Proof. Obvious.

Theorem 4.6. *A subset H of X is $(1,2)^{\star}$-$\breve{g}$-open $\iff K \subseteq \tau_{1,2}$-$int(H)$ whenever K is $(1,2)^{\star}$-$\mathscr{G}$-closed and $K \subseteq H$.*

Proof. Suppose that $K \subseteq \tau_{1,2}$-$int(H)$ such that K is $(1,2)^{\star}$-$\mathscr{G}$-closed and $K \subseteq H$. Let $H^c \subseteq U$ where U is $(1,2)^{\star}$-$\mathscr{G}$-open. Then $U^c \subseteq H$ and U^c is $(1,2)^{\star}$-$\mathscr{G}$-closed. Therefore $U^c \subseteq \tau_{1,2}$-$int(H)$ by hypothesis. Since $U^c \subseteq \tau_{1,2}$-$int(H)$, we have $(\tau_{1,2}$-$int(H))^c \subseteq U$.

i.e., $\tau_{1,2}$-$cl(H^c) \subseteq U$, since $\tau_{1,2}$-$cl(H^c) = (\tau_{1,2}$-$int(H))^c$. Thus H^c is $(1,2)^{\star}$-$\breve{g}$-closed. i.e., H is $(1,2)^{\star}$-$\breve{g}$-open.

Conversely, suppose that H is $(1,2)^{\star}$-$\breve{g}$-open such that $K \subseteq H$ and K is $(1,2)^{\star}$-$\mathscr{G}$-closed. Then K^c is $(1,2)^{\star}$-$\mathscr{G}$-open and $H^c \subseteq K^c$. Therefore, $\tau_{1,2}$-$cl(H^c) \subseteq K^c$ by definition of $(1,2)^{\star}$-$\breve{g}$-closedness and so $K \subseteq \tau_{1,2}$-$int(H)$, since $\tau_{1,2}$-$cl(H^c) = (\tau_{1,2}$-$int(H))^c$.

Lemma 4.7. *For an $x \in X$, $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H) \iff K \cap H \neq \phi$ for every $(1,2)^{\star}$-$\breve{g}$-open set K containing x.*

Proof. Let $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H)$ for any $x \in X$. To prove $K \cap H \neq \phi$ for every $(1,2)^{\star}$-$\breve{g}$-open set K containing x. Prove the result by contradiction. Suppose there exists a $(1,2)^{\star}$-$\breve{g}$-open set K containing x such that $K \cap H = \phi$. Then $H \subseteq K^c$ and K^c is $(1,2)^{\star}$-$\breve{g}$-closed. We have $(1,2)^{\star}$-$\breve{g}$-$cl(H) \subseteq K^c$. This shows that $x \notin (1,2)^{\star}$-$\breve{g}$-$cl(H)$ which is a contradiction. Hence $K \cap H \neq \phi$ for every $(1,2)^{\star}$-$\breve{g}$-open set K containing x.

Conversely, let $K \cap H \neq \phi$ for every $(1,2)^{\star}$-$\breve{g}$-open set K containing x. To prove $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H)$. We prove the result by contradiction. Suppose $x \notin (1,2)^{\star}$-$\breve{g}$-$cl(H)$. Then there exists a $(1,2)^{\star}$-$\breve{g}$-closed set S containing H such that $x \notin S$. Then $x \in S^c$ and S^c is $(1,2)^{\star}$-$\breve{g}$-open. Also $S^c \cap H = \phi$, which is a contradiction to the hypothesis. Hence $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H)$.

Lemma 4.8. *For any $H \subseteq X$, $\tau_{1,2}$-$int(H) \subseteq (1,2)^{\star}$-$\breve{g}$-$int(H) \subseteq H$.*

Proof. It follows from Proposition 4.4(1).

Theorem 4.9. *Let H be any subset of a space X. Then*

1. *$((1,2)^{\star}$-$\breve{g}$-$int(H))^c = (1,2)^{\star}$-$\breve{g}$-$cl(H^c)$.*
2. *$(1,2)^{\star}$-$\breve{g}$-$int(H) = ((1,2)^{\star}$-$\breve{g}$-$cl(H^c))^c$.*
3. *$(1,2)^{\star}$-$\breve{g}$-$cl(H) = ((1,2)^{\star}$-$\breve{g}$-$int(H^c))^c$.*

Proof.

1. Let $x \in ((1,2)^{\star}$-$\breve{g}$-$int(H))^c$. Then $x \notin (1,2)^{\star}$-$\breve{g}$-$int(H)$. That is, every $(1,2)^{\star}$-$\breve{g}$-open set U containing x is such that $U \nsubseteq H$. That is, every $(1,2)^{\star}$-$\breve{g}$-open set U containing x is such that $U \cap H^c \neq \phi$. By Lemma 4.7, $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H^c)$ and therefore $((1,2)^{\star}$-$\breve{g}$-$int(H))^c \subseteq (1,2)^{\star}$-$\breve{g}$-$cl(H^c)$.
   Conversely, let $x \in (1,2)^{\star}$-$\breve{g}$-$cl(H^c)$. Then by Lemma 4.7, every $(1,2)^{\star}$-$\breve{g}$-open set U containing x is such that $U \cap H^c \neq \phi$. That is, every $(1,2)^{\star}$-$\breve{g}$-open set U containing x is such that $U \nsubseteq H$. This implies by Definition 4.1(2), $x \notin (1,2)^{\star}$-$\breve{g}$-$int(H)$. That is, $x \in ((1,2)^{\star}$-$\breve{g}$-$int(H))^c$ and so $(1,2)^{\star}$-$\breve{g}$-$cl(H^c) \subseteq ((1,2)^{\star}$-$\breve{g}$-$int(H))^c$. Thus $((1,2)^{\star}$-$\breve{g}$-$int(H))^c = (1,2)^{\star}$-$\breve{g}$-$cl(H^c)$.
2. Follows by taking complements in (1).
3. Follows by replacing H by H^c in (1).

5. Conclusion

The notions of sets and functions in bitopological spaces and fuzzy topological spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences. By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all bitopological sets and functions defined will have many possibilities of applications in digital topology and computer graphics.

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