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A proposed rational numerical one-step integrator of order eight for initial value problems

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Abstract
In this paper, we derive a one-step rational numerical integrator of order eight for solving stiff and non-stiff initial value problems (IVP). It is proved that the proposed method is consistent and convergent. It is also found that the method is L-stable. Numerical result shows that the proposed method is efficient and accurate.

Keywords
Rational numerical integrator, Convergent, Stability.

AMS Subject Classification
65L04, 65L05

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1. Introduction

The development of numerical solvers for ordinary differential equations (ODE) is a wide and mature area of research. In particular for non-stiff ODE, many successful approaches have been studied and used, which provide excellent efficiency. But for stiff ODE a more careful treatment is required.

Stiff equations arise in a system of ODE $y^1 = f(y(t))$ when Eigen values of its Jacobian matrix $\partial f/\partial t$ differ in orders of magnitude. Stiff problem has got a wide variety of applications including the study of spring and damping system, the analysis of control system and problems in the chemical kinetics, biochemistry, weather prediction, mathematical biology and electronics.

When the familiar classical sixth order Runge-Kutta (RK6) method [2] and the implicit Adam-Bash forth Moulton (ABM) predictor-corrector method are used for solving stiff equations, they become very inefficient since the step size is controlled by stability requirement rather than accuracy requirement [1]. They are based on the approximation of the solution by a polynomial, an approach that is too expensive when high accuracy is required.

Although most of the numerical analysts were confident that the implicit methods work better in producing results for stiff problems, some of the explicit methods which recently was proven.

The aim of this paper is to present a new L-stable explicit one step eighth order method for numerically solving the stiff ODE. The method done for seventh order is found in [15].

We consider the initial value problem (IVP)

$$\begin{aligned} &y^{'} = f(x,y), \quad y(x_0) = y_0; \\ &\quad y, f \in R^M \text{ and } x \in [a,b], \ a,b \in R \end{aligned} \tag{1.1}$$

whose solution may contain singularities. The points of singularities of the solution of the IVP (1.1) are the poles of the rational interpolant. It is assumed that $f(x,y)$ satisfy the Lipschitz condition.

The rational nonlinear schemes for the numerical solution of (1.1) are given in Lambert and Shaw [9], Lambert [10], Luke et al [11] and Fatunla [3–6], Fatunla and Aashikpelokai [7], Ikhile [8], Otunta and Ikhile [14] and Otunta and Nwachukwu [15]. These schemes are based on rational function interpolation since rational functions are more accurate than polynomials in representing a function in the neighbourhood of singularities.

Otunta and Ikhile [12, 13], Fatunla and Aashikpelokai [7], Otunta and Nwachukwu [14, 15] constructed schemes of order three, four, five, and six, seven respectively.

In this paper, we develop a one-step rational integrator of order eight with a better accuracy of solution.

Definition 1.1. ***(Stiff IVP)***
A system of IVP of the form (1.1) is said to be stiff if the eigen value λ_i of the Jacobian matrix $\left[\frac{\partial f}{\partial y}\right]$ at every integration point x have negative real parts and differ greatly in magnitude.

Also, the eigen values λ_t satisfy the following conditions .

(i) $Re(\lambda_t) < 0,\ t = 1,2,....,m$ *and*

(ii) $\frac{\max_t |\lambda_t|}{\min_t |\lambda_t|} = S > 1$; *S is the stiffness ratio.*

Definition 1.2. ***(Stability Function)***
The stability function $R(z)$ of a numerical method is the rational function that satisfies $y_{n+1} = R(h\lambda)y_n$, where y_{n+1} is the approximation generated when the numerical formula is applied to the Dahlquist's simple test problem $y' = \lambda y, \lambda \in C$, the complex plane.

Definition 1.3. ***(Stability Domain)***
The stability domain, S of a Runge-Kutta method is given by, $S = \{z \in C : R(z) \leq 1\}$ where $R(z)$ is the stability function of the method.

Definition 1.4. ***(A-Stability)***
A numerical method is A-stable if the numerical solution of test problem $y' = \lambda y, \lambda \in C$ is bounded in the entire left half-plane, C. This is equivalent to $C \subseteq S$ or in terms of the stability function that $\forall y \in R : R(iy) \leq 1$ and $R(z)$ is analytic for $Re(z) < 0$.

Definition 1.5. ***(L-Stability)***
A given one-step method is said to be L-Stable if it is A-stable and in addition,

$$\lim_{Re(\bar{h})\to\infty} |S(\bar{h})| = 0$$

where $S(\bar{h})$ is the stability function.

