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Pell graceful labeling of graphs

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Abstract

In this paper, we introduce a new concept of Pell graceful labeling as follows. An injective function f from $V(G)$ into $\{0,1,2,...,p_q\}$ is Pell graceful if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{p_1, p_2, ..., p_q\}$. A graph $G(p,q)$ which admits a Pell graceful labeling is called a Pell graceful graph, where p_q is the q^{th} Pell number in the Pell sequence. Here, Pell graceful labeling of some family of graphs are obtained. Its non-existence are established.

Keywords

Pell sequence, Pell graceful labeling, Pell graceful graph.

AMS Subject Classification

05C78

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Contents

1. Introduction

All graphs considered here are simple, finite and undirected. The terms not defined in this paper are used as in Harary [2]. This paper presents results on graph labeling.

Labeled graphs find their applications in Coding Theory and Communication Network Addressing.

Rosa introduced the concept of graceful labeling f of a (p,q) graph G as follows: f is a graceful labeling if f is an injection from $V(G)$ to the set $\{0,1,2,...,q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

By following Acharya and Hegde, a new type of labeling called Fibonacci graceful labeling is introduced. The numbers $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, ...$ is the sequence of Fibonacci numbers.

As an extension to Fibonacci graceful labeling, in this paper, we introduce Pell graceful labeling. Here, Pell graceful labeling of certain families of graphs are discussed. Also its non-existence are established.

2. Main Results

Definition 2.1. *Let $G(p,q)$ be a graph. A injective function f from $V(G)$ into $\{0,1,2,...,F_q\}$, where F_q is the q^{th} Fibonacci number is said to be* ***Fibonacci graceful*** *if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, ..., F_q\}$.*

If a graph $G(p,q)$ admits a Fibonacci graceful labeling then G is called a ***Fibonacci graceful graph****.*

As an extension to Fibonacci graceful labeling, we introduce Pell graceful labeling.

Definition 2.2. *Let $G(p,q)$ be a graph. An injective function f from $V(G)$ into $\{0,1,2,...,p_q\}$ where p_q is the q^{th} Pell number in the Pell sequence is said to be* ***Pell graceful*** *if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{p_1, p_2, ..., p_q\}$.*

If a graph $G(p,q)$ admits a Pell graceful labeling then G is called a ***Pell graceful graph****.*

Remark 2.3. *The classic Pell sequence is obtained as follows:*

$$p_0 = 0,$$

$$p_1 = 1$$

and

$$p_{n+1} = 2p_n + p_{n-1} \text{ for all } n \geq 1.$$

(i.e.) $\{0,1,2,5,12,29,70,...\}$ *is the Pell sequence.*

Illustration 2.1

In Fig. 2.1, we provide an example of a Pell graceful labeling of a graph.

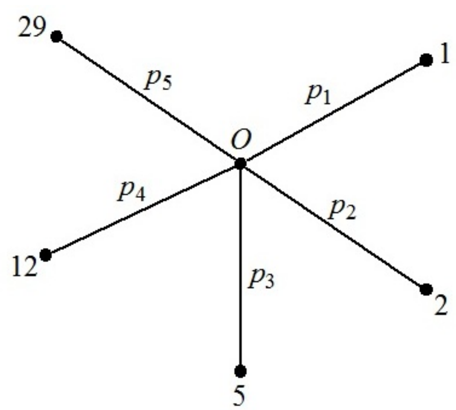

Fig. 2.1

Observation 2.1

1. The edge number p_q has to be induced from the adjacent vertex labels 0 and p_q. Therefore 0 and p_q are ought to be vertex labels in a Pell graceful graph.

2. All vertices adjacent to the vertex labeled with 0 have to receive Pell numbers as their labels.

3. If $\{l_1(=0), l_2, l_3, ..., l_n(= p_q)\}$ is a set of vertex labels of a Pell graceful graph then changing each label l_i to $p_q - l_i$ also gives a Pell graceful labeling of the graph.

Theorem 2.4. *The cycle C_3 is not a Pell graceful graph.*

Proof. Suppose C_3 is a Pell graceful graph, then there exists an injective function $f : V(G) \to \{0,1,2,...,p_q\}$ such that the induced edge labels are Pell numbers $\{p_1, p_2, p_3\} = \{1,2,5\}$.

Let u, v, w be the vertices of the cycle C_3. By Observation 2.1 (1), $f(u) = 0, f(v) = p_3 = 5$.

Let $f(w) = x$. Then $f^*(uv) = p_3$ as in Fig. 2.2.

