Malaya Journal of Matematik, Vol. 7, No. 3, 468-471, 2019
https://doi.org/10.26637/MJM0703/0017

Numerical investigation of the fuzzy integro-differential equations by He′s Homotopy perturbation method

S. Sekar[1]* and A. S. Thirumurugan[2]

Abstract
In this paper, the authors introduced the He's Homotopy Perturbation Method (HHPM) for solving the Fuzzy Integro-Differential equations (FIDE) [13]. The obtain discrete solutions were compared with another method taken from the literature [13]. The new method has a lower computational cost which effects the time consumption. The numerical example was given to highlight the efficiency of this method.

Keywords
Integro-Differential Equations, Fuzzy Integro-Differential Equations, Single-term Haar wavelet series, He's Homotopy Perturbation Method.

AMS Subject Classification
41A45, 41A46, 41A58.

[1] *Department of Mathematics, Chikkanna Government Arts College, Tiruppur – 641 602, Tamil Nadu, India.*
[2] *Department of Mathematics, Mahendra Arts and Science College, Kalipatti, Namakkal-637 501, Tamil Nadu, India.*
***Corresponding author**: [1] sekar_nitt@rediffmail.com; [2]thirumuruganas08@gmail.com
Article History: Received **14** April **2019**; Accepted **09** June **2019**

Contents

1. Introduction

The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) in both theoretical and numerical points of view have been developed in recent years. Prior to discussing fuzzy integro-differential equations (FIDE) and their numerical treatments, it is necessary to present a brief introduction of the previous works about FDE and FIE. When a physical system is modelled under the differential sense; it finally gives a fuzzy differential equation, a fuzzy integral equation or a fuzzy integro-differential equation and hence, the solution of integro-differential equations have a major role in the fields of science and engineering. Nonlinear integro-differential equations are usually hard to solve analytically and exact solutions are scarce. Therefore, they have been of great interest by several authors [S. Abbasbandy and T. Allahviranloo [2, 3]; T. Allahviranloo and N. A. Kiani [7]; A. Arikoglu and I. Ozkol [8].

If we take He's homotopy perturbation method [14–21] getting applied to the wide class problems in physics, biology and chemical reactions. If provides this result in a computable terms of rapid convergent series. The specialty is integro-differential equations using fuzzy initial value problems have been taken for research by many mathematicians and physicists. He's homotopy perturbation method plays an important role in both the analysis and numerical result of integro-differential equations [16, 19, 21]. He's homotopy perturbation method can have a significant impact on what is looked upon a practical approach and on the types of problems that can be answered. However, working with fuzzy integro-differential equations places special demands on He's homotopy perturbation method codes. Although some cases can be answered analytically, the majority of fuzzy integro-differential equations are too complicated to have analytical results.

In this way, Allahviranloo et al. [7] proposed FDTM for solving first order fuzzy differential equation under strongly H-differentiability. Moreover, Arikoglu et al. [8] has been proposed differential transform method for solving integro-differential equations. Integro-differential equations in fuzzy setting are a natural way to model uncertainty of dynamical systems. On the other hand, integro-differential equations in crisp sense has been studied by some numerical methods that can be found in [M.I. Berenguer et al. [9], J. Pour-Mahmoud et al. [12]] for example.

We study the solution of fuzzy integro-differential equations in parametric form by homotopy analysis method (HAM) [SJ. Liao [10, 11]] that is the analytic approach to get series solutions of various types of linear and nonlinear equations. Recently Abbasbandy [1] found the solution of Kawahara equation, generalized Zakharov equation [S. Abbasbandy, E. Babolian and M. Ashtiani [4]], MHD Falkne-Skan flow [S. Abbasbandy and T. Hayat [5]] and nonlinear boundary value problems [S. Abbasbandy and E. Shivanian [6]]. In this article we developed numerical methods for FIDE to get the discrete solutions via He's Homotopy Perturbation method which was studied by Sekar et al. [14–21]. The problems are solved in two methods; one is single-term Haar wavelet series method (STHWS) which is presented previously by Sekar et al. [13] and another one is He's Homotopy Perturbation method which has been developed exclusively to deal with fuzzy integro-differential equations. The approximate solutions obtained are compared with the exact solutions of the fuzzy integro-differential equations, and are found to be very accurate. Error graphs for approximate and exact solutions are presented in a graphical form to highlight the high accuracy and the wide applicability of He's Homotopy Perturbation method approach and the same is demonstrated with numerical examples.

