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Some results on smooth fuzzy subspaces and Hausdorff spaces

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Abstract

In this work, first we prove some interesting results in the context of smooth fuzzy subspaces through bases. In follow, we define the concept of (α,ℓ)-Hausdorff spaces and prove that the intersection of finitely many (α,ℓ)-Hausdorff topologies is again an (α,ℓ)-Hausdorff topology in contrast with the crisp theory; we also prove that product of finitely many (α,ℓ)-Hausdorff spaces is (α,ℓ)-Hausdorff. Finally, we define and discuss the concept of ℓ-Hausdorffness of space.

Keywords

Smooth fuzzy topological spaces, Subspaces, Hausdorffness.

AMS Subject Classification

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Contents

1. Introduction

Following Chang[3], Šostak[11] redefined the concept of fuzzy topology as a fuzzy subset of I^X, satisfying some properties. Later, Ramadan[7] generalized the concept in the name of "smooth fuzzy topological spaces". Peeters, Park, Min, Kim, Abbas, Kalaivani and Roopkumar[1, 4–6] are some of the others who studied the concept in Šostak's sense.

Fang Jin-ming and Yue Yue-li[13] defined the concept of a basis for a given smooth fuzzy topology in 2006. Two necessary and sufficient conditions for a given function $\mathscr{B} : I^X \to [0,1]$ to be a basis for a smooth fuzzy topology were given by them. In [12] the concept of basis is defined and discussed in a way different from the one available in [13]. The approach in [12] is much easier and we follow this approach throughout this paper.

In 1992, Ramadan[8] defined the concept of subspace of a smooth fuzzy topological space. Werner Peeters[6] and Abbas[1] are some others who developed the concept further. In 2004, Abbas generalized the concept of subspaces, for a smooth fuzzy topology defined on a general fuzzy subset. The concept of Hausdorffness was studied by Azad, Yueli Yue, Jinming Fang, Rekha Srivastava and many others[2, 9, 10, 14]. But the definition for Hausdorffness which we are going to give is entirely different from ones available in the literature.

In Section 3, we follow the approach of Abbas[1] and prove some interesting results on the context subspaces, using the concept of basis for a smooth fuzzy topology defined in [12]. In Section 4, we define the concept of (α,ℓ)-Hausdorff space and prove that the intersection of two (α,ℓ)-Hausdorff topologies is again an (α,ℓ)-Hausdorff topology in contrast with the crisp theory. In follow, we prove that product of two (α,ℓ)-Hausdorff spaces is again (α,ℓ)-Hausdorff. Finally we define and discuss the concept of ℓ-Hausdorffness of space.

2. Preliminaries

For any non empty set X, a function $\mu : X \to [0,1]$ is called a fuzzy subset of X. As usual, I^X and I denotes the family of all fuzzy subsets of X and $[0,1]$; 0_X and 1_X denotes the characteristic function of $\emptyset$ and X respectively. The union $\underset{\lambda\in J}{\vee}\mu_\lambda$ and intersection $\underset{\lambda\in J}{\wedge}\mu_\lambda$ of a collection $\{\mu_\lambda : \lambda \in J\}$ of fuzzy sets of X, where J is an arbitrary indexing set, are defined as:

$$\left(\bigvee_{\lambda\in J}\mu_\lambda\right)(x)=\sup_{\lambda\in J}\mu_\lambda(x) \text{ and } \left(\bigwedge_{\lambda\in J}\mu_\lambda\right)(x)=\inf_{\lambda\in J}\mu_\lambda(x).$$

For any two fuzzy subsets A and B of X, with $A\geq B$, the complement $(A-B)$ of B in A is defined as: for all $x\in X$, $(A-B)(x)=A(x)-B(x)$.

Definition 2.1. *[7, 11] Let μ be a fuzzy subset of a nonempty set X and let $\Im_\mu=\{A\in I^X/A\leq\mu\}$. Let $\mathscr{T}:\Im_\mu\to[0,1]$ be a mapping that satisfies:*

i. $\mathscr{T}(\mu)=1$

ii. $\mathscr{T}(0_X)=1$

iii. $\mathscr{T}(A\wedge B)\geq\mathscr{T}(A)\wedge\mathscr{T}(B)$ *for any two fuzzy subsets* $A,B\in\Im_\mu$

iv. $\mathscr{T}(\vee A_\lambda)\geq\wedge\mathscr{T}(A_\lambda)$ *for any collection* $\{A_\lambda\}_{\lambda\in\Lambda}$, *where* $A_\lambda\in\Im_\mu$.

Then $\mathscr{T}$ is called a smooth fuzzy topology on μ and the pair $(\mu,\mathscr{T})$ is called a smooth fuzzy topological space; for any $A\in\Im_\mu$, $\mathscr{T}(A)$ is called the degree of openness of the A. Let $\mathscr{C}:\Im_\mu\to[0,1]$ be the mapping defined by $\mathscr{C}(A)=\mathscr{T}(\mu-A)$. Then $\mathscr{C}(A)$ is called the degree of closedness of A.

