Malaya Journal of Matematik, Vol. 7, No. 3, 545-549, 2019
https://doi.org/10.26637/MJM0703/0029

Numerical solution of generalised Pantograph equation using natural continuous extension fourth order Runge-Kutta method

K. Ponnammal[1]* and R. Sayeelakshmi[2]

Abstract
In this paper, we have solved Generalised pantograph equation which is special delay differential equation (DDE) using Natural Continuous Extension Runge-Kutta two stage fourth order Method (NCERKM). A modest effort is taken to derive NCERKM quadrature formula. Cubic Hermite Interpolation is incorporated to estimate the delay term. Numerical Results are given for various coefficients arrived.

Keywords
Delay Differential Equation, Runge-Kutta Method, Continuous Extension, Interpolation.

AMS Subject Classification
34A45.

[1] *PG and Research Department of Mathematics, Periyar E. V. R. College, Tiruchirappalli-620023, Tamil Nadu, India.*
[2] *Department of Mathematics, Mookambigai Engineering College, Tiruchirappalli-622502, Tamil Nadu, India.*
***Corresponding author**: [1] ponnammal_k@yahoo.co.in; [2] sayeelakshmi10@gmail.com
Article History: Received **17** April **2019**; Accepted **11** July **2019**

Contents

1. Introduction

A big and very significant aspect of this is the study of numerical solutions of functional equations in contemporary mathematics. It enables the development of computer systems by considering different functional equations, obtaining numerical outcomes, and finding approximate alternatives. In recent years, numerical solutions have been applied to a wide range of techniques. The British Railways had to make the 1960ś electric locomotive run quicker. A device named pantograph, which collects current from an overhead wire, was a significant thing built for this purpose. Additionally, J. R. Ockendon and A. B. Tayler, [5] researched on an electric locomotive the movement of the pantograph head. They found a unique delay differential equation(DDE) of the type in the solution method for this issue. A special delay differential equation is of the form

$$\frac{dy}{dt} = ay(t) + by(\lambda t) \tag{1.1}$$

where a, b and λ are real constants and $0<\lambda<1, t \in (0,\infty)$. In 1971, the pantograph was mathematically modelled. The pantograph equation has become a major instance of delay differential equations in latest years. In particular, analytic solutions for pantograph delay differential equations can rarely be achieved, so numerical methods have gained increasing attention. The generalized pantograph equations solved numerically in the past years using the technique of Adomian decomposition, Taylor method, and Bessel matrix based on collocation points. Researchers implemented the differential transform technique to get the solution as Taylor expansion. Salih Yalcinbas, Huseyin Hilmi Sorkun and Mehmet, [4] found a numerical method for solutions of pantograph type differential equations with variable coefficients using Bernstein polynomials In addition, the approximate solutions of generalized pantograph equations obtained using the homotopy method and variational iteration methods. Ali, H. Brunner and T. Tang, [6] used the Galerkin methods for solutions of pantograph delay differential equations. Researchers developed the collocation methods to solve the functional delay differential equation. In this work, Natural Continuous Extension Fourth Order Runge-Kutta method quadrature formula is derived. The comparison of certain fourth order Runge-Kutta Butcher coefficients to the numerical solution of first order pantograph equation is investigated. Cubic Hermite Interpolation is used to approximate the delay argument.

2. Methods

Bellen and Zennaro [1] stated the definitions of Natural Continuous Extension(NCE) for all Runge-Kutta(RK) processes.

2.1 continuous Runge-Kutta method for ordinary differential equation

Given mesh $\Delta = (t_1, t_2, \dots, t_N = t_f)$, a v-stage R-K method for the numerical solution of the ODE.

