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Existence of continuous solutions for nonlinear functional differential and integral inclusions

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Abstract

In this article, we establish the existence of a positive continuous solution of the functional integral inclusion of fractional order

$$x(t) \in p(t) + I^{\alpha} F_1(t, I^{\beta} f_2(t, x(\varphi(t)))), \quad t \in [0,1], \quad \alpha, \beta \in (0,1).$$

The study holds in the case when the set-valued function has Lipschitz selections.
As an application, we study the initial-value problem of the arbitrary fractional order differential inclusion

$$\frac{dx}{dt} \in F_1(t, D^{\gamma} x(t)), \quad a.e, \quad t \in [0,1], \quad \gamma > 0$$

where $F_1(t, x(t))$ is a Lipschitz set-valued function defined on $[0,1] \times R^+$.

Keywords

Set-valued function, functional Integral inclusion, fixed point theorem, Lipschitz selections.

AMS Subject Classification

47B38, 47H30.

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Contents

1. Introduction

The topic of differential and integral inclusions is of much interest in the subject of set-valued analysis. Differential Equations and Control Processes, the existence theorems for the inclusions problems are generally obtained under the assumptions that the set-valued function is either lower or upper semicontinuous in the domain of its definitions (see [1] and [12]) and for the discontinuity of the set-valued function (see [6]). The integral inclusions have been studied by B.C. Dhage and D. $O^{'}$Regan (see [5] and [12]) for the existence results under Caratheodory Condition of F.

Consider the functional integral inclusion of fractional order

$$x(t) \in p(t) + F_1(t, I^{\alpha} f_2(t, x(\varphi(t)))), \ t \in [0,1], \ \alpha \in (0,1). \tag{1.1}$$

In [8], the authors proved the existence of global integrable solutions for the nonlinear functional integral inclusion (1.1), where the set-valued map $F_1 : (0,1) \times R^+ \to 2^{R^+}$ has nonempty closed values which are satisfying Caratheodory and growth conditions.

Recently, the existence of positive monotonic continuous and integrable solutions of the mixed type integral inclusion

$$x(t) \in p(t) + \int_0^1 k(t,s) F_1(s, I^{\beta} f_2(s, x(s))) ds, \ t \in [0,1], \ \beta > 0 \tag{1.2}$$

has been studied in [9, 10] by using Schauder's and Nonlinear Alternative of Leray-Shauder type fixed-point Theorem. As a generalization of previous results the authors (see [2]) proved the existence of positive integrable solution for the nonlinear functional integral inclusion

$$x(t) \in p(t) + I^{\alpha} F_1(t, I^{\beta} f_2(t, x(\varphi(t)))\; t \in [0,1],\; \alpha, \beta \in (0,1). \tag{1.3}$$

Here, we are going to study the existence of positive continuous solutions for the integral inclusion (1.3), where the set-valued map $F_1 : (0,1) \times R^+ \to 2^{R^+}$ satisfies Lipschitiz condition.
As an application, the initial-value problem of the arbitrary (fractional) Order differential inclusion

$$\frac{dx(t)}{dt} \in p(t) + I^{\alpha} F_1(t, , D^{\gamma} x(t)),\;\; t \in [0,1], \quad \gamma > 0, \tag{1.4}$$

$$x(0) = x_{\circ} \tag{1.5}$$

will be also studied.

2. Preliminaries

In this section, we recall some definitions and basic results of fractional calculus, which will be used throughout the paper.

Let $L^1(I)$ be the class of Lebesgue integrable function on the Interval $I = [a,b]$,
$0 \leq a < b < \infty$ and Let $\Gamma(.)$ be the gamma function.

Definition 2.1. *The Riemann-Liouville of fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by (cf.[14], [15],[16] , and [17])*

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds$$

and when $a = 0$, we have $I^{\alpha} f(t) = I_0^{\alpha} f(t)$.

Definition 2.2. *The (Caputo) fractional order derivative D^{α}, $\alpha \in (0,1]$ of the absolutely continuous function g is defined as (see [4], [15], [16], and [17])*

$$D_a^{\alpha} g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t) \quad , \quad t \in [a,b].$$

For further properties of fractional calculus operator (see [4], [15], [16], and [17])

Definition 2.3. *Let X and Y be two non-empty sets, a set-valued (multivalued) map $F : X \to Y$ is a function associate to any element $x \in X$ a subset $F(x)$ of Y, called the (image) valued of F at x.*

Let F be a set-valued map defined on a Banach space E, f is called a Selection of F if $f(x) \in F(x)$ for every $x \in E$ and we denote by

$$S_F = \{f : f(x) \in F(x), x \in E\}$$

the set of all selections of F (For the properties of the selection of F see [3, 11, 18]).

Definition 2.4. *A set-valued map F from $I \times E$ to family of all nonempty closed subsets of E is called Lipschitzian if there exists $L > 0$, such that for all $t \in I$ and all $x_1, x_2 \in E$, we have*

$$h(F(t,x_1), F(t,x_2)) \leq L \|x_1 - x_2\| \tag{2.1}$$

where, $h(A,B)$ is the Hausdorff distance between the two subsets $A, B \in I \times E$ (For the properties of the Hausdorff distance see ([1])).