Definition 2.2. *[12] Let $\mathscr{B}:\Im_\mu\to[0,1]$ be a function. Consider the following conditions*

B1 *If $x\in X$, $\varepsilon>0$ and $\delta>0$, then there exists $A\in\Im_\mu$ such that $A(x)\geq\mu(x)-\delta$ and $\mathscr{B}(A)\geq1-\varepsilon$,*

B2 *If $x\in X$, $A,B\in\Im_\mu$, $\varepsilon>0$ and $\delta>0$, then there exists $C\in\Im_\mu$ such that $C\leq A\wedge B$, $C(x)\geq(A(x)\wedge B(x))-\delta$, and $\mathscr{B}(C)\geq(\mathscr{B}(A)\wedge\mathscr{B}(B))-\varepsilon$.*

Any function $\mathscr{B}$ satisfying the condition **B1** *is called a subbasis and any subbasis satisfying the condition* **B2** *is called a basis for a smooth fuzzy topology on μ.*

Definition 2.3. *[12] A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of non-zero fuzzy subsets of a fuzzy set A is said to be an inner cover for A if $\bigvee_{\lambda\in\Lambda}A_\lambda=A$.*

Definition 2.4. *[12] Let $\mathscr{B}$ be a basis for a smooth fuzzy topology on μ. Define the smooth fuzzy topology $\mathscr{T}$ from $\Im_\mu$ to $[0,1]$ generated by $\mathscr{B}$ as follows:*
Define $\mathscr{T}(A)=1$ if $A=0_X$; otherwise, define

$$\mathscr{T}(A)=\sup_{\Lambda\in\Gamma}\left\{\inf_{A_\lambda\in\mathfrak{C}_\Lambda}\{\mathscr{B}(A_\lambda)\}\right\},$$

where $\{\mathfrak{C}_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\mathfrak{C}_\Lambda=\{A_\lambda\}_{\lambda\in\Lambda}$ of A.

The following two theorems together give a characterisation for a function $\mathscr{B}:\Im_\mu\to[0,1]$ to be a basis for a smooth fuzzy topology.

Theorem 2.5. *[12] Let $\mathscr{T}$ be a smooth fuzzy topology on μ and let $\mathscr{B}:\Im_\mu\to[0,1]$ be a function satisfying*

i. $\mathscr{T}(A)\geq\mathscr{B}(A)$ *for all* $A\in\Im_\mu$

ii. *if $\delta,\varepsilon>0$, $x\in X$, $A\in\Im_\mu$, then there exists $B\in\Im_\mu$ such that $B(x)\geq A(x)-\delta$, $B\leq A$ and $\mathscr{B}(B)\geq\mathscr{T}(A)-\varepsilon$.*

Then $\mathscr{B}$ is a basis for $\mathscr{T}$.

Theorem 2.6. *[12] If $\mathscr{B}$ is a basis for a given smooth fuzzy topology $\mathscr{T}$ on μ, then*

i. $\mathscr{T}(A)\geq\mathscr{B}(A)$ *for all* $A\in\Im_\mu$.

ii. *if $x\in X$, $A\in\Im_\mu$, $\delta,\varepsilon>0$, then there exists $B\in\Im_\mu$ such that $B(x)\geq A(x)-\delta$, $B\leq A$ and $\mathscr{B}(B)\geq\mathscr{T}(A)-\varepsilon$.*

Definition 2.7. *[12] Let $\mathscr{T}_\mu$ and $\mathscr{T}_\nu$ be smooth fuzzy topologies on μ and ν respectively. A basis $\mathscr{B}$ for the smooth fuzzy topology on $\mu\times\nu$ is defined as a function $\mathscr{B}$ from $\Im_{\mu\times\nu}$ to $[0,1]$ as follows:*

Let $E\in\Im_{\mu\times\nu}$. If E cannot be written as $A\times B$ for any $A\in\Im_\mu$ and $B\in\Im_\nu$, then define $\mathscr{B}(E)=0$. Otherwise define

$$\mathscr{B}(E)=\sup_{\lambda\in\Lambda}\{\inf\{\mathscr{T}_\mu(A_\lambda),\mathscr{T}_\nu(B_\lambda)\}\}$$

where $\{A_\lambda\times B_\lambda\}_{\lambda\in\Lambda}$ is the collection of all possible ways of writing E as $E=A_\lambda\times B_\lambda$, where $A_\lambda\in\Im_\mu$, $B_\lambda\in\Im_\nu$.

The smooth fuzzy topology that $\mathscr{B}$ generates is called the smooth fuzzy product topology on $\mu\times\nu$.

Theorem 2.8. *[12] Let $\mathscr{T}_\mu$ and $\mathscr{T}_\nu$ be smooth fuzzy topologies on μ and ν respectively. Let $\mathscr{B}_\mu,\mathscr{B}_\nu$ be bases for the smooth fuzzy topologies $\mathscr{T}_\mu,\mathscr{T}_\nu$ respectively. Define a function $\mathscr{B}_{\mu\times\nu}:\Im_{\mu\times\nu}\to[0,1]$ as follows:*

Let $E\in\Im_{\mu\times\nu}$. If E cannot be written as $A\times B$ for any $A\in\Im_\mu$ and $B\in\Im_\nu$, then define $\mathscr{B}_{\mu\times\nu}(E)=0$. Otherwise define

$$\mathscr{B}_{\mu\times\nu}(E)=\sup_{\lambda\in\Lambda}\{\inf\{\mathscr{B}_\mu(A_\lambda),\mathscr{B}_\nu(B_\lambda)\}\}$$

where $\{A_\lambda\times B_\lambda\}_{\lambda\in\Lambda}$ is the collection of all possible ways of writing E as $E=A_\lambda\times B_\lambda$, where $A_\lambda\in\Im_\mu$, $B_\lambda\in\Im_\nu$.

then $\mathscr{B}_{\mu\times\nu}$ is a basis for the product topology on $\mu\times\nu$.

In [1] S. E. Abbas quoted that for given $\mathscr{T}_\mu$ and $\nu\in\mathscr{A}_\mu$, one can define $(\mathscr{T}_\mu)_\nu$, the smooth ν-topology induced over ν by $\mathscr{T}_\mu$, in the obvious way. Following him Roopkumar and Kalaivani gave the definition of a subspace smooth fuzzy topology for general smooth fuzzy topological space $(\mu,\mathscr{T})$ in [4].