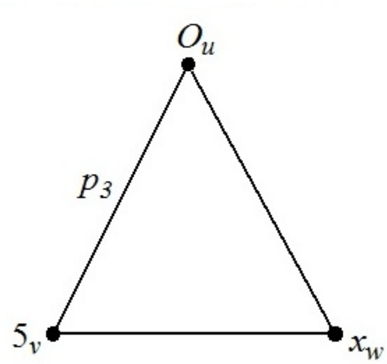

Fig. 2.2

By Observation 2.1(2), x has to receive the Pell number 1 or 2.

If $x = 1$, then $f^*(vw) = 4$ which is not a Pell number, $a \Rightarrow\Leftarrow$.

If $x = 2$, then also $f^*(vw) = 3$ which is not a Pell number, $a \Rightarrow\Leftarrow$ to f is a Pell graceful labeling.

Hence, C_3 is not a Pell graceful graph.

□

Observation 2.2

The absolute difference of any two Pell numbers (except 1 and 2) is not a Pell number.

Theorem 2.5. *The wheel W_3 is not a Pell graceful graph.*

Proof. Suppose W_3 is a Pell graceful graph then there exists an injective function $f : V(G) \to \{0,1,2,...,p_q\}$ such that the induced edge labels are Pell numbers $\{p_1, p_2, ..., p_6\} = \{1,2,5,12,29,70\}$.

Let u_1, u_2, u_3, u_4 be the vertices of W_3 and the ordinary labeling of W_3 is denoted as in Fig. 2.3.

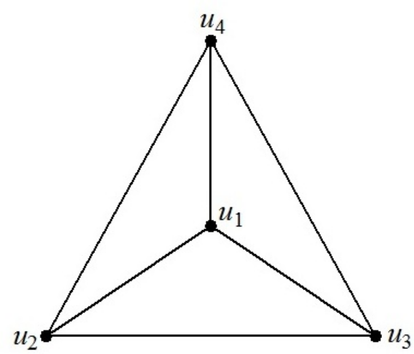

Fig. 2.3

Case 1: $f(u_1) = 0$

By Observation 2.2(2), the vertices u_2, u_3, u_4 are ought to be labeled with Pell numbers.

By Observation 2.2(1), assume without loss of generality that $f(u_2) = 70$.

Then u_3 and u_4 cannot hold 1 and 2 as their vertex labels. For, otherwise

$$f^*(u_1u_3) = f^*(u_3u_4) = 1, a \Rightarrow\Leftarrow .$$

Now by Observation 2.2(3), f cannot be a Pell graceful labeling.

Case 2: $f(u_2) = 0$ **(say)**

Then either $f(u_1), f(u_3)$ or $f(u_4) = 70$.

Let $f(u_1) = 70, f(u_3) = x, f(u_4) = y$ (say).

By Observation 2.2(2), x and y are Pell numbers and cannot be from the set $\{1,2\}$.

Again by Observation 2.2(3), f cannot be a Pell graceful labeling.

Hence, W_3 is not a Pell graceful graph.

□

Theorem 2.6. *Every path P_n of length n is Pell graceful for all $n \geq 1$.*

Proof. Let P_n be a path of length n.

Let $\{v_0, v_1, ..., v_n\}$ be its vertex set and $\{e_1, e_2, ..., e_n\}$ be its edge set where $e_i = v_{i-1}v_i$ for $i = 1, 2, ..., n$ as denoted in Fig. 2.4.

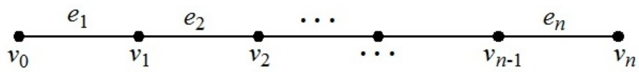

Fig. 2.4: Ordinary labeling of P_n

We know that

$$|V(P_n)| = n + 1,$$

$$|E(P_n)| = n.$$

First we label the vertices as follows:

Consider the following labeling f on $V(P_n)$.

$$f(v_0) = 0$$

$$f(v_i) = p_n - p_{n-1} + p_{n-2} ... + (-1)^{i-1} + p_{n-(i-1)}$$

for $i = 1, 2, ..., n$.

Here, the induced edge labels are distinct Pell numbers.

Hence, the path P_n is Pell graceful for all $n \geq 3$. □

Illustration 2.2

The Pell graceful labeling of P_8 is given in Fig. 2.5.

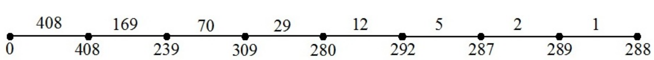

Fig. 2.5: Pell graceful labeling of P_8

Illustration 2.3

The Pell graceful labeling of P_9 is given in Fig. 2.6.