2. Derivation of the Rational Numerical Integrator

Recently, Otunta and Ikhile [15] considered the following rational function approximation for the IVP (1.1).

$$y(x) = A + \frac{xP_{k-1}(x)}{1+\sum_{j=1}^{k} b_j x^j},\ k > 0, \tag{2.1}$$

where

$$P_k(x) = \sum_{j=0}^{k} a_j x^j,\ k \geq 1 \tag{2.2}$$

We therefore consider the one-step scheme

$$y_{n+1} = A + \frac{x_{n+1}P_{k-1}(x_{n+1})}{1+\sum_{j=1}^{k} b_j x_{n+1}^j},\ k \geq 1, \tag{2.3}$$

Thus, we interpolate the theoretical solution of (1.1) by

$$y(x) = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4} \tag{2.4}$$

The resultant one step scheme is given by

$$y_{n+1} = \frac{a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4}{1 + b_1x_{n+1} + b_2x_{n+1}^2 + b_3x_{n+1}^3 + b_4x_{n+1}^4} \tag{2.5}$$

We write (2.5) as,

$$y_{n+1} = \left(a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4\right) \left[1 + \sum_{r=1}^{\infty}(-1)^r\left(\sum_{j=1}^{4} b_j x_{n+1}^j\right)^r\right] \tag{2.6}$$

and superimposing it on

$$y(x_{n+1}) = \sum_{j=1}^{\infty} \frac{h^j y_n^{(j)}}{j!}; \qquad y_n^{(a)} = y_a \tag{2.7}$$

we get,

$$\begin{aligned}&\left(a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4\right)\\ &\quad = \left(1 + b_1x_{n+1} + b_2x_{n+1}^2 + b_3x_{n+1}^3 + b_4x_{n+1}^4\right)\\ &\left(y_n + hy_n^I + \frac{h^2y_n^{II}}{2!} + \frac{h^3y_n^{III}}{3!} + \frac{h^4y_n^{IV}}{4!}\right.\\ &\quad \left. + \frac{h^5y_n^{V}}{5!} + \frac{h^6y_n^{VI}}{6!} + \frac{h^7y_n^{VII}}{7!} + \frac{h^8y_n^{VIII}}{8!}\right) + 0(h^9)\end{aligned} \tag{2.8}$$

We obtain the method parameters from (2.8) as :

$$a_0 = y_n \tag{2.9}$$

$$a_1 = y_nb_1 + \frac{hy_n^I}{x_{n+1}} \tag{2.10}$$

$$a_2 = y_nb_2 + \frac{hy_n^I}{x_{n+1}}b_1 + \frac{h^2y_n^{II}}{2!x_{n+1}^2} \tag{2.11}$$

$$a_3 = y_nb_3 + \frac{hy_n^I}{x_{n+1}}b_2 + \frac{h^2y_n^{II}}{2!x_{n+1}^2}b_1 + \frac{h^3y_n^{III}}{3!x_{n+1}^3} \tag{2.12}$$

$$a_4 = y_nb_4 + \frac{hy_n^I}{x_{n+1}}b_3 + \frac{h^2y_n^{II}}{2!x_{n+1}^2}b_2 + \frac{h^3y_n^{III}}{3!x_{n+1}^3}b_1 + \frac{h^4y_n^{IV}}{4!x_{n+1}^4} \tag{2.13}$$

By Writing (2.13) as a combination of (2.9) – (2.12), we arrive at a system of simultaneous equations where for each positive integer m, the term $\frac{h^m y_n^m}{m! x_{n+1}^m}$ is a real number.

The system, in matrix form, is as shown below:

$$\begin{bmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & hy_n^I \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{h^2 y_n^{II}}{2!} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{h^3 y_n^{III}}{3!} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{h^4 y_n^{IV}}{4!} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} \\ \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} \\ \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} \\ \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} \end{bmatrix}$$

This is of the form $AX = B$, we may use cramer's rule to obtain b_1, b_2, b_3, b_4 so that,

$$b_1 = \frac{T_1}{T} \tag{2.14}$$

$$b_2 = \frac{T_2}{T} \tag{2.15}$$

$$b_3 = \frac{T_3}{T} \tag{2.16}$$

$$b_4 = \frac{T_4}{T} \tag{2.17}$$

where

$$T = \begin{vmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & hy_n^I \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{h^2 y_n^{II}}{2!} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{h^3 y_n^{III}}{3!} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{h^4 y_n^{IV}}{4!} \end{vmatrix}$$

$$T_1 = \begin{vmatrix} \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & hy_n^I \\ \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{h^2 y_n^{II}}{2!} \\ \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{h^3 y_n^{III}}{3!} \\ \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{h^4 y_n^{IV}}{4!} \end{vmatrix}$$

$$T_2 = \begin{vmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & hy_n^I \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{h^2 y_n^{II}}{2!} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{h^3 y_n^{III}}{3!} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{h^4 y_n^{IV}}{4!} \end{vmatrix}$$

$$T_3 = \begin{vmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} & hy_n^I \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} & \frac{h^2 y_n^{II}}{2!} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} & \frac{h^3 y_n^{III}}{3!} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} & \frac{h^4 y_n^{IV}}{4!} \end{vmatrix}$$