Definition 2.9. *[4] Let $(\mu,\mathscr{T}_\mu)$ be a smooth fuzzy topological space and let $\nu\in\Im_\mu$. The function $\mathscr{T}_\nu:\Im_\nu\to[0,1]$ defined by $\mathscr{T}_\nu(A)=\sup\{\mathscr{T}_\mu(B)/B\wedge\nu=A,B\in\Im_\mu\}$ is a smooth fuzzy subspace topology induced over ν by $\mathscr{T}_\mu$, with this smooth fuzzy topology ν is called smooth fuzzy subspace of μ.*

3. Some Results on Subspaces

In [1] S. E. Abbas quoted that for given smooth fuzzy topology $\mathscr{T}_\mu$ on μ and $\nu \in \mathscr{A}_\mu$, one can define $(\mathscr{T}_\mu)_\nu$, the smooth ν-topology induced over ν by $\mathscr{T}_\mu$, in the obvious way. Following him Roopkumar and Kalaivani gave the definition of a subspace smooth fuzzy topology for general smooth fuzzy topological space $(\mu, \mathscr{T})$ in [4]. We adopt the definition given by Kalaivani and Roopkumar[4] and prove some results.

Definition 3.1. *[4] Let $(\mu, \mathscr{T}_\mu)$ be a smooth fuzzy topological space and let $\nu \in \Im_\mu$. The function $\mathscr{T}_\nu : \Im_\nu \to [0,1]$ defined by $\mathscr{T}_\nu(A) = \sup\{\mathscr{T}_\mu(B)/B \wedge \nu = A, B \in \Im_\mu\}$ is a smooth fuzzy subspace topology induced over ν by $\mathscr{T}_\mu$, with this smooth fuzzy topology ν is called smooth fuzzy subspace of μ.*

Lemma 3.2. *Let $(\nu, \mathscr{T}_\nu)$ be a smooth fuzzy subspace of $(\mu, \mathscr{T}_\mu)$. If $\mathscr{B}_\mu$ is a basis for $\mathscr{T}_\mu$, then the function $\mathscr{B}_\nu : \Im_\nu \to [0,1]$ defined as*

$$\mathscr{B}_\nu(A) = \sup\{\mathscr{B}_\mu(B)/B \wedge \nu = A, B \in \Im_\mu\}$$

is a basis for the smooth fuzzy subspace topology on ν.

Proof. First we claim that $\mathscr{T}_\nu(A) \geq \mathscr{B}_\nu(A)$ for all $A \in \Im_\nu$.

Let $A \in \Im_\nu$, then

$$\begin{aligned} \mathscr{T}_\nu(A) &= \sup\{\mathscr{T}_\mu(B)/B \in \Im_\mu, B \wedge \nu = A\} \\ &\geq \sup\{\mathscr{B}_\mu(B)/B \in \Im_\mu, B \wedge \nu = A\} \\ &= \mathscr{B}_\nu(A). \end{aligned}$$

Now we claim that for any $A \in \Im_\mu$, $x \in X$, $\delta > 0$ and $\varepsilon > 0$, there exists $B \in \Im_\mu$ such that $B(x) \geq A(x) - \delta$, $B \leq A$ and $\mathscr{B}_\nu(B) \geq \mathscr{T}_\nu(A) - \varepsilon$. By definition of $\mathscr{T}_\nu$, we have

$$\mathscr{T}_\nu(A) = \sup\{\mathscr{T}_\mu(C)/C \in \Im_\mu, C \wedge \nu = A\}.$$

Then for any given $\varepsilon > 0$, there exists $C \in \Im_\mu$ such that $C \wedge \nu = A$ and

$$\mathscr{T}_\mu(C) \geq \mathscr{T}_\nu(A) - \frac{\varepsilon}{2}.$$

Now, since $\mathscr{B}_\mu$ is the basis for the smooth fuzzy topology $\mathscr{T}_\mu$ we have,

$$\mathscr{T}_\mu(C) = \sup_{\Lambda \in \Gamma} \left\{ \inf_{E_\lambda \in \mathfrak{C}_\Lambda} \{\mathscr{B}_\mu(E_\lambda)\} \right\},$$

where $\{\mathfrak{C}_\Lambda\}_{\Lambda \in \Gamma}$ is the collection of all possible inner covers $\mathfrak{C}_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ of C. Let $\mathfrak{C}_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ be an inner cover for C such that

$$\inf_{E_\lambda \in C_\Lambda} \{\mathscr{B}_\mu(E_\lambda)\} \geq \mathscr{T}_\mu(C) - \frac{\varepsilon}{2}.$$

Thus there exists some $E_0 \in \{E_\lambda\}_{E_\lambda \in C_\Lambda}$ such that

$$E_0(x) \geq C(x) - \delta \text{ and } E_0 \leq C.$$

Since $C \wedge \nu = A$ and $\{E_\lambda\}_{E_\lambda \in C_\Lambda}$ is an inner cover for C, we have, $E_0 \wedge \nu \leq A$. Let $B = E_0 \wedge \nu$, then by the Definition of $\mathscr{B}_\nu$ we have $\mathscr{B}_\nu(B) \geq \mathscr{B}_\mu(E_0)$ and

$$\begin{aligned} B(x) &= (E_0 \wedge \nu)(x) \\ &= E_0(x) \wedge \nu(x) \\ &\geq (C(x) - \delta) \wedge \nu(x) \\ &\geq (C(x) - \delta) \wedge (\nu(x) - \delta) \\ &= (C(x) \wedge \nu(x)) - \delta \\ &= A(x) - \delta. \end{aligned}$$