$$\frac{dy}{dt} = g(t, y(t)), y(t_0) = y_0 \tag{2.1}$$

has the form

$$y^i_{n+1} = y_n + h_{n+1}\sum_{j=1}^{v} a_{ij} g(t^j_{n+1}, y^j_{n+1}) \tag{2.2}$$

$$y_{n+1} = y_n + h_{n+1}\sum_{i=1}^{v} b_i g(t^i_{n+1}, y^i_{n+1}) \tag{2.3}$$

where $t^i_{n+1} = t_n + c_i h_{n+1}, \quad c_i = \sum_{i=1}^{v} a_{ij}, \quad i = 1, 2 \dots, v,$ $h_{n+1} = t_{n+1} - t_n$ and v is referred to as the number of stages. The $b_i\prime s$ are called quadrature weights and $c_i\prime s$ are called abscissa. The one-step RK method interpolants provided in (2.2) and (2.3) are created step by step by using information from the underlying mesh interval $[t_n, t_{n+1}]$ only, potentially by including some additional phases, i.e. some additional evaluations of the g(t, y) in equation (2.1). Interpolants obtained from no additional stages are called first-class interpolants. In each sub-interval of the mesh the value acquired from continuous extension $\eta(t)$ is described by a one-step continuous quadrature rule of the form

$$\eta(t_n + \theta h_{n+1}) = y_n + h_{n+1}\sum_{i=1}^{v} b_i(\theta) g(t^i_{n+1}, y^i_{n+1}) \tag{2.4}$$

(or) in the K- Notation

$$\eta(t_n + \theta h_{n+1}) = y_n + h_{n+1}\sum_{i=1}^{v} b_i(\theta) K^i_{n+1} \tag{2.5}$$

where the $b_i(\theta)\prime s$ are polynomials of suitable degree $\leq \delta$ satisfying

$$b_i(0) = 0 \text{ and } b_i(1) = b_i, i = 1, 2 \dots, v \tag{2.6}$$

So as to satisfy the continuity conditions

$$\eta(t_n) = y_n, \eta(t_{n+1}) = y_{n+1}. \tag{2.7}$$

3. Continuous extension of Runge-Kutta methods for delay differential equations

The first order delay differential equation has the form

$$y'(t) = f(t, y, y(t-\tau)) \quad \text{for} \quad t > t_0 \tag{3.1}$$

$y(t) = \psi(t) \quad$ for $\quad t \leq t_0$ $\psi(t)$ is the history initial function, the function $\tau(t, y(t))$ is called the delay, $(t-\tau)$ called the delay argument, the value of $y(t - \tau(t, y(t)))$ is the solution of the delay term. The delay is classified as constant delay, time dependent and state dependent. The conventional technique to solve the DDE is to resolve the local issues step by step

$$\begin{aligned} &\omega^t_{n+1} = f(t, \omega_{n+1}(t), y(t - \tau(t, \omega_{n+1}(t))) \text{ for } t_n < t < t_{n+1} \\ &\omega_{n+1} = y_n \end{aligned} \tag{3.2}$$

$$x(s) = \begin{cases} \psi(s) & \text{for} \quad s \leq t_0 \\ \eta(s) & \text{for} \quad t_0 \leq s \leq t_n \\ \omega_{n+1}(s) & \text{for} \quad t_n \leq s \leq t_{n+1} \end{cases} \tag{3.3}$$

and $\eta(s)$ is the continuous approximate solution computed up to t_n. The overall method for DDE is presented as

$$Y^i_{n+1} = y_n + h_{n+1}\sum_{i=1}^{v} a_{ij} f(t^i_{n+1}, y^i_{n+1}, \eta(t^i_{n+1}, t - \tau(t^i_{n+1}, Y^i_{n+1})), \quad i = 1, 2, \dots s \tag{3.4}$$

$$\begin{aligned} &\eta(t_n + \theta h_{n+1}) \\ &= y_n + h_{n+1}\sum_{i=1}^{v} b_i(\theta) f(t^i_{n+1}, Y^i_{n+1}, \eta(t^i_{n+1}, t - \tau(t^i_{n+1}, Y^i_{n+1})), \\ &0 \leq \theta \leq 1. \end{aligned} \tag{3.5}$$