$$T_4 = \begin{vmatrix} \frac{h^4 y_n^{IV}}{4!x_{n+1}^3} & \frac{h^3 y_n^{III}}{3!x_{n+1}^2} & \frac{h^2 y_n^{II}}{2!x_{n+1}} & \frac{-h^5 y_n^{V}}{5!x_{n+1}^4} \\ \frac{h^5 y_n^{V}}{5!x_{n+1}^3} & \frac{h^4 y_n^{IV}}{4!x_{n+1}^2} & \frac{h^3 y_n^{III}}{3!x_{n+1}} & \frac{-h^6 y_n^{VI}}{6!x_{n+1}^4} \\ \frac{h^6 y_n^{VI}}{6!x_{n+1}^3} & \frac{h^5 y_n^{V}}{5!x_{n+1}^2} & \frac{h^4 y_n^{IV}}{4!x_{n+1}} & \frac{-h^7 y_n^{VII}}{7!x_{n+1}^4} \\ \frac{h^7 y_n^{VII}}{7!x_{n+1}^3} & \frac{h^6 y_n^{VI}}{6!x_{n+1}^2} & \frac{h^5 y_n^{V}}{5!x_{n+1}} & \frac{-h^8 y_n^{VIII}}{8!x_{n+1}^4} \end{vmatrix}$$

$$\begin{aligned} T = \frac{h^{16}}{3456x_{n+1}^6}\Bigg[& \frac{y_n^{IV}}{4}\Bigg[\frac{y_n^{IV}}{6}\left(\frac{y_n^{IV}y_n^{IV}}{4} - \frac{y_n^{V}y_n^{III}}{5}\right) - \frac{y_n^{III}}{15}\left(\frac{y_n^{V}y_n^{IV}}{2} - \frac{y_n^{VI}y_n^{III}}{3}\right) + \frac{y_n^{II}}{10}\left(\frac{y_n^{V}y_n^{V}}{5} - \frac{y_n^{VI}y_n^{IV}}{6}\right)\Bigg] \\ & - \frac{y_n^{III}}{15}\Bigg[\frac{y_n^{V}}{2}\left(\frac{y_n^{IV}y_n^{IV}}{4} - \frac{y_n^{V}y_n^{III}}{5}\right) - \frac{y_n^{III}}{3}\left(\frac{y_n^{VI}y_n^{IV}}{4} - \frac{y_n^{VII}y_n^{III}}{7}\right) + \frac{y_n^{II}}{4}\left(\frac{y_n^{VI}y_n^{V}}{5} - \frac{y_n^{VII}y_n^{IV}}{7}\right)\Bigg] \\ & + \frac{y_n^{II}}{20}\Bigg[\frac{y_n^{V}}{5}\left(\frac{y_n^{V}y_n^{IV}}{2} - \frac{y_n^{VI}y_n^{III}}{3}\right) - \frac{y_n^{IV}}{3}\left(\frac{y_n^{VI}y_n^{IV}}{4} - \frac{y_n^{VII}y_n^{III}}{7}\right) + \frac{y_n^{II}}{5}\left(\frac{y_n^{VI}y_n^{VI}}{6} - \frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg] \\ & - \frac{y_n^{I}}{10}\Bigg[\frac{y_n^{V}}{10}\left(\frac{y_n^{V}y_n^{V}}{5} - \frac{y_n^{VI}y_n^{IV}}{6}\right) - \frac{y_n^{IV}}{12}\left(\frac{y_n^{VI}y_n^{V}}{5} - \frac{y_n^{VII}y_n^{IV}}{7}\right) + \frac{y_n^{III}}{15}\left(\frac{y_n^{VI}y_n^{VI}}{6} - \frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg]\Bigg] \end{aligned}$$

and therefore

$$T = \frac{h^{16}U}{3456x_{n+1}^6} \tag{2.18}$$

where

$$U = U_1 + U_2 + U_3 + U_4$$

$$U_1 = \frac{y_n^{IV}}{4}\Bigg[\frac{y_n^{IV}}{6}\left(\frac{y_n^{IV}y_n^{IV}}{4} - \frac{y_n^{V}y_n^{III}}{5}\right) - \frac{y_n^{III}}{15}\left(\frac{y_n^{V}y_n^{IV}}{2} - \frac{y_n^{VI}y_n^{III}}{3}\right) + \frac{y_n^{II}}{10}\left(\frac{y_n^{V}y_n^{V}}{5} - \frac{y_n^{VI}y_n^{IV}}{6}\right)\Bigg]$$

$$U_2 = \frac{y_n^{III}}{15}\Bigg[\frac{y_n^{V}}{2}\left(\frac{y_n^{IV}y_n^{IV}}{4} - \frac{y_n^{V}y_n^{III}}{5}\right) - \frac{y_n^{III}}{3}\left(\frac{y_n^{VI}y_n^{IV}}{4} - \frac{y_n^{VII}y_n^{III}}{7}\right) + \frac{y_n^{II}}{4}\left(\frac{y_n^{VI}y_n^{V}}{5} - \frac{y_n^{VII}y_n^{IV}}{7}\right)\Bigg]$$