Thus $B(x) \geq A(x) - \delta$ and $B \leq A$. Now as

$$\mathscr{T}_\nu(A) - \frac{\varepsilon}{2} \leq \mathscr{T}_\mu(C) \leq \inf_{E_\lambda \in C_\Lambda} \{\mathscr{B}_\mu(E_\lambda)\} + \frac{\varepsilon}{2},$$

we have

$$\begin{aligned} \mathscr{T}_\nu(A) &\leq \inf_{E_\lambda \in C_\Lambda} \{\mathscr{B}_\mu(E_\lambda)\} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \inf_{E_\lambda \in E_\Lambda} \{\mathscr{B}_\mu(E_\lambda)\} + \varepsilon \\ &\leq \mathscr{B}_\mu(E_0) + \varepsilon \\ &\leq \mathscr{B}_\nu(B) + \varepsilon. \end{aligned}$$

Thus

$$\mathscr{B}_\nu(B) \geq \mathscr{T}_\nu(A) - \varepsilon.$$

Hence by Theorem 2.5, $\mathscr{B}_\nu$ is a basis for the smooth fuzzy subspace topology on ν. □

Theorem 3.3. *Let $(\nu, \mathscr{T}_\nu)$ be a smooth fuzzy subspace of $(\mu, \mathscr{T}_\mu)$ and let $A \in \Im_\nu$.*

i. If $\mathscr{T}_\nu(A) > \alpha$ and $\mathscr{T}_\mu(\nu) > \alpha$, then $\mathscr{T}_\mu(A) > \alpha$.

ii. If $\mathscr{T}_\nu(A) = \alpha$ and $\mathscr{T}_\mu(\nu) = \alpha$, then $\mathscr{T}_\mu(A) = \alpha$.

iii. If $\mathscr{T}_\nu(A) = \alpha$ and $\mathscr{T}_\mu(\nu) > \alpha$, then $\mathscr{T}_\mu(A) = \alpha$.

iv. If $\mathscr{T}_\nu(A) > \alpha$ and $\mathscr{T}_\mu(\nu) = \alpha$, then $\mathscr{T}_\mu(A) \geq \alpha$.

Proof. (i) Let $\mathscr{T}_\nu(A) = \beta$. Since $\beta > \alpha$, there exists $B \in \Im_\mu$ such that $B \wedge \nu = A$ and $\mathscr{T}_\mu(B) > \alpha$. Thus we have

$$\mathscr{T}_\mu(A) = \mathscr{T}_\mu(B \wedge \nu) \geq \mathscr{T}_\mu(B) \wedge \mathscr{T}_\mu(\nu) > \alpha \wedge \alpha = \alpha.$$

(ii) Let $A \in \Im_\nu$ be such that

$$\mathscr{T}_\nu(A) = \sup\{\mathscr{T}_\mu(B)/B \wedge \nu = A,\ B \in \Im_\mu\} = \alpha.$$

Let $\varepsilon > 0$. Then there exists $B \in \Im_\mu$ be such that $B \wedge \nu = A$ and $\mathscr{T}_\mu(B) \geq \mathscr{T}_\nu(A) - \varepsilon = \alpha - \varepsilon$. Then it follows that

$$\mathscr{T}_\mu(A) = \mathscr{T}_\mu(B \wedge \nu) \geq \mathscr{T}_\mu(B) \wedge \mathscr{T}_\mu(\nu) = (\alpha - \varepsilon) \wedge \alpha = \alpha - \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we have $\mathscr{T}_\mu(A) \geq \alpha$. Now suppose $\mathscr{T}_\mu(A) > \alpha$, then $\mathscr{T}_\nu(A) > \alpha$, which leads to a contradiction. Thus it follows that $\mathscr{T}_\mu(A) = \alpha$.

Similarly, we can prove (iii) and (iv). Note that both strict inequality and equality may hold in (iv).

For example, let $X=\{a,b,c\}$ and let $\mu(x)=1$ for all $x\in X$. Define $\mathscr{T}_\mu:\Im_\mu\to[0,1]$ as follows:

$$\mathscr{T}_\mu(A)=\begin{cases}1 & \text{if } A=1_X \text{ or } A=0_X\\ \frac{1}{2} & \text{if } A=\chi_{\{a,b\}} \text{ or } A=\chi_{\{a\}} \text{ or } A=\chi_{\{b\}}\\ \frac{1}{4} & \text{if } A=\chi_{\{a,c\}}\\ 0 & \text{otherwise}\end{cases}$$

Let $\nu(x)=\chi_{\{a,c\}}$ and let $\mathscr{T}_\nu$ be the subspace smooth fuzzy topology on ν. Then

$$\mathscr{T}_\nu(A)=\begin{cases}1 & \text{if } A=\nu \text{ or } A=0_X\\ \frac{1}{2} & \text{if } A=\chi_{\{a\}}\\ 0 & \text{otherwise}\end{cases}$$

Let $A=\chi_{\{a\}}$ and $\alpha=\frac{1}{4}$, then $\mathscr{T}_\nu(A)>\alpha$, $\mathscr{T}_\mu(\nu)=\alpha$ and $\mathscr{T}_\mu(A)>\alpha$.

Let $X=\{a,b,c\}$ and let $\mu(x)=1$ for all $x\in X$. Define $\mathscr{T}_\mu:\Im_\mu\to[0,1]$ as follows:

$$\mathscr{T}_\mu(A)=\begin{cases}1 & \text{if } A=1_X\\ \frac{3}{4} & \text{if } A=\chi_{\{a,b\}}\\ \frac{1}{2} & \text{if } A=\chi_{\{a\}} \text{ or } A=\chi_{\{b\}} \text{ or } A=\chi_{\{a,c\}}\\ 0 & \text{otherwise}\end{cases}$$

Let $\nu(x)=\chi_{\{a,c\}}$ and let $\mathscr{T}_\nu$ be the subspace smooth fuzzy topology on ν. Then

$$\mathscr{T}_\nu(A)=\begin{cases}1 & \text{if } A=\nu \text{ or } A=0_X\\ \frac{3}{4} & \text{if } A=\chi_{\{a\}}\\ 0 & \text{otherwise}\end{cases}$$