The method are called the RK method for DDE. The coefficients (A, b) are the underlying discrete RK method, whereas (A, b(θ)) are the interpolants. The pair created the discrete RK method and interpolants is called the underlying continuous RK method. In the mesh interval $[t_n, t_{n+1}]$, the equation (3.4) and (3.5) takes the form

$$\eta(t_n + \theta h_{n+1}) = y_n + h_{n+1}\sum_{i=1}^{v} b_i(\theta) f(t^i_{n+1}, Y^i_{n+1}, y^{i-}_{n+1}), \quad 0 \leq \theta \leq 1$$

$$(3.6)$$

$$Y_{n+1}^{i} = y_n + h_{n+1} \sum_{i=1}^{v} a_{ij} f(t_{n+1}^{i}, Y_{n+1}^{i}, Y_{n+1}^{\bar{i}}) \tag{3.7}$$

where the spurious stages $Y_{n+1}^{\bar{}}{}^{i}$ are given by

$$Y_{n+1}^{\bar{}}{}^{i} = y_n + h_{n+1} \sum_{j=1}^{v} b_j(\theta) f(t_{n+1}^{j}, Y_{n+1}^{j}, Y_{n+1}^{\bar{}} ji) \tag{3.8}$$

if by

$$t_{n+1}^{i} - \tau(t_{n+1}^{i} - Y_{n+1}^{i}) > t_n \text{ and by } Y_n\bar{+}1^{i} = \eta(t_{n+1}^{i} - Y_{n+1}^{i}) \tag{3.9}$$

Otherwise, the system of equations (3.6) , (3.7) and (3.9) has to be solved only for the stage values, $Y_{n+1}^{\bar{j}}, j = 1, 2, \ldots, v$, the system enlarged by equation (3.8) for some it has to be solved also for the relevant spurious stage value $Y_{n+1}^{\bar{i}}$.

4. Natural continuous extension first order interpolant for Runge-Kutta method

Bellen and Zennaro [1] developed the Natural Continuous Extension(NCE) of first class interpolants of RK method. The interpolant $\eta(t)$ in(2.4) of order p is a NCE of the RK method (2.3) of degree q if the polynomials, $i = 1, 2, \ldots, v$ are such that $b_i(\theta)$ satisfies the additional asymptotic orthogonality condition as

$$\left\| \int_{t_n}^{t_{n+1}} G(t)[y\prime(t) - \eta\prime(t)] \right\| = O(h^{p+1}), \tag{4.1}$$

G satisfies

$\eta(t_0) = y_0, \eta(t_0 + h) = y_1$ hold. The RK method is accurate of order p (≥ 1) satisfies $|y(t_0 + h) - y\prime| = O(h^{p+1})$. The approximate solution find iteratively on a mesh $\delta = (t_0, t_1, t_2 \ldots t_N = t_f)$ of the interval (t_0, t_1), such that

$$max \|y\prime(t) - \eta\prime(t)\| = O(h^q). \tag{4.2}$$

It is to be noticed that the collocation polynomial for any one-step first interpolant method is as NCE of degree q = v. The Theorems 1 to 3 are given by Butcher [2].

Theorem 1:

Every RK process (2.3) and (2.5) of order p has a NCE of minimal degree $q = \frac{p+1}{2}$.

Theorem 2:

If the interpolant (2.4) of order (and degree) q is an NCE of the RK method 2.3 and 2.4 of order p, then $q \geq \lfloor \frac{p+1}{2} \rfloor$.

Theorem 3:

Every Runge-Kutta method of (2.2) of order p$\geq$1 has a continuous extension η of order q=1 $\ldots \lfloor \frac{p+1}{2} \rfloor$ The polynomial $b_i(\theta)$ satisfies the condition

$$b_i(\theta) = 0 \text{ and } \int_0^1 b_i\prime(\theta) d\theta = b_i c_i^r \quad r = 0, 1, \ldots, q \tag{4.3}$$

The NCE of RK method is not unique and (4.3) gives a rule to get one. Table 1 and 2 show the coefficient of Butcher [2],[3]. The order condition indicated in Table 1 is used to achieve the NCE of RK method. Table 3 reflects tableau for NCE RK coefficient of order p=4, $c_1 = 0$, $c_2 = c_3 = \frac{1}{2}, c_4 = 1$ and $b_3 \neq 0$

$$a_{41} = \frac{c_3^2(12c_2^2 - 12c_2 + 4) + c_3(12c_2^2 - 15c_2 + 5) + 4c_2^2 - 6c_2 + 2}{2c_2c_3[3 - 4(c_2 + c_3) + 6c_2c_3]}$$

$$a_{42} = \frac{(-4c_3^2 + 5c_3 + c_2 + 2)(1 - c_3)}{2c_2(c_3 - c_2)[3 - 4(c_2 + c_3) + 6c_2c_3]}$$