$$
\begin{aligned}
U_3 = \frac{y_n^{II}}{20}\Big[&\frac{y_n^{V}}{5}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&-\frac{y_n^{IV}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{5}\Big(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
U_4 = -\frac{y_n^{I}}{10}\Big[&\frac{y_n^{V}}{10}\Big(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\Big)\\
&-\frac{y_n^{IV}}{12}\Big(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\Big)\\
&+\frac{y_n^{III}}{15}\Big(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big]
\end{aligned}
$$

we have,

$$
\begin{aligned}
T_1 = \frac{h^{17}}{69120x_{n+1}^{7}}\Big[&\frac{-y_n^{V}}{1}\Big[\frac{y_n^{IV}}{6}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&\quad-\frac{y_n^{III}}{15}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&\quad+\frac{y_n^{II}}{10}\Big(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{30}\Big)\Big]\\
&-\frac{y_n^{III}}{3}\Big[\frac{-y_n^{VI}}{3}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&\quad-\frac{y_n^{III}}{21}\Big(\frac{y_n^{IV}y_n^{VII}}{1}-\frac{y_n^{VII}y_n^{III}}{2}\Big)\\
&\quad+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VII}y_n^{V}}{5}-\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\Big]\\
&+\frac{y_n^{II}}{2}\Big[\frac{y_n^{VI}}{45}\Big(\frac{y_n^{V}y_n^{IV}}{20}-\frac{y_n^{VI}y_n^{III}}{1}\Big)\\
&\quad-\frac{y_n^{IV}}{42}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}-\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&\quad+\frac{y_n^{II}}{35}\Big(\frac{-y_n^{VII}y_n^{VI}}{3}-\frac{y_n^{VIII}y_n^{V}}{4}\Big)\Big]\\
&-\frac{y_n^{I}}{3}\Big[\frac{-y_n^{VI}}{10}\Big(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\Big)\\
&\quad-\frac{y_n^{IV}}{24}\Big(\frac{-y_n^{VII}y_n^{V}}{5}-\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\\
&\quad+\frac{y_n^{III}}{35}\Big(\frac{-y_n^{VII}y_n^{VI}}{3}+\frac{y_n^{VIII}y_n^{V}}{4}\Big)\Big]\Big]
\end{aligned}
$$

and therefore

$$
T = \frac{h^{17}V}{69120x_{n+1}^{7}} \tag{2.19}
$$

where

$$
V = V_1 + V_2 + V_3 + V_4
$$

$$
\begin{aligned}
V_1 = \frac{-y_n^{V}}{1}\Big[&\frac{y_n^{IV}}{6}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&-\frac{y_n^{III}}{15}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&+\frac{y_n^{II}}{10}\Big(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{30}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
V_2 = -\frac{y_n^{III}}{3}\Big[&\frac{-y_n^{VI}}{3}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&-\frac{y_n^{III}}{21}\Big(\frac{y_n^{IV}y_n^{VII}}{1}-\frac{y_n^{VII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VII}y_n^{V}}{5}-\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
V_3 = \frac{y_n^{II}}{2}\Big[&\frac{y_n^{VI}}{45}\Big(\frac{y_n^{V}y_n^{IV}}{20}-\frac{y_n^{VI}y_n^{III}}{1}\Big)\\
&-\frac{y_n^{IV}}{42}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}-\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{II}}{35}\Big(\frac{-y_n^{VII}y_n^{VI}}{3}-\frac{y_n^{VIII}y_n^{V}}{4}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
V_4 = -\frac{y_n^{I}}{3}\Big[&\frac{-y_n^{VI}}{10}\Big(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\Big)\\
&-\frac{y_n^{IV}}{24}\Big(\frac{-y_n^{VII}y_n^{V}}{5}-\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\\
&+\frac{y_n^{III}}{35}\Big(\frac{-y_n^{VII}y_n^{VI}}{3}+\frac{y_n^{VIII}y_n^{V}}{4}\Big)\Big]
\end{aligned}
$$

we have,

$$
\begin{aligned}
T_2 = \frac{h^{18}}{207360x_{n+1}^{8}}\Big[&\frac{y_n^{IV}}{4}\Big[\frac{-y_n^{IV}}{3}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&\quad-\frac{y_n^{III}}{21}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&\quad+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VII}y_n^{V}}{5}+\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\Big]\\
&+\frac{y_n^{V}}{5}\Big[\frac{y_n^{V}}{2}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&\quad-\frac{y_n^{III}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&\quad+\frac{y_n^{II}}{4}\Big(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\Big)\Big]\\
&+\frac{y_n^{II}}{1}\Big[\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&\quad+\frac{y_n^{VI}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&\quad+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\Big]\\
&-\frac{hy_n^{I}}{5}\Big[\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{V}}{5}+\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\\
&\quad+\frac{y_n^{VI}}{12}\Big(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\Big)\\
&\quad+\frac{y_n^{III}}{21}\Big(\frac{-y_n^{VIII}y_n^{VI}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\Big]\Big]
\end{aligned}
$$