If $A=\chi_{\{a\}}$ and $\alpha=\frac{1}{2}$, then $\mathscr{T}_\nu(A)>\alpha$, $\mathscr{T}_\mu(\nu)=\alpha$ and $\mathscr{T}_\mu(A)=\alpha$. □

Corollary 3.4. *Let $(\nu,\mathscr{T}_\nu)$ be a smooth fuzzy subspace of $(\mu,\mathscr{T}_\mu)$ and let $A\in\Im_\nu$. Then $\mathscr{T}_\mu(A)\geq\mathscr{T}_\nu(A)\wedge\mathscr{T}_\mu(\nu)$.*

Theorem 3.5. *Let $(A,\mathscr{T}_A)$ and $(B,\mathscr{T}_B)$ be smooth fuzzy subspaces of $(\mu,\mathscr{T}_\mu)$ and $(\nu,\mathscr{T}_\nu)$. Then the smooth fuzzy product topology on $A\times B$ is same as the smooth fuzzy topology that $A\times B$ inherits as the subspace of $\mu\times\nu$.*

Proof. Let $\mathscr{B}_\mu$, $\mathscr{B}_\nu$ be bases for the topologies $\mathscr{T}_\mu$, $\mathscr{T}_\nu$. Let

$$\mathscr{B}_A(G)=\sup\{\mathscr{B}_\mu(C)/C\wedge A=G, C\in\Im_\mu\}$$

and

$$\mathscr{B}_B(H)=\sup\{\mathscr{B}_\nu(D)/D\wedge B=H, D\in\Im_\nu\},$$

then by Lemma 3.2 $\mathscr{B}_A$, $\mathscr{B}_B$ are the bases for the topologies $\mathscr{T}_A$, $\mathscr{T}_B$. Let $\mathscr{B}^p_{A\times B}$ be a function from $\Im_{A\times B}$ to $[0,1]$ defined as follows: Let $E\in\Im_{A\times B}$. If E cannot be written as $A\times B$ for any $A\in\Im_A$ and $B\in\Im_B$, then define $\mathscr{B}^p_{A\times B}(E)=0$. Otherwise define

$$\mathscr{B}^p_{A\times B}(E)=\sup_{\lambda\in\Lambda}\{\inf\{\mathscr{B}_A(A_\lambda),\mathscr{B}_B(B_\lambda)\}\}$$

where $\{A_\lambda\times B_\lambda\}_{\lambda\in\Lambda}$ is the collection of all possible representations of E as $E=A_\lambda\times B_\lambda$, where $A_\lambda\in\Im_A$, $B_\lambda\in\Im_B$. Then by Theorem 2.8, we have $\mathscr{B}^p_{A\times B}$ is a basis for the product topology on $A\times B$. Let $\mathscr{B}^p_{\mu\times\nu}$ be a function from $\Im_{\mu\times\nu}$ to $[0,1]$ defined as follows: Let $E\in\Im_{\mu\times\nu}$. If E cannot be written as $C\times D$ for any $C\in\Im_\mu$ and $D\in\Im_\nu$, then define $\mathscr{B}^p_{\mu\times\nu}(E)=0$. Otherwise define

$$\mathscr{B}^p_{\mu\times\nu}(E)=\sup_{\lambda\in\Lambda}\{\inf\{\mathscr{B}_\mu(C_\lambda),\mathscr{B}_\nu(D_\lambda)\}\}$$

where $\{C_\lambda\times D_\lambda\}_{\lambda\in\Lambda}$ is the collection of all representations of E as $E=C_\lambda\times D_\lambda$, where $C_\lambda\in\Im_\mu$, $D_\lambda\in\Im_\nu$. Then again by Theorem 2.8, $\mathscr{B}^p_{\mu\times\nu}$ is a basis for the product topology on $\mu\times\nu$. Let $\mathscr{B}^s_{A\times B}$ be a function from $\Im_{A\times B}$ to $[0,1]$ defined as

$$\mathscr{B}^s_{A\times B}(E)=\sup\{\mathscr{B}^p_{\mu\times\nu}(F)/F\wedge(A\times B)=E,\ F\in\Im_{\mu\times\nu}\}.$$

Then by Lemma 3.2 $\mathscr{B}^s_{A\times B}$ is a basis for the smooth fuzzy subspace topology on $A\times B$.

Now we prove that, $\mathscr{B}^s_{A\times B}(E)=\mathscr{B}^p_{A\times B}(E)$ for all subsets E in $\Im_{A\times B}$. Assume that E is not of the form $G\times H$ for any pair (G,H), where $G\in\Im_A$ and $H\in\Im_B$. Then by our assumption it follows that $\mathscr{B}^p_{A\times B}(E)=0$. On the other hand, if E is not of the form $G\times H$ for any pair (G,H), then there exists no F of the form $C\times D$ for some $C\in\Im_\mu$ and $D\in\Im_\nu$ so that $F\wedge(A\times B)=E$. But for all such F's, we have $\mathscr{B}^p_{\mu\times\nu}(F)=0$. Thus, by the definition of $\mathscr{B}^s_{A\times B}$, it follows that $\mathscr{B}^s_{A\times B}(E)=0$ and hence

$$\mathscr{B}^s_{A\times B}(E)=\mathscr{B}^p_{A\times B}(E).$$

Suppose, E is of the form $G\times H$ for some fuzzy subsets $G\in\Im_A$, $H\in\Im_B$.