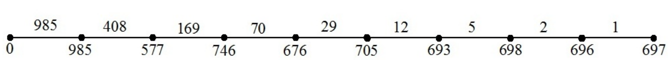

Fig. 2.6: Pell graceful labeling of P_9

Definition 2.7. *An olive tree O_n is a collection of n paths joined in one of the end vertices, where the i^{th} path has length i.*

Theorem 2.8. *Olive trees O_n are Pell graceful for all $n \geq 3$.*

Proof. Let O_n be the olive tree having n paths of length $1, 2, ..., n$ adjoined at one vertex v_0.

Let the vertices of O_n be $\{v_0, v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2(n-1)}, v_{31}, v_{32}, ..., v_{3(n-2)}, ..., v_{n(1)}\}$.

The ordinary labeling of O_n be as given in Fig. 2.7.

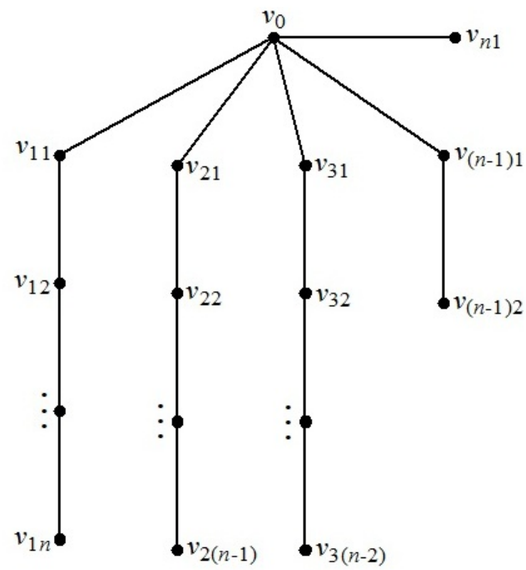

Fig. 2.7: Ordinary labeling of olive trees O_n

Clearly

$$|V(O_n)| = \frac{n^2 + n + 2}{2}$$

and

$$|E(O_n)| = \frac{n(n+1)}{2} = m$$

Consider the following labeling f for O_n

$$f(v_0) = 0$$
$$f(v_{i1}) = p_{m+1-i} \qquad i = 1, 2, ..., n$$
$$f(v_{i2}) = f(v_{i1}) - p_{m+1-n-i} \qquad i = 1, 2, ..., n-1$$
$$f(v_{i3}) = f(v_{i2}) - p_{m+2-2n-i} \qquad i = 1, 2, ..., n-2$$
$$f(v_{i4}) = f(v_{i3}) - p_{m+4-3n-i} \qquad i = 1, 2, ..., n-3$$

Continuing in this manner, subtracting the appropriate Pell numbers from $f(v_{i4}), f(v_{i5}), ...,$ we get the label for $v_{i5}, v_{i6}, ...$

Thus, the induced edge labels are distinct Pell numbers. Hence, the olive trees O_n are Pell graceful for all $n \geq 3$. □

Illustration 2.4

Pell graceful labeling of O_3 is given in Fig. 2.8

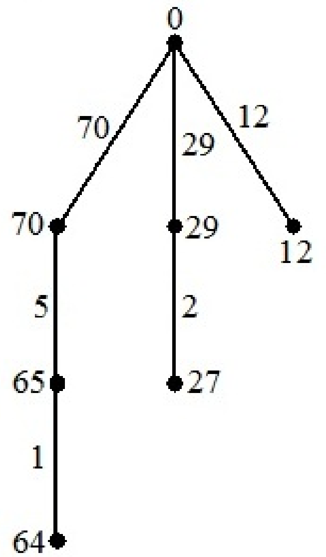

Fig. 2.8: Pell graceful labeling of O_3

Illustration 2.5

Pell graceful labeling of O_4 is given in Fig. 2.9.

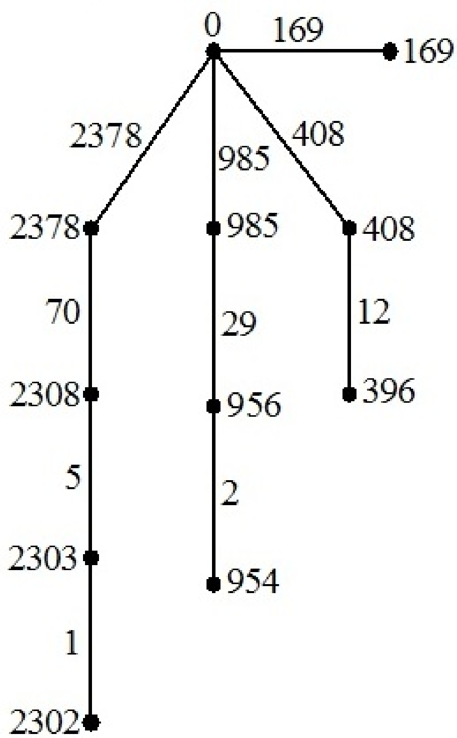

Fig. 2.9: Pell graceful labeling of O_4

Theorem 2.9. *The Combs $P_n \odot K_1$ are Pell graceful for all $n \geq 3$.*

Proof. Let the vertices of $P_n \odot K_1$ be $\{u_1, u_2, ..., u_{n+1}, v_1, v_2, ..., v_{n+1}\}$ and the edges of $P_n \odot K_1$ be as denoted in Fig. 2.10.