and therefore

$$
T_2 = \frac{h^{18}S}{207360x_{n+1}^{8}} \tag{2.20}
$$

where

$$
S = S_1 + S_2 + S_3 + S_4
$$

$$
\begin{aligned}
S_1 = \frac{y_n^{IV}}{4}\Big[&\frac{-y_n^{IV}}{3}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&-\frac{y_n^{III}}{21}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VII}y_n^{V}}{5}+\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
S_2 = \frac{y_n^{V}}{5}\Big[&\frac{y_n^{V}}{2}\Big(\frac{y_n^{IV}y_n^{IV}}{4}-\frac{y_n^{V}y_n^{III}}{5}\Big)\\
&-\frac{y_n^{III}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{4}\Big(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
S_3 = \frac{y_n^{II}}{1}\Big[&\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{VI}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
S_4 = -\frac{hy_n^{I}}{5}\Big[&\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{V}}{5}+\frac{y_n^{VIII}y_n^{IV}}{8}\Big)\\
&+\frac{y_n^{VI}}{12}\Big(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\Big)\\
&+\frac{y_n^{III}}{21}\Big(\frac{-y_n^{VIII}y_n^{VI}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\Big]
\end{aligned}
$$

we have,

$$
\begin{aligned}
T_3 = \frac{h^{19}}{414720x_{n+1}^{9}}\Big[&\frac{y_n^{IV}}{48}\Big[\frac{y_n^{IV}}{42}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{VI}}{15}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&+\frac{y_n^{II}}{35}\Big(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\Big)\Big]\\
&-\frac{y_n^{III}}{15}\Big[\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{VI}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{IV}}{7}\Big)\Big]\\
&-\frac{y_n^{V}}{10}\Big[\frac{y_n^{V}}{5}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&-\frac{y_n^{IV}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{5}\Big(\frac{y_n^{VI}y_n^{VI}}{8}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big]\\
&-\frac{y_n^{I}}{10}\Big[\frac{y_n^{V}}{35}\Big(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\Big)\\
&-\frac{y_n^{IV}}{21}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\\
&-\frac{y_n^{VI}}{15}\Big(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big]\Big].
\end{aligned}
$$

Hence we obtain

$$
T_3 = \frac{h^{19}W}{414720x_{n+1}^{9}}, \tag{2.21}
$$

where

$$
W = W_1 + W_2 + W_3 + W_4
$$

$$
\begin{aligned}
W_1 = \frac{y_n^{IV}}{48}\Big[&\frac{y_n^{IV}}{42}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{VI}}{15}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&+\frac{y_n^{II}}{35}\Big(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
W_2 = -\frac{y_n^{III}}{15}\Big[&\frac{y_n^{V}}{14}\Big(\frac{-y_n^{VII}y_n^{IV}}{1}+\frac{y_n^{VIII}y_n^{III}}{2}\Big)\\
&+\frac{y_n^{VI}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{7}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{IV}}{7}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
W_3 = -\frac{y_n^{V}}{10}\Big[&\frac{y_n^{V}}{5}\Big(\frac{y_n^{V}y_n^{IV}}{2}-\frac{y_n^{VI}y_n^{III}}{3}\Big)\\
&-\frac{y_n^{IV}}{3}\Big(\frac{y_n^{VI}y_n^{IV}}{4}-\frac{y_n^{VII}y_n^{III}}{7}\Big)\\
&+\frac{y_n^{II}}{5}\Big(\frac{y_n^{VI}y_n^{VI}}{8}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
W_4 = -\frac{y_n^{I}}{10}\Big[&\frac{y_n^{V}}{35}\Big(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\Big)\\
&-\frac{y_n^{IV}}{21}\Big(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\Big)\\
&-\frac{y_n^{VI}}{15}\Big(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\Big)\Big].
\end{aligned}
$$

We now find,

$$
\begin{aligned}
T_4 = \frac{h^{20}}{414720x_{n+1}^{10}}\Bigg[& \frac{y_n^{IV}}{12}\Bigg[\frac{y_n^{IV}}{14}\left(\frac{y_n^{IV}y_n^{VIII}}{8}+\frac{y_n^{V}y_n^{VII}}{5}\right) \\
& -\frac{y_n^{III}}{35}\left(\frac{-y_n^{V}y_n^{VII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\right) \\
& -\frac{y_n^{VI}}{10}\left(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\right)\Bigg] \\
& -\frac{y_n^{III}}{15}\Bigg[\frac{y_n^{V}}{14}\left(\frac{y_n^{IV}y_n^{VIII}}{8}+\frac{y_n^{V}y_n^{VII}}{5}\right) \\
& -\frac{y_n^{III}}{21}\left(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\right) \\
& -\frac{y_n^{VI}}{12}\left(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\right)\Bigg] \\
& +\frac{y_n^{II}}{10}\Bigg[\frac{y_n^{V}}{7}\left(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\right) \\
& -\frac{y_n^{IV}}{42}\left(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\right) \\
& -\frac{y_n^{VI}}{30}\left(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg] \\
& +\frac{y_n^{V}}{10}\Bigg[\frac{y_n^{V}}{10}\left(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\right) \\
& -\frac{y_n^{IV}}{12}\left(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\right) \\
& +\frac{y_n^{III}}{15}\left(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg]\Bigg]
\end{aligned}
$$