Now to compute $\mathscr{B}^p_{A\times B}(E)$, first we collect all pairs (G,H) such that $G\in\Im_A$, $H\in\Im_B$ and $E=G\times H$ and compute $\inf\{\mathscr{B}_A(G),\mathscr{B}_B(H)\}$ and then we find the supremum of all these numbers obtained from all such pairs (G,H). To compute $\mathscr{B}_A(G)$, we collect all possible members $\{C_{G,\lambda}\}_{\lambda\in\Lambda}$ in $\Im_\mu$ such that $C_{G,\lambda}\wedge A=G$ and compute $\sup\{\mathscr{B}_\mu(C_{G,\lambda})\}$. Similarly, to compute $\mathscr{B}_B(H)$, we collect all possible members $\{D_{H,\gamma}\}_{\gamma\in\Gamma}$ in $\Im_\nu$ such that $D_{H,\gamma}\wedge B=H$ and compute $\sup\{\mathscr{B}_\nu(D_{H,\gamma})\}$. Now

$$\begin{aligned}\mathscr{B}^p_{A\times B}(E) &= \sup\{\inf\{\mathscr{B}_A(G),\mathscr{B}_B(H)\}\}\\ &= \sup\{\inf\{\sup\{\mathscr{B}_\mu(C_{G,\lambda})\},\sup\{\mathscr{B}_\nu(D_{H,\gamma})\}\}\}\\ &= \sup\{\sup\{\inf\{\mathscr{B}_\mu(C_{G,\lambda}),\mathscr{B}_\nu(D_{H,\gamma})\}\}\}\\ &= \sup\{\inf\{\mathscr{B}_\mu(C_{G,\lambda}),\mathscr{B}_\nu(D_{H,\gamma})\}\}\end{aligned}$$

Thus we have

$$\mathscr{B}^p_{A\times B}(E)=\sup\{\mathscr{B}_\mu(C)\wedge\mathscr{B}_\nu(D)\},$$

where the supremum is taken over all possible pairs (C,D) such that $E=(C\wedge A)\times(D\wedge B)$.

Now to compute $\mathscr{B}^s_{A\times B}(E)$; first we collect all possible members $\{F\}$ in $\Im_{\mu\times\nu}$ such that $F\wedge(A\times B)=E$ and find

$\sup\{\mathscr{B}^p_{\mu\times\nu}(F)\}$. Suppose F is not of the form $C\times D$ for any $C\in\mathfrak{I}_\mu$, $D\in\mathfrak{I}_\nu$ then by definition of $\mathscr{B}^p_{\mu\times\nu}$ we have,

$$\mathscr{B}^p_{\mu\times\nu}(F)=0.$$

So, $\mathscr{B}^s_{A\times B}(E)=\sup\{\mathscr{B}^p_{\mu\times\nu}(C\times D)\}$, where the supremum is taken over all pairs (C,D) such that $E=(C\times D)\wedge(A\times B)$. But since $\mathscr{B}^p_{\mu\times\nu}(C\times D)$ is equal to $\sup\{\mathscr{B}_\mu(C_q)\wedge\mathscr{B}_\nu(D_r)\}$ where the supremum is taken over all pairs (C_q,D_r) such that $C_q\times D_r=C\times D$. It follows that,

$$\mathscr{B}^s_{A\times B}(E)=\sup\{\mathscr{B}_\mu(C')\wedge\mathscr{B}_\nu(D')\},$$

where the supremum is taken over all possible pairs (C',D') such that $(C'\times D')\wedge(A\times B)=E$. Now since $(C'\times D')\wedge(A\times B)$ is equal to $(C'\wedge A)\times(D'\wedge B)$, it is clear to see that, $\mathscr{B}^s_{A\times B}(E)=\sup\{\mathscr{B}_\mu(C')\wedge\mathscr{B}_\nu(D')\}$, where the supremum is taken over all possible pairs (C',D') such that $E=(C'\wedge A)\times(D'\wedge B)$, as desired. □

Theorem 3.6. *Let $(\mu,\mathscr{T}_\mu)$ be a smooth fuzzy topological space and let $B<A<\mu$. Then the smooth fuzzy subspace topology induced over B by the smooth fuzzy topology on A is same as the smooth fuzzy subspace topology induced over B by the smooth fuzzy topology on μ.*

Proof. Let $\mathscr{B}_\mu$ be a basis for $\mathscr{T}_\mu$. Let $\mathscr{T}_{A_\mu}$ be the smooth fuzzy subspace topology induced over A by $\mathscr{T}_\mu$ with basis $\mathscr{B}_{A_\mu}$ as defined in Lemma 3.2 and let $\mathscr{T}_{B_\mu}$ and $\mathscr{T}_{B_A}$ be the smooth fuzzy subspace topologies induced over B by $\mathscr{T}_\mu$ and $\mathscr{T}_{A_\mu}$ respectively. Let $\mathscr{B}_{B_\mu}$ and $\mathscr{B}_{B_A}$ be the bases for smooth fuzzy subspace topologies $\mathscr{T}_{B_\mu}$ and $\mathscr{T}_{B_A}$ as defined in Lemma 3.2. Now we prove that $\mathscr{B}_{B_\mu}=\mathscr{B}_{B_A}$. Let $E\in\mathfrak{I}_B$, then by the definitions of $\mathscr{B}_{B_\mu}$ and $\mathscr{B}_{B_A}$ we have

$$\mathscr{B}_{B_\mu}(E)=\sup\{\mathscr{B}_\mu(F)/F\wedge B=E,F\in\mathfrak{I}_\mu\}$$

and

$$\mathscr{B}_{B_A}=\sup\{\mathscr{B}_{A_\mu}(G)/G\wedge B=E,G\in\mathfrak{I}_A\}.$$