2. He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [16, 19, 21]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t, p) : \Omega \times [0,1] \to \Re$, which satisfies:

$$H(U, p) = (1-p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, \tag{2.2}$$

where $p \in [0,1], t \in \Omega$

or

$$H(U, p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, \tag{2.3}$$

where $p \in [0,1], t \in \Omega$. where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t, p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \ldots \tag{2.4}$$

Setting $p = 1$, results in the approximate solution of Eq.(2.1)

$$U(t) = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + \ldots$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (2.3), we obtain

$$\left.\begin{aligned} U(t) = U(0) + \textstyle\int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt \\ -p[\textstyle\int_0^t (NU(t) - f(t))dt] \end{aligned}\right\} \tag{2.5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \ldots$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\left.\begin{aligned} &p^0 : U_0(t) = u_0 + \textstyle\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ &p^1 : U_1(t) = -\textstyle\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ &p^2 : U_2(t) = -\textstyle\int_0^t NU_1(t)dt \\ &\vdots \\ &p^j : U_j(t) = -\textstyle\int_0^t NU_{j-1}(t)dt \end{aligned}\right\} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (2.7) results in $U_1(t) = U_2(t) = U_3(t) = \ldots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$$

can be used in Eq. (2.8), where $a_0, a_1, a_2, \ldots$ are known coefficients and α_n are unknown ones, which must be computed.

3. General format for the Fuzzy Integro-Differential Equations

Now we consider the Fuzzy Integro-Differential Equations is of the form

$$\frac{dX(t)}{dt} = f(t, X(t), \int_{t_0}^{t} k(t,s,X(s))ds)), t \in T = [t_0, b],$$
$$X(t_0) = X_0$$

where $f : T \times L_2 \times L_2 \to L_2, k : T^2 \times L_2 \to L_2$ are m.s. continuous fuzzy mappings with respect to $t, s, t_0, b \in R, X_0 \in L_2$.

4. Numerical Example for the Fuzzy Integro-Differential Equations

In this section, the exact solution and approximated solution obtained by He's Homotopy Perturbation method and single-term Haar wavelet series method with $n = 10$. To show the efficiency of the He's Homotopy Perturbation method, we have considered the following problem taken from [M. Zeinali, S. Shahmorad and K. Mirnia [22] and Sekar et al. [13]], with step size $\pi/20$ along with the exact solutions.

The absolute errors between them are tabulated and are presented in Table 1. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected step size and are presented in Figure 1 for the following problem, using three dimensional effects.

4.1 Example

Consider the fuzzy number A along with the r-cuts $[A]^r = [r^2 + r, 4 - r^3 - r]$ for $r \in [0,1]$. Let the functions $k : [0,1] \times [0,1] \to R$ and $f : [0,1] \to R_F$ be given by

$$k(x,t) = 0.1 sin(\tfrac{x}{2} sin(t))$$
$$f(x) = (\tfrac{1}{2} cos(\tfrac{x}{2}) - 0.1 sin^2(\tfrac{x}{2}) + \tfrac{0.1}{3} sin(\tfrac{x}{2}) sin(\tfrac{3x}{2})).A$$

Then the fuzzy integro-differential equation

$$y'(x) = f(x) + \int_0^x k(x,t) y(t) dt, x \in [0, \tfrac{\phi}{2}], y(0) = 0$$

Has the exact solution

$$y(x) = sin(\tfrac{x}{2}).A$$

To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of $n = 10$ and are presented in Figure 1 for the following problem to highlight the efficiency of the He's Homotopy Perturbation Method, using three dimensional effects.

Table 1. Example 4.1 – Error Calculations of x

	x	STHWS Error	HHPM Error
1	$\pi/20$	0.5356E-11	0.5356E-13
2	$2\pi/20$	0.9213E-10	0.9213E-12
3	$3\pi/20$	0.6001E-10	0.6001E-12
4	$4\pi/20$	0.9991E-09	0.9991E-11
5	$5\pi/20$	0.5862E-09	0.5862E-11
6	$6\pi/20$	0.1634E-09	0.1634E-11
7	$7\pi/20$	0.9268E-08	0.9268E-10
8	$8\pi/20$	0.7567E-08	0.7567E-10
9	$9\pi/20$	0.4327E-08	0.4327E-10
10	$\pi/2$	0.1002E-08	0.1002E-10

Figure 1. Error estimation of Example 4.1 at $\underline{y}(t,r)$

5. Conclusions

The obtained results (approximate solutions) of the FIDE show that the He's Homotopy Perturbation Method works well for finding the state vector. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in the He's Homotopy Perturbation Method when compared to the single-term Haar wavelet series method discussed by Sekar et al. [13], which yields a small error, along with the exact solutions. From the Figure 1, it can be predicted that the error is very less in He's Homotopy Perturbation Method when compared to the single-term Haar wavelet series method [13]. Hence the He's Homotopy Perturbation Method is more suitable for studying the FIDE. Finally, in this article, it is concluded that from the Table and Figure, which indicate the error to be almost, less with the FIDE using He's Homotopy Perturbation Method. Hence, it can be said that the He's Homotopy Perturbation Method is more suitable to study the FIDE.