and therefore

$$T_4 = \frac{h^{20}R}{414720x_{n+1}^{10}}, \tag{2.22}$$

where

$$R = R_1 + R_2 + R_3 + R_4$$

$$
\begin{aligned}
R_1 = \frac{y_n^{IV}}{12}\Bigg[& \frac{y_n^{IV}}{14}\left(\frac{y_n^{IV}y_n^{VIII}}{8}+\frac{y_n^{V}y_n^{VII}}{5}\right) \\
& -\frac{y_n^{III}}{35}\left(\frac{-y_n^{V}y_n^{VII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\right) \\
& -\frac{y_n^{VI}}{10}\left(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
R_2 = -\frac{y_n^{III}}{15}\Bigg[& \frac{y_n^{V}}{14}\left(\frac{y_n^{IV}y_n^{VIII}}{8}+\frac{y_n^{V}y_n^{VII}}{5}\right) \\
& -\frac{y_n^{III}}{21}\left(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\right) \\
& -\frac{y_n^{VI}}{12}\left(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
R_3 = \frac{y_n^{II}}{10}\Bigg[& \frac{y_n^{V}}{7}\left(\frac{-y_n^{V}y_n^{VIII}}{4}+\frac{y_n^{VI}y_n^{VII}}{3}\right) \\
& -\frac{y_n^{IV}}{42}\left(\frac{-y_n^{VI}y_n^{VIII}}{8}+\frac{y_n^{VII}y_n^{VII}}{7}\right) \\
& -\frac{y_n^{VI}}{30}\left(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
R_4 = \frac{y_n^{V}}{10}\Bigg[& \frac{y_n^{V}}{10}\left(\frac{y_n^{V}y_n^{V}}{5}-\frac{y_n^{VI}y_n^{IV}}{6}\right) \\
& -\frac{y_n^{IV}}{12}\left(\frac{y_n^{VI}y_n^{V}}{5}-\frac{y_n^{VII}y_n^{IV}}{7}\right) \\
& +\frac{y_n^{III}}{15}\left(\frac{y_n^{VI}y_n^{VI}}{6}-\frac{y_n^{VII}y_n^{V}}{7}\right)\Bigg].
\end{aligned}
$$

Substituting (2.18) and (2.19) in (2.14), (2.18) and (2.20) in (2.15), (2.18) and (2.21) in (2.16) and (2.18) and (2.22) in (2.17) respectively, we get

$$b_1 = \frac{hV}{20Ux_{n+1}} \tag{2.23}$$

$$b_2 = \frac{h^2S}{30Ux_{n+1}^2} \tag{2.24}$$

$$b_3 = \frac{h^3W}{120Ux_{n+1}^3} \tag{2.25}$$

$$b_4 = \frac{h^4R}{60Ux_{n+1}^4}. \tag{2.26}$$

Substituting b_1, b_2, b_3, b_4 from (2.23) - (2.26) in (2.9) - (2.13), we obtain equations.

(2.27) - (2.30) as follows:

$$a_1 = \frac{h\left[Vy_n + 20Uy_n^I\right]}{20Ux_n} \tag{2.27}$$

$$a_2 = \frac{h^2\left[2Sy_n + 3Vy_n^I + 30Uy_n^{II}\right]}{60Ux_{n+1}^2} \tag{2.28}$$

$$a_3 = \frac{h^3\left[Wy_n + 4Sy_n^I + 3Vy_n^{II} + 20y_n^{III}\right]}{120Ux_{n+1}^3} \tag{2.29}$$

$$a_4 = \frac{h^4\left[2Ry_n + Wy_n^I + 2Sy_n^{II} + 5Uy_n^{IV}\right]}{120Ux_{n+1}^4}, \tag{2.30}$$

where

$$
\begin{aligned}
A &= Vy_n + 20Uy_n^I \\
B &= 2Sy_n + 3Vy_n^I + 30Uy_n^{II} \\
C &= Wy_n + 4Sy_n^I + 3Vy_n^{II} + 20y_n^{III} \\
D &= 2Ry_n + Wy_n^I + 2Sy_n^{II} + 5Uy_n^{IV}.
\end{aligned}
$$

Substituting the values of $a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 in (2.4), we obtain

$$y_{n+1} = \frac{120Uy_n + 6hA + 2h^2B + h^3C + Dh^4}{120U + 6hV + 4h^2S + h^3W + 2h^4R}, \tag{2.31}$$

where U,V,W,S,R are taken as in (2.18) – (2.22) respectively and equation (2.31) gives the desired rational numerical integrator.

3. Convergence and Consistency of the method

We prove here the convergence of the method by showing that the method is consistent and stable.