$$a_{43} = \frac{(1 - 2c_2)(1 - c_3)(1 - c_2)}{c_3 - (c_3 - c_2)[3 - 4(c_2 + c_3) + 6c_2c_3]}$$

$$b_1 = \frac{1 - 2(c_2 + c_3) + 6c_2c_3}{12c_2c_3}$$

$$b_2 = \frac{2c_3 - 1}{12c_2(c_3 - c_2)(1 - c_2)}$$

$$b_3 = \frac{1 - 2c_2}{2c_2(c_3 - c_2)(1 - c_2)}$$

$$b_4 = \frac{3 - 4(c_2 + c_3) + 6c_2c_3}{12c_2(1 - c_2)(1 - c_3)}.$$

Table 1. Order Condition for Continuous RK Method

1	$\sum_{i=1}^{v} b_i(\theta)$	=	θ
2	$\sum_{i=1}^{v} b_i(\theta) c_i$	=	$\frac{1}{2}\theta^2$
3	$\sum_{i=1}^{v} b_i(\theta) c_i^2$	=	$\frac{1}{3}\theta^2$
4	$\sum_{i=1}^{v} b_i(\theta) c_i^3$	=	$\frac{1}{4}\theta^4$
	$\sum_{i,j=1}^{v} b_i(\theta) c_i a_{ij} c_j$	=	$\frac{1}{8}\theta^4$
	$\sum_{i,j=1}^{v} b_i(\theta) c_i a_{ij} c_j^2$	=	$\frac{1}{12}\theta^2$
	$\sum_{i,j,k=1}^{v} b_i(\theta) a_{ij} c_k$	=	$\frac{1}{12}\theta^2$

Table 2. Runge-Kutta Butcher Coefficient Tableau

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & \frac{3b_3-1}{6b_3} & \frac{1}{6b_3} & & \\ 1 & 0 & 1-3b_3 & 3b_3 & \\ \hline & \frac{1}{6} & \frac{2}{3}-b_3 & b_3 & \frac{1}{6} \end{array}$$

From Theorem 1 it follows that NCE of minimal degree $q = \lfloor \frac{p+1}{2} \rfloor$ = 2 for $p = 4$ is derived. The second degree polynomial stated as $b_i(\theta) = \xi_i\theta^2 + \eta_i\theta$ where $\eta_i = b_i - \xi$i=1,2,3 and 4 fulfills the order condition of order 2 in Table 1, Then the equations are given as, $\sum_{i=1}^{v} b_i(\theta) = \theta$ and $\sum_{i=1}^{v} b_i(\theta) c_i = \frac{1}{2}\theta^2$

Putting i = 1,2,3,4 in equation $b_i(\theta)$, we get

$b_1(\theta) = \xi_1\theta^2 + (b_1 - \xi_1)\theta$

$b_2(\theta) = \xi_2\theta^2 + (b_2 - \xi_2)\theta$

$b_3(\theta) = \xi_3\theta^2 + (b_3 - \xi_3)\theta$
$b_4(\theta) = \xi_4\theta^2 + (b_4 - \xi_4)\theta.$
To find $b_4(\theta)$

$$b_4(\theta)c_4 = \frac{1}{2}\theta^2 - b_2(\theta)c_2 - b_3(\theta)c_3$$
$$b_4(\theta) = \left(\frac{1}{2c_4} - \frac{\xi_2 c_2}{c_4} - \frac{\xi_3 c_3}{c_4}\right)\theta^2 - \frac{b_2 c_2}{c_4}\theta + \frac{\xi_2 c_2}{c_4}\theta - \frac{b_3 c_3}{c_4}\theta + \frac{\xi_3 c_3}{c_4}\theta$$

Putting $\lambda = \frac{\xi_2}{c_4}, \quad \mu = \frac{\xi_3}{c_4}$ in the above equation, we have

$$b_4(\theta) = \left(\frac{1}{2c_4} - \lambda c_2 - \mu c_3\right)\theta^2 + b_4\theta - \left(\frac{1}{2c_4} - \lambda c_2 - \mu c_3\right)\theta$$
$$b_4(\theta) = \left(\frac{1}{2c_4} - \lambda c_2 - \mu c_3\right)\theta^2 - \frac{\theta}{c_4}(b_2c_2 + b_3c_3) + \lambda c_2\theta + \mu c_3\theta$$
$$b_4(\theta) = \left(\frac{1}{2c_4} - \lambda c_2 - \mu c_3\right)\theta^2 + b_4\theta - \left(\frac{1}{2c_4} - \lambda c_2 - \mu c_3\right)\theta$$