Acknowledgment

The authors gratefully acknowledge the Dr. G. Balaji, Professor of Mathematics & Head, Department of Science & Humanities, Al-Ameen Engineering College, Erode — 638 104, for encouragement and support. The authors also heartfelt thank to Dr. M. Vijayarakavan, Associate Professor, Department of Mathematics, VMKV Engineering College, Salem – 636 308, Tamil Nadu, India, for his kind help and suggestions.

References

[1] S. Abbasbandy, Homotopy analysis method for the Kawahara equation, *Nonlinear Analysis: Real World Applications*, 11 (2010), 307-312.

[2] S. Abbasbandy, T. Allah Viranloo, Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method of Order 2, *J. Sci. I. A. U (JSIAU)*, 11 (2001), 3176-3188.

[3] S. Abbasbandy, T. Allah Viranloo, Numerical solutions of fuzzy differential equations by Taylor method, *Journal of Computational Methods and Applied Mathematics*, 2 (2002), 113-124.

[4] S. Abbasbandy, E. Babolian, M. Ashtiani, Numerical solution of the generalized Zakharov equation by homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simulat.*, 14 (2009), 4114-4121.

[5] S. Abbasbandy, T. Hayat, Solution of the MHD Falkner-Skan flow by homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simulat.*, 14 (2009), 3591-3598.

[6] S. Abbasbandy, E. Shivanian, Prediction of multiplicity of solutions of nonlinear boundary value problems: Novel application of homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simulat.*, 15 (2010), 3830-3846.

[7] T. Allahviranloo, N. A. Kiani, Solving fuzzy differential equations by differential transformation method, *Information Science, Elsevier*, 179 (2009), 956-966.

[8] A. Arikoglu, I. Ozkol, Solution of boundary value problems for integro-differential equations by using differential transform method, *Applied Mathematics and Computation*, 168 (2005), 1145-1158.

[9] M.I. Berenguer, M.A. Fortes, A.I. Garralda Guillem, M. Ruiz Gallan, Linear Volterra integro-differential equation and Schauder bases, *Appl. Math. Comput*, 159 (2004), 495-507.

[10] S.J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. thesis. Shanghai Jiao Tong University, (1992).

[11] S.J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, CRC Press, Boca Raton: Chapman Hall, (2003).

[12] J. Pour-Mahmoud, M.Y. Rahimi-Ardabili, S. Shahmorad, Numerical Solution of the System of Fredholm Integro-Differential Equations by the Tau Method, *Applied Mathematics Computation*, 168(1) (2005), 465-478.

[13] S. Sekar and C. Jaisankar, Numerical Analysis of the Fuzzy Integro-Differential Equations using Single-Term Haar Wavelet Series, *International Journal of Mathematics Trends and Technology*, 5(2) (2014), 168-175.

[14] S. Sekar and A. Sakthivel. Numerical Investigations of linear first order Fuzzy Differential Equations using He's Homotopy Perturbation Method. *IOSR Journal of Mathematics*, 11 (2015), 33-38.

[15] S. Sekar and A. Sakthivel, Numerical Investigations of linear second order fuzzy differential equations using He's Homotopy Perturbation Method. *International Journal of Advanced Science and Engineering Research*, 1 (2016), 1112-1117.

[16] S. Sekar and A. Sakthivel, Numerical investigation of the hybrid fuzzy differential equations using He's homotopy perturbation method, *Malaya Journal of Mathematik*, 5(2) (2017), 475-482.

[17] S. Sekar and A. Sakthivel, Numerical Investigation of the Fuzzy Differential Inclusions using HHPM, Mathematica Aeterna, *International Journal for Pure and Applied Mathematics*, 7(2) (2017), 105-112.

[18] S. Sekar and A. Sakthivel, Numerical Investigation of the n^{th} order Fuzzy Differential Equations using He's Homotopy Perturbation Method, *International Journal of Pure and Applied Mathematics*, 116(4) (2017), 899-911.

[19] S. Sekar and A. S. Thirumurugan, Numerical Investigations of integro-differential equations using He's Homotopy Perturbation Method. *International Journal of Advanced Science and Engineering Research*, 1(1) (2016), 1006-1011.

[20] S. Sekar and A. S. Thirumurugan, Numerical Investigation of the Nonlinear Integro-Differential Equations using He's Homotopy Perturbation Method, *Malaya Journal of Mathematik*, 5(2) (2017), 389-394.

[21] S. Sekar and A. S. Thirumurugan, Numerical Investigation of the weakly singular Volterra integro-differential equations using He's Homotopy perturbation method, *Malaya Journal of Mathematik*, 6(1) (2018), 129-132.

[22] M. Zeinali, S. Shahmorad and K. Mirnia, Fuzzy integro-differential equations: discrete solution and error estimation, *Iranian Journal of Fuzzy System*, 10(1) (2013), 107-122.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