Theorem 3.1. *The one-step, eighth-order method* (2.31) *is consistent and convergent.*

Proof. A one-step numerical method of the form $y_{n+1} - y_n = h\phi(x_n, y_n; h)$ is convergent if and only if it is consistent as given in [2].

If we subtract yn from both sides of (2.31), we obtain

$$\begin{aligned} y_{n+1} - y_n = \Big[& 6h(A - Vy_n) + 2h^2(B - 2Sy_n) \\ & + h^3(C - Wy_n) + h^4(D - 2Ry_n) \Big] \\ & \div \Big[120U + 6hV + 4h^2S + h^3W + 2h^4R \Big] \end{aligned}$$

and

$$\begin{aligned} y_{n+1} - y_n = \Big[& h\lambda^{17} y_n^2 \Big(840 + 6720h\lambda \\ & + 3290(\lambda h)^2 - 4831(\lambda h)^3 \Big) \Big] \\ & \div \Big[\lambda^{16} y_n \Big(840 - 26460\lambda y_n \\ & + 16380(\lambda h)^2 - 10(\lambda h)^3 + 4831(\lambda h)^4 \Big) \Big]. \end{aligned}$$

Then,

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = y_n^{'}.$$

This implies that

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = y_n^1 \cong f(x_n, y_n),$$

and therefore the method is consistent with the initial value problem (1.1). Hence the method is convergent. □

Theorem 3.2. *The method* (2.31) *is L-stable.*

Proof. If we apply to the method (2.31), the well-known Dahquist stability test equation

$$y^{'} = \lambda y, \quad y(x_0) = y_0$$

and $Re(\lambda) < 0$ We get the following values:

$$U_1 = \frac{1}{7200}\lambda^{16} y_n; \quad U_2 = \frac{-1}{4200}\lambda^{16} y_n;$$
$$U_3 = \frac{1}{8400}\lambda^{16} y_n; \quad U_4 = \frac{-1}{63000}\lambda^{16} y_n;$$

and

$$U = \frac{1}{252000}\lambda^{16} y_n,$$

$$V_1 = \frac{-1}{72}\lambda^{17} y_n; \quad V_2 = \frac{43}{2520}\lambda^{17} y_n;$$
$$V_3 = \frac{-73}{12600}\lambda^{17} y_n; \quad V_4 = \frac{1}{8400}\lambda^{17} y_n;$$

and

$$V = \frac{-63}{25200}\lambda^{17} y_n,$$

$$S_1 = \frac{-1}{1120}\lambda^{18} y_n; \quad S_2 = \frac{1}{1400}\lambda^{18} y_n;$$
$$S_3 = \frac{1}{392}\lambda^{18} y_n; \quad S_4 = \frac{-1}{19600}\lambda^{18} y_n;$$

and

$$S = \frac{13}{5600}\lambda^{18} y_n,$$

$$W_1 = \frac{1}{2520}\lambda^{19} y_n; \quad W_2 = \frac{-1}{5880}\lambda^{19} y_n;$$
$$W_3 = \frac{-1}{4200}\lambda^{19} y_n; \quad W_4 = \frac{1}{176400}\lambda^{19} y_n;$$

and

$$W = \frac{-1}{176400}\lambda^{19} y_n,$$

$$R_1 = \frac{7}{4800}\lambda^{20} y_n; \quad R_2 = \frac{-23}{19600}\lambda^{20} y_n;$$
$$R_3 = \frac{377}{352800}\lambda^{20} y_n; \quad R_4 = \frac{1}{63000}\lambda^{20} y_n;$$

and

$$R = \frac{4831}{3528000}\lambda^{20} y_n,$$

By making use of all the values U,V,W,S,R in the following expression, we obtain,

$$\begin{aligned} y_{n+1} - y_n = h\lambda y_n \Big(& 840 - 6720\lambda h \\ & + 3290(\lambda h)^2 - 4831(\lambda h)^3 \Big) \\ & \div \Big(840 - 26460\lambda h - 16380(\lambda h)^2 \\ & - 10(\lambda h)^3 + 4831(\lambda h)^4 \Big) \end{aligned}$$

Setting $\bar{h} = \lambda h$ and

$$S(\bar{h}) = \frac{840 - 6720\bar{h} + 3290(\bar{h})^2 - 4831(\bar{h})^3}{840 - 26460\bar{h} - 16380(\bar{h})^2 - 10(\bar{h})^3 + 4831(\bar{h})^4}$$

we find that,

$$\lim_{Re(\bar{h}) \to \infty} S(\bar{h}) = 0.$$

Therefore the method is L-stable. □

4. Numerical Experiment

We consider the following IVP:

$$\begin{aligned} y_1^{'} &= -1002y_1 + 1000y_2^2, \\ y_2^{'} &= y_1 - y_2(1+y_2), \\ y_1^{'}(0) &= 1, \quad y_2^{'}(0) = 1, \end{aligned} \tag{4.1}$$

The exact solution is

$$y_1 = e{-}2x, y_2 = e{-}x.$$

For the stiff system (4.1), the Euler's method is stable only if $h \leq 1.6791 \times 10^{-3}$ and RK method for (4.1) is stable only if $h \leq 2.6234 \times 10{-}3$. The problem has been integrated on the interval $[0,1]$ and the results using MATLAB are presented in Table 4.1 for adaptive step size h.