To find $b_3(\theta)$

$$b_3(\theta)c_3 = \frac{1}{2}\theta^2 - b_2(\theta)c_2 - b_4(\theta)c_4$$
$$b_3(\theta) = \frac{1}{2c_3}\theta^2 - b_2(\theta)c_2c_3 - b_4(\theta)c_4c_3$$

Substituting $b_2(\theta)$ and $b_4(\theta)$

$$b_3(\theta) = -\frac{1}{c_3}[c_2b_2 + c_4b_4] + c_4\mu\theta^2 + \frac{\theta}{2c_3} - c_4\mu$$
$$= -c_4\mu\theta^2 + -c_4\mu\theta + b_3\theta$$

To find $b_2(\theta)$

$$b_2(\theta) = (\frac{1}{2}\theta^2 - b_4(\theta)c_2 - b_3(\theta)c_4$$
$$= -\frac{b_3\theta c_3}{c_2} + \lambda c_4\theta^2 - \frac{b_4\theta c_4}{c_2} + \frac{\theta}{2c_2} - \lambda\theta c_4$$
$$= b_2\theta + \lambda c_4\theta^2 - \lambda c_4\theta$$

To find $b_1(\theta)$

$$b_1(\theta) = \theta - b_2(\theta) - b_3(\theta) - b_4(\theta)$$
$$b_1(\theta) = \theta(1 - b_2 - b_3 - b_4) + \theta^2\left(-c_4(\lambda+\mu) - \frac{1}{2c_4} + \lambda c_2 + \mu c_3\right) + \theta\left(c_4(\lambda+\mu) + \frac{1}{2c_4} - \lambda c_2 + \mu c_3\right)$$

For the finding the NCEs of order q=2, put r = 1 in $\int_0^1 \theta b_i\prime(\theta)d\theta = b_ic_i$, $\xi = 3b_i(2c_i - 1)$.
We know that

$$b_i(\theta) = \xi_i\theta^2 + (b_i - \xi_i)\theta$$
$$b_i(\theta) = 3b_i(2c_i - 1)\theta^2 + (4b_i - 6b_ic_i)\theta$$

and therefore,
$b_i(\theta) = 3b_i(2c_i - 1)\theta^2 + 2b_i(2 - 3c_i)\theta$

Table 3. Butcher tableau for a RK method with 4^{th} order

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$		
1	0	−3	4	
	$\frac{1}{6}$	-$\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{6}$

where $\lambda = \frac{3(2c_2-1)}{c_4}b_3, \mu = \frac{3(2c_3-1)}{c_4}b_3$
Fourth order Runge-Kutta Coefficients are presented in Table. 3.

Case 1:
The continuous extension coefficient becomes
$b_1(\theta) = -\frac{1}{2}\theta^2 + \frac{2}{3}\theta$
$b_2(\theta) = -\frac{2}{3}\theta$
$b_3(\theta) = -\frac{4}{3}\theta$
$b_4(\theta) = -\frac{1}{2}\theta^2 - \frac{1}{3}\theta.$
Similarly
Case 2: For $c_1 = 0, c_3 = 0, c_2 = \frac{1}{2}, c_4 = 1$
$b_1(\theta) = \frac{7}{2}\theta^2 - \frac{14}{3}\theta$
$b_2(\theta) = \frac{2}{3}$
$b_3(\theta) = -4\theta^2 + \frac{16}{3}\theta$
$b_4(\theta) = \frac{1}{2}\theta^2 - \frac{1}{3}\theta.$
Case 3:
For $c_1 = 0, c_3 = 0, c_2 = \frac{1}{2}, c_4 = 1$
$b_1(\theta) = -\frac{3}{6}\theta^2 + \frac{14}{6}\theta$
$b_2(\theta) = -\frac{21}{6}\theta^2 + \frac{14}{6}\theta$
$b_3(\theta) = \frac{2}{3}\theta$
$b_4(\theta) = 4\theta^2 - \frac{8}{3}\theta.$
Case 4:
For $c_1 = 0, c_3 = \frac{2}{3}, c_2 = \frac{1}{3}, c_4 = 1$
$b_1(\theta) = -\frac{3}{8}\theta^2 + \frac{1}{2}\theta$
$b_2(\theta)) = -\frac{3}{8}\theta^2 + \frac{3}{4}\theta$
$b_3(\theta)) = \frac{3}{8}\theta^2$
$b_4(\theta)) = -\frac{3}{8}\theta^2 - \frac{1}{4}\theta.$