The following Theorem [[1], Sect.9, Chap. 1, Th. 1] assume the existence of Lipschitzian selection.

Theorem 2.5. *Let M be a metric space and F be Lipschitzian set-valued function from M into the nonempty compact convex subsets of R^n. Assume, moreover, that for some $\lambda > 0$, $F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball on R^n. Then there exists a constant c and a single-valued function $f : M \to R^n$, $f(x) \in F(x)$ for $x \in M$, this function is Lipschitzian with constant L.*

3. Main results

In this section, we deal with the existence of positive continuous solutions for the fractional integral inclusion (1.3). Now, we consider the following assumptions to establish the existence results:

(i) The function $p(t) : [0,1] \to R^+$ is continuous.

(ii) The function $f_2 : [0,1] \times R^+ \to R^+$ is continuous in its two arguments and there exists a constants c, such that

$$|f_2(t, x_1(t)) - f_2(t, x_2(t))| \leq \; c\; |x_1(t) - x_2(t)|$$

for every $x_1,\; x_2 \; \in R^+$ and $t \in [0,1]$.

(iii) Let $F_1(t, x(t)) : [0,1] \times R^+ \to 2^{R^+}$ be a Lipschitzian set-valued map with nonempty compact convex subset of 2^{R^+}.

(iv) The function $\phi : (0,1) \to (0,1)$ is continuous.

(v) $L\, c < \Gamma(\alpha+1)\Gamma(\beta+1)$.

It is clear that from Theorem 2.5 and assumption (iii), the set of Lipschitiz selection of F_1 is non empty. So, the solution of the single valued integral equation

$$x(t) = p(x) + I^{\alpha} F_1(t, I^{\beta} f_2(t, x(\varphi(t))),\;\; t \in [0,1] \tag{3.1}$$

where $f_1 \in S_{F_1}$, is a solution of (1.3).
It must be noted that f_1 satisfied the Lipschitiz selection

$$|f_1(t,x) - f_1(s,y)| \leq \; L\; (|t-s|, |x-y|).$$

Now, for the existence of a unique continuous solution of the functional integral inclusion (1.3) we have the following theorem.

Theorem 3.1. *Let assumptions (i)-(v) be satisfied, then the inclusion (1.3) has a unique positive solution $x \in C[0,1]$.*

Proof. Let $A : C[0,1] \to C[0,1]$ be the operator defined by

$$Ax(t) = p(t) + I^{\alpha} f_1(t, I^{\beta} f_2(t, x(\varphi(t)))). \tag{3.2}$$

Let $x_1, x_2 \in C[0,1]$. Then, in view of our assumptions, we have

$$\begin{aligned}
&|Ax_1(t) - Ax_2(t)| \\
&\leq |I^{\alpha} f_1(t, I^{\beta} f_2(t, x_1(\varphi(t)))) \\
&- I^{\alpha} f_1(t, I^{\beta} f_2(t, x_2(\varphi(t))))| \\
&\leq |\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [f_1(s, I^{\beta} f_2(s, x_1(\varphi(s)))) \\
&- f_1(s, I^{\beta} f_2(s, x_2(\varphi(s))))]|ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^{\beta} f_2(s, x_1(\varphi(s)))) \\
&- f_1(t, I^{\beta} f_2(s, x_2(\varphi(s))))|ds
\end{aligned}$$

using Lipschitz condition for f_1, we obtain:

$$\begin{aligned}
&|Ax_1(t) - Ax_2(t)| \\
&\leq L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |I^{\beta} f_2(s, x_1(\varphi(s))) \\
&- I^{\beta} f_2(s, x_2(\varphi(s)))|ds \\
&\leq L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |f_2(\tau, x_1(\varphi(\tau))) \\
&- f_2(\tau, x_2(\varphi(\tau)))|d\tau\, ds
\end{aligned}$$

using Lipschitz condition for f_2, we obtain:

$$\begin{aligned}
&|Ax_1(t) - Ax_2(t)| \\
&\leq L\, c \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |x_1(\varphi(\tau)) \\
&- x_2(\varphi(\tau))|d\tau\, ds \\
&\leq Lc \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^{\beta}}{\Gamma(\beta+1)} |x_1(\varphi(\tau)) - x_2(\varphi(\tau))|ds \\
&\leq \frac{Lc}{\Gamma(\beta+1)} \|x_1 - x_2\| \int_0^t \frac{(t-s)^{\alpha-1}(s)^{\beta}}{\Gamma(\alpha)} ds \\
&\leq \frac{L\, c}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|x_1 - x_2\|.
\end{aligned}$$

Then, from assumption (v) we get

$$\|Ax_1 - Ax_2\| \quad < \quad \|x_1 - x_2\|.$$

Hence the map $A : C[0,1] \to C[0,1]$, defined by (3.2), is a contraction, then it has a fixed point $x(t) = Ax(t)$.
Therefore, there exists a unique solution $x \in C[0,1]$ of the integral equation (3.1), from which we deduce that solution satisfy the integer inclusion (1.3), so there exists a solution $x \in C[0,1]$ for inclusion (1.3). □