Let $\mathscr{B}_{B_\mu}(E)=\lambda$ and $\varepsilon>0$, then there exists a $F\in\mathfrak{I}_\mu$ such that $\mathscr{B}_\mu(F)\geq\lambda-\varepsilon$ and $F\wedge B=E$. As $F\wedge B=E$ it clearly follows that $(F\wedge A)\wedge B=E$ and $F\wedge A\leq A$. Now since $\mathscr{B}_\mu(F)\geq\lambda-\varepsilon$ and by the definition of $\mathscr{B}_{A_\mu}$, it follows that $\mathscr{B}_{A_\mu}(F\wedge A)\geq\lambda-\varepsilon$. Let $G=F\wedge A$, then $\mathscr{B}_{A_\mu}(G)\geq\lambda-\varepsilon$, $G\wedge B=E$ and $G\leq A$. Thus for each $\varepsilon>0$, there exists a $G\in\mathfrak{I}_A$ such that $\mathscr{B}_{A_\mu}(G)\geq\lambda-\varepsilon$, $G\wedge B=E$ and $G\leq A$. This implies,

$$\mathscr{B}_{B_A}(E)=\sup\{\mathscr{B}_{A_\mu}(G)/G\wedge B=E,G\in\mathfrak{I}_A\}\geq\lambda.$$

Now suppose that

$$\mathscr{B}_{B_A}(E)=\sup\{\mathscr{B}_{A_\mu}(G)/G\wedge B=E,G\in\mathfrak{I}_A\}>\lambda.$$

Let $\mathscr{B}_{B_A}(E)=\delta>\lambda$. Choose $\varepsilon>0$ such that $\delta>\delta-\varepsilon>\lambda$, then by definition of $\mathscr{B}_{B_A}$ we can find a $G\in\mathfrak{I}_A$ such that $\mathscr{B}_{A_\mu}(G)\geq\delta-\frac{\varepsilon}{2}>\lambda$ and $G\wedge B=E$. Now by the definition of $\mathscr{B}_{A_\mu}$, we can find a $H\in\mathfrak{I}_\mu$ such that $H\wedge A=G$ and

$$\mathscr{B}_\mu(H)=\delta-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}>\lambda.$$

But since $H\wedge B=E$, it follows that, there exists a $H\in\mathfrak{I}_\mu$ such that $H\wedge B=E$ and

$$\mathscr{B}_\mu(H)=\delta-\varepsilon>\lambda.$$

This leads to a contradiction that $\mathscr{B}_{B_\mu}(E)=\lambda$, and hence our assumption $\mathscr{B}_{B_A}(E)>\lambda$ is wrong. Thus it follows that $\mathscr{B}_{B_A}(E)=\lambda$ and therefore $\mathscr{B}_{B_\mu}=\mathscr{B}_{B_A}$. □

Result 3.7. *Let $\mathscr{T}_\mu$ and $\mathscr{T}_\mu{}'$ be two smooth fuzzy topologies on μ and let $\nu<\mu$. Let $(\nu,\mathscr{T}_\nu)$ $(\nu,\mathscr{T}_\nu{}')$ be smooth fuzzy subspace topological spaces of $(\mu,\mathscr{T}_\mu)$ and $(\mu,\mathscr{T}_\mu{}')$ respectively. If $T_\mu\leq T'_\mu$, then $T_\nu\leq T'_\nu$.*

Proof. Let $E\in\mathfrak{I}_\nu$, then

$$\begin{aligned}\mathscr{T}_\nu(E) &= \sup\{\mathscr{T}_\mu(G)/G\wedge\nu=E,G\in\mathfrak{I}_\mu\}\\ &\leq \sup\{\mathscr{T}'_\mu(G)/G\wedge\nu=E,G\in\mathfrak{I}_\mu\}\\ &= \mathscr{T}'_\nu(E).\end{aligned}$$

□

In the above result the equality may also hold.

Example 3.8. *Let $X=\{1,2,3,4\}$ and let $\mu(x)=1$ for all $x\in X$. Now, define a function $\mathscr{T}_\mu$ from $\mathfrak{I}_\mu$ to $[0,1]$ as follows:*

$$\mathscr{T}_\mu(E)=\begin{cases}1 & \text{if } E=\mu \text{ or } E=0_X\\ \frac{1}{2} & \text{if } E=\chi_{\{1\}}\\ \frac{1}{4} & \text{if } E=\chi_{\{1,2\}}\\ 0 & \text{otherwise}\end{cases}$$

Then $\mathscr{T}_\mu$ is a smooth fuzzy topology on μ. Let $\mathscr{T}'_\mu$ be a function from $\mathfrak{I}_\mu$ to $[0,1]$ defined as

$$\mathscr{T}'_\mu(E)=\begin{cases}1 & \text{if } E=\mu \text{ or } E=0_X\\ \frac{1}{2} & \text{if } E=\chi_{\{2,3,4\}} \text{ or } \chi_{\{1\}}\\ \frac{1}{4} & \text{if } E=\chi_{\{3,4\}} \text{ or } \chi_{\{1,2\}}\\ 0 & \text{otherwise}\end{cases}$$

Then $\mathscr{T}'_\mu$ is a smooth fuzzy topology on μ. Let

$$\nu(x)=\begin{cases}1 & \text{if } x=1\\ 0 & \text{otherwise}\end{cases}$$

then clearly it can be seen that $\nu<\mu$. Let $\mathscr{T}_\nu$ be the smooth fuzzy subspace topology induced over ν by μ. Then for $E\in\mathfrak{I}_\nu$ we have

$$\mathscr{T}_\nu(E)=\begin{cases}1 & \text{if } E=\nu \text{ or } E=0_X\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathscr{T}'_\nu(E)=\begin{cases}1 & \text{if } E=\nu \text{ or } E=0_X\\ 0 & \text{otherwise}\end{cases}$$

Thus $\mathscr{T}_\nu=\mathscr{T}'_\nu$.

4. (α,ℓ)-Hausdorff Spaces

In this section, we define the concept of (α,ℓ)-Hausdorff spaces and prove some interesting results; in particular we prove that the intersection of two (α,ℓ)-Hausdorff topologies is again an (α,ℓ)-Hausdorff topology in contrast with the crisp theory.