The errors have been defined as the maximum of the absolute errors on the nodal points in the integration interval.

$$E_{MAX} = ||y(t_j) - y_j||$$

x	Exact solution		Rational Numerical Integrator		$\|Error\| \times 10^{-9}$	
	y_1	y_2	y_1	y_2	y_1	y_2
0.00	1.0000000000	1.0000000000	1.0000000000	1.0000000000	0.0	0.0
0.10	0.8187307531	0.9048374180	0.8187307553	0.9048374208	2.2	2.8
0.20	0.6703200460	0.8187307510	0.6703200498	0.8187307554	3.8	4.4
0.30	0.5488116361	0.7408182207	0.5488116406	0.7408182246	4.5	3.9
0.40	0.4493289641	0.6703200460	0.4493289688	0.6703200527	4.7	6.7
0.50	0.3678794412	0.6065306597	0.3678794474	0.6065306643	6.2	4.6
0.60	0.3011942119	0.5488116361	0.3011942184	0.5488116410	6.5	4.9
0.70	0.2465969639	0.4965853038	0.2465969683	0.4965853107	4.4	6.9
0.80	0.201896518	0.4493289641	0.201896573	0.4493289697	5.5	5.6
0.90	0.1652988882	0.4065696597	0.1652988920	0.4065696646	3.8	4.9
1.00	0.1353352832	0.3678794412	0.1353352874	0.3678794477	4.2	6.5

Table 4.1 Numerical results of (4.1) using Rational Numerical Integrator

5. Conclusion

The numerical results obtained through our proposed one-step rational numerical integrator exhibits efficiency and accuracy in the test problem. In addition, the proposed method is L-stable. The consistency and convergency is also achieved. Further examination of stiff problems in application areas using this method can be carried out.

References

[1] R.L.Burden, J.D. Faires, *Numerical Analysis*, 5-th Edition, Boston, PWS Publishing Company, (1993).

[2] J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations*, 2^{nd} revised edition, England John Wiley & Sons, (2008).

[3] S.O. Fatunla, Numerical Treatment of Singular / Discontinuous IVPs, *Journal Computers Math. Appl.*, 12b(5/6)(1986), 109–115.

[4] S.O. Fatunla, *Numerical Methods for IVPs in ODES*, Academic Press, Boston, USA, (1988).

[5] S.O. Fatunla, On the numerical solution of Singular IVPs, *ABACUS*, 19(2)(1990), 121–130.

[6] S.O. Fatunla, Recent Development in the Numerical Treatment of Singular; Discontinuous IVPS, *Scientific Computing (ed. S.O. Fatunla), Ada + Jane Press*, (1994), 46–60.

[7] S.O. Fatunla and U.S.L.J. Aashikpelokhai, A Fifth order L-stable Numerical integrate, *Scientific Computing (S.O. Fatunla Ed)*, (1994), 68–86.

[8] M.N.O. Ikhile, Coefficients for studying one-step rational schemes for IVPs in ODEs:I, *Computers and Mathematics with Applications*, 41(2001), 769–781.

[9] J.D. Lambert and B. Shaw, On the numerical solution of $y' = f(x,y)$ by a class of formulae based on rational approximation, *Math. Comp.*, 19(1965), 456–462.

[10] J.D. Lambert, Nonlinear methods for stiff system of ODES, *Conference on the Numerical Solutions of ODEs, Dundee*, (1974), 77–88.

[11] Y.L. Luke, Fair, W. and Wimp, J. Predictor - Corrector formulas based on Rational Interpolants, *Journal Computer and Maths with Applications*, 1(1975), 302–307.

[12] F.O. Otunta and M.N.O. Ikhile, Stability and Convergence of a Class of Variable order non-linear one-step Rational Integrators of IVPs in ODEs, *International Journal of Computer Mathematics*, 62(1996), 199–208.

[13] F.O. Otunta and M.N.O. Ikhile, Efficient Rational one-step Numerical Integrators for Initial Value Problems in Ordinary Differential Equations, *International Journal of Computer Mathematics*, 72(1999), 49-61.

[14] F.O. Otunta and G.G. Nwachukwu, An R [2, 3; 1:5] Rational one-step Integrators for Initial Value Problems

in Ordinary Differential Equations, *Nigerian Journal of Mathematics and Applications*, 16(2)(2003), 146–158.

[15] F.O. Otunta and G.C. Nwachukwu, Rational one-step Numerical Integrator for Initial Value Problems in Ordinary Differential Equations, *Journal of the Nigerian Association of Mathematical Physics*, 9(2005), 285–294.

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