5. Results and Discussions

Natural continuous extension Runge-Kutta method with interpolation used for finding approximate solutions of delay differential equations. A first order Pantograph delay Differential equation is applied to NCERKM. In this study, a pantograph equation

$$y\prime(t) = \frac{1}{2}y(t) + \frac{1}{2}e^{\frac{t}{2}}y(\frac{t}{2}), y(0) = 1, 0 \le t \le 1 \qquad (5.1)$$

which has the exact solution $y(t) = e^t$ considered [4].

The Butcher coefficients for Runge-Kutta approximation methods is derived, combined with continuous extensions, are applied to the Pantograph delay differential equations with fixed delay. Cubic Hermite interpolation is used to approximate the delay term. We compared the computed results with the exact solution of the problems where the exact solution of the problems is known. From the numerical results obtained we conclude that these proposed methods are well suited for finding the numerical solution of Pantograph DDE.

Table 4. Results of NCERKM for equation (5.1)

t	CASE 1	CASE 2	CASE 3	CASE 4	Exact Solution
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.103823876	1.103823872	1.107155872	1.103823871	1.105170918
0.2	1.215664315	1.215665071	1.222763828	1.215665069	1.221402758
0.3	1.336071905	1.336072770	1.347413791	1.336072767	1.349858808
0.4	1.465630474	1.465631385	1.481734219	1.465631380	1.491824698
0.5	1.604961780	1.604962734	1.626394520	1.604962728	1.648721271
0.6	1.754727347	1.754728346	1.782107610	1.754728339	1.822118800
0.7	1.915630845	1.915631891	1.949632632	1.915631882	2.013752707
0.8	2.088420672	2.088421766	2.129777839	2.088421755	2.225540928
0.9	2.273892687	2.273893832	2.323403649	2.273893819	2.459603111
1	2.472893112	2.472894311	2.531425889	2.472894296	2.718281828

Conclusion

We have derived coefficients of RK approximation methods aligning with continuous extensions. The continuous extension coefficients can be obtained for different abscissa and four such cases are given here. We have used the proposed NCERKM incorporated with cubic Hermite interpolation, which estimates delay term , to find the solution of the pantograph equation (5.1) in all the four cases. We observe from the Table.4 that the exact solution coincides well with the approximations in case (iii) coefficients in compared with all the other three cases for this particular Pantograph DDE. The closeness of the approximations with the exact solution shows the efficiency of the method. Further studies can be made with the proposed method with other delay differential equations.

References

[1] A. Bellen and M. Zennaro,(2003), *Numerical Methods for Delay Differential Equations*, Clarendon Press, (2003).

[2] J.C. Butcher, Numerical methods for ordinary differential equations in the 20^{th} century, *Journal of Computational and Applied Mathematics*, (2000),125–129.

[3] J.C. Butcher, *Numerical Methods for Ordinary Differential Equations*, Wiley and Sons, England, (2008).

[4] Salih Yalcinbas, Huseyin Hilmi Sorkun and Mehmet Sezer, A numerical method for solutions of pantograph type differential equations with variable coefficients using Bernstein polynomials,*New Trends in Mathematical Sciences*, 2018, 179–195.

[5] J. R. Ockendon and A. B. Tayler,The dynamic of a current collection system for an electric locomotive, *Proc. Roy. Soc. London Ser.*, (1971), 447–468.

[6] Ali, H. Brunner and T. Tang, A spectral method pantograph-type delay differential equations and its convergence analysis , *Journal of Computational Mathematics*, 27(2-3)(2009), 254–265.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