Corollary 3.2. *The solution of inclusion (1.3) is continuously depends on the S_{F_1} of all Lipschitzian selections of F_1.*

Proof. Let $h_1(t, x(t))$ and $h_2(t, x(t))$ be two different Lipschitzian selections of $F_1(t, x(t))$, such that

$$|h_1(t, x(t)) - h_2(t, x(t)| < \varepsilon, \quad \varepsilon > 0, \ t \in [0,1]$$

then for the two corresponding solutions $x_{h_1}(t)$ and $x_{h_2}(t)$ of (1.3) we have.

$$\begin{aligned}
&x_{h_1}(t) - x_{h_2}(t) \\
&= I^{\alpha} h_1(t, I^{\beta} f_2(t, x_{h_1}(\varphi(t))) - I^{\alpha} h_2(t, I^{\beta} f_2(t, x_{h_2}(\varphi(t)))
\end{aligned}$$

$$\begin{aligned}
&|x_{h_1}(t) - x_{h_2}(t)| \\
&\leq |I^{\alpha} h_1(t, I^{\beta} f_2(t, x_{h_1}(\varphi(t))) - I^{\alpha} h_2(t, I^{\beta} f_2(t, x_{h_2}(\varphi(t)))| \\
&\leq |I^{\alpha} h_1(t, I^{\beta} f_2(t, x_{h_1}(\varphi(t))) - I^{\alpha} h_1(t, I^{\beta} f_2(t, x_{h_2}(\varphi(t)))| \\
&+ |I^{\alpha} h_1(t, I^{\beta} f_2(t, x_{h_2}(\varphi(t))) - I^{\alpha} h_2(t, I^{\beta} f_2(t, x_{h_2}(\varphi(t)))| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |h_1(s, I^{\beta} f_2(s, x_{h_1}(\varphi(s))) \\
&- h_1(s, I^{\beta} f_2(s, x_{h_2}(\varphi(s)))|ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |h_1(s, I^{\beta} f_2(s, x_{h_2}(\varphi(t))) \\
&- h_2(s, I^{\beta} f_2(s, x_{h_2}(\varphi(s))))|ds \\
&\leq L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ |I^{\beta} f_2(s, x_{h_1}(\varphi(s))) \\
&- I^{\beta} f_2(s, x_{h_2}(\varphi(s)))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varepsilon\, ds \\
&\leq L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |f_2(\tau, x_{h_1}(\varphi(\tau))) \\
&- f_2(\tau, x_{h_2}(\varphi(\tau)))|d\tau ds + \frac{\varepsilon}{\Gamma(\alpha+1)} \\
&\leq Lc \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^{\beta}}{\Gamma(\beta+1)} |x_{h_1}(\varphi(s)) - x_{h_2}(\varphi(s))|ds \\
&+ \frac{\varepsilon}{\Gamma(\alpha+1)} \\
&\leq \frac{L\, c}{\Gamma(\beta+1)} \ \|x_{h_1} - x_{h_2}\| \int_0^t \frac{(t-s)^{\alpha-1}(s)^{\beta}}{\Gamma(\alpha)} ds + \frac{\varepsilon}{\Gamma(\alpha+1)}
\end{aligned}$$

$$\|x_{h_1} - x_{h_2}\| \leq \frac{L\, c}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|x_{h_1} - x_{h_2}\| + \frac{\varepsilon}{\Gamma(\alpha+1)}$$

$$\|x_{h_1} - x_{h_2}\| \leq (1 - \frac{L\, c}{\Gamma(\alpha+1)\Gamma(\beta+1)})^{-1} \frac{\varepsilon}{\Gamma(\alpha+1)} = \delta(\varepsilon).$$

From the above estimate, we drive the following inequality:

$$\|x_{h_1} - x_{h_2}\| \leq \delta(\varepsilon)$$

which proves the continuous dependence of the solutions on the set S_{F_1} of all Lipschitzian selections of F_1. This completes the proof. □

4. Differential inclusion

Consider now the initial value problem of the differential inclusion (1.4) with the initial data (1.5).

Theorem 4.1. *Let assumptions of Theorem 3.1 be satisfied, then the initial value problem* (1.4)-(1.5) *has a unique positive solution* $x \in C([0,1])$.

Proof. Let $y(t) = \frac{dx(t)}{dt}$, then the inclusion (1.4), will be

$$y(t) \in p(t) + I^{\alpha} F_1(t, I^{1-\gamma} y(t)). \tag{4.1}$$

Letting $\phi(t) = t$, $f_2(t,x) = x$ and $\beta = 1-\gamma$ and applying Theorem 3.1 on the functional inclusion (4.1) we deduce that there exists a positive continuous solution $y \in C[0,1]$ of the functional inclusion (4.1) and this solution depends continuously on the set S_{F_1}.
This implies that the existence of a solution $x \in C^1[0,1]$,

$$x(t) = x_\circ + \int_0^t y(s)ds$$

of the initial-value problem (1.4)-(1.5).

□

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