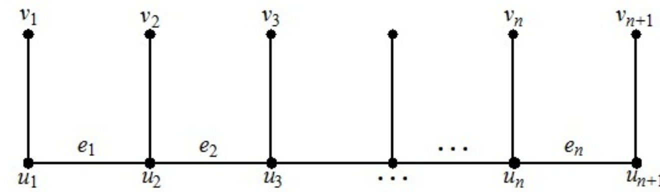

Fig. 2.10: Ordinary labeling of $P_n \odot K_1$

We know that

$$|V(P_n \odot K_1)| = 2n + 2 = p$$

and

$$|E(P_n \odot K_1)| = 2n + 1 = q.$$

We first label the vertices of $P_n \odot K_1$ as follows.
Define $f : V(P_n \odot K_1) \to \{0, 1, 2, ..., p_q\}$ by

Case 1: for $n = 3$

$$f(u_2) = 0$$
$$f(v_1) = p_2 + 1$$
$$f(v_2) = p_4$$
$$f(v_3) = 28$$
$$f(u_{2i-1}) = p_{2i-1} \quad \text{for } i = 1, 2$$

Case 2: for $n \geq 4$

$$f(u_{n+1}) = 0$$
$$f(u_n) = p_{2n+1}$$
$$f(u_i) = p_{2n+1} - \sum_{j=0}^{(n-1)-i} p_{2n-2j}$$
for $i = 1, 2, ..., n-1$
$$f(v_i) = p_{2n+1} - p_2 - \sum_{j=0}^{(n-1)-i} p_{2n-2j}$$
for $i = 1$
$$f(v_i) = p_{2n+1} - p_{2i-1} - \sum_{j=0}^{(n-1)-i} p_{2n-2j}$$
for $i = 2, 3, ..., n-1$
$$f(v_n) = p_{2n+1} - p_{2i-1}$$
$$f(v_{n+1}) = 1$$

The induced edge labels are distinct Pell numbers. Hence, the Combs $P_n \odot K_1$ are Pell graceful for all $n \geq 3$. □

Illustration 2.6

Pell graceful labeling of $P_4 \odot K_1$ is given in Fig. 2.11.

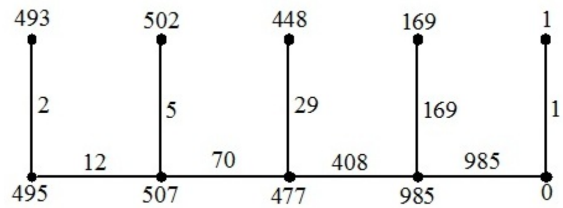

Fig. 2.11: Pell graceful labeling of $P_4 \odot K_1$.

Illustration 2.7

Pell graceful labeling of $P_5 \odot K_1$ is given in Fig. 2.12.

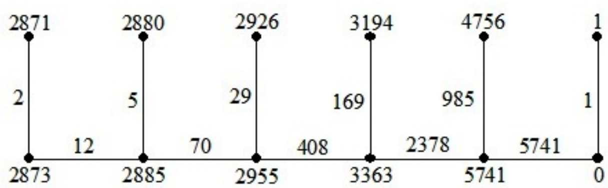

Fig. 2.12: Pell graceful labeling of $P_5 \odot K_1$.

Theorem 2.10. *complete graph K_n is not a Pell graceful for all $n \geq 3$.*

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertex set of K_n. Clearly, each vertex v_i is adjacent to every other vertex $v_j \forall j$.

Let v_1 be labelled with 0. Then, in order to receive p_q as edge label, by Observation 2.5(1).

$f(v_j) = p_q$ for some j say $j = 2$. (i.e.)

$$f(v_1) = 0,$$
$$f(v_2) = p_q$$

Now
$$f^*(v_1 v_2) = p_q.$$

By Observation 2.5(2), every vertex of K_n has to be a Pell number.

By Observation 2.7, the difference is not a Pell number, which is a contradiction to the fact that

$$f^*(E(G)) = \{p_1, p_2, ..., p_q\}.$$

Hence, K_n is not a Pell graceful graph for all n.

□

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