Definition 4.1 ((α,ℓ)-Hausdorff)**.** *Let $\alpha \in (0,1)$ and $\ell \in (0,1]$. A smooth fuzzy topological space $(\mu,\mathscr{T}_\mu)$ is said to be (α,ℓ)-Hausdorff if $\forall\ x,y \in X$ with $x \neq y$, there exist $A,B \in \Im_\mu$ such that $\mathscr{T}_\mu(A) > \alpha$, $\mathscr{T}_\mu(B) > \alpha$, $A(x) \geq \ell$, $B(y) \geq \ell$ and $(A\wedge B)(z) < \ell$, $\forall\, z \in X$.*

Note that if a smooth fuzzy topological space $(\mu,\mathscr{T}_\mu)$ is (α,ℓ)-Hausdorff for all $\ell \in (0,1]$, then $\mu(x) = 1_X$.

Result 4.2. *Let $(\nu,\mathscr{T}_\nu)$ subspace of an (α,ℓ)-Hausdorff space $(\mu,\mathscr{T}_\mu)$ such that $\nu(x) \geq \ell$, $\forall\, x \in X$. Then $(\nu,\mathscr{T}_\nu)$ is also an (α,ℓ)-Hausdorff space.*

Proof. As the result can be proved in a usual way, we skip the proof. □

Now we prove an interesting result, that the intersection of finitely many (α,ℓ)-Hausdorff topologies is a (α,ℓ)-Hausdorff topology in contrast with the crisp theory.

Theorem 4.3. *Let $(\mu,\mathscr{T}_{\mu_i})$ be (α_i,ℓ_i)-Hausdorff for all $i = 1,2,\ldots,n$ and let $\mathscr{T}_\mu = \overset{n}{\underset{i=1}{\wedge}}\{\mathscr{T}_{\mu_i}\}$. Then $(\mu,\mathscr{T}_\mu)$ is an (α,ℓ)-Hausdorff space, where $\alpha = \min\{\alpha_i\}$ and $\ell = \min\{\ell_i\}$.*

Proof. Let $x \neq y \in X$. Now since $(\mu,\mathscr{T}_{\mu_i})$ is (α_i,ℓ_i)-Hausdorff for all i, there exist $A_i,B_i \in \Im_{\mu_i}$ such that $\mathscr{T}_{\mu_i}(A_i) > \alpha$, $\mathscr{T}_{\mu_i}(B_i) > \alpha$, $A_i(x) \geq \ell_i$, $B_i(y) \geq \ell_i$ and $(A_i \wedge B_i)(z) < \ell_i$ for all $z \in X$. Let $A = \min\{A_i\}$ and $B = \min\{B_i\}$, then it follows that

$$\mathscr{T}_\mu(A) = \overset{n}{\underset{i=1}{\wedge}} \mathscr{T}_{\mu_i}(A) \geq \overset{n}{\underset{i=1}{\wedge}}\{\overset{n}{\underset{i=1}{\wedge}} \mathscr{T}_{\mu_i}(A_i)\} > \min\{\alpha_i\} = \alpha$$

and analogously, it is easy to see that, $\mathscr{T}_\mu(B) > \alpha$. Now, by construction of A and B, we have $A(x) \geq \ell$, $B(y) \geq \ell$ and $(A\wedge B)(z) < \ell\ \forall\, z \in X$. Thus $(\mu,\mathscr{T}_\mu)$ is an (α,ℓ)-Hausdorff space. □

Corollary 4.4. *The intersection of finitely many (α,ℓ)-Hausdorff topologies is (α,ℓ)-Hausdorff.*

Theorem 4.5. *Let μ_1 and μ_2 be fuzzy subsets of X and Y respectively. Let $(\mu_1,\mathscr{T}_{\mu_1})$ be an (α_1,ℓ_1)-Hausdorff space and $(\mu_2,\mathscr{T}_{\mu_2})$ an (α_2,ℓ_2)-Hausdorff space. Let $\mu = \mu_1 \times \mu_2$, then the product space $(\mu,\mathscr{T}_\mu)$ is (α,ℓ)-Hausdorff where $\alpha = \alpha_1 \wedge \alpha_2$ and $\ell = \ell_1 \wedge \ell_2$.*

Proof. Let (x_1,y_1) and (x_2,y_2) be two distinct points on $X\times Y$. If $x_1 \neq x_2$, then there exist $A,B \in \Im_{\mu_1}$ such that $\mathscr{T}_{\mu_1}(A) > \alpha$, $\mathscr{T}_{\mu_1}(B) > \alpha$ $A(x_1) \geq \ell_1$, $B(x_2) \geq \ell_1$ and $(A\wedge B)(z) < \ell_1$ for all $z \in X$. Let $C = A\times \mu_2$ and $D = B\times \mu_2$. Then clearly it follows that, $\mathscr{T}_\mu(C) > \alpha$, $\mathscr{T}_\mu(D) > \alpha$ and $(C\wedge D) < \ell$. Note that,

$$C(x_1,y_1) = (A\times \mu_2)(x_1,y_1) = A(x_1)\wedge \mu_2(y_1) \geq \ell_1 \wedge \ell_2 = \ell.$$

Analogously, it can be seen that, $D(x_2,y_2) \geq \ell$. Thus the product space $(\mu,\mathscr{T}_\mu)$ is (α,ℓ)-Hausdorff. If $y_1 \neq y_2$, then the argument is similar to the above. □

Corollary 4.6. *The product of two (α,ℓ)-Hausdorff spaces is (α,ℓ)-Hausdorff.*

Definition 4.7. *The ℓ-Hausdorffness of space is defined as the supremum of all α for which the space is (α,ℓ)-Hausdorff.*

Let $(X,\mathscr{T})$ be a crisp Hausdorff space. This space can be viewed as a smooth fuzzy topological space by declaring $\mathscr{T}(A) = 1$ if A is the characteristic function of an open set and zero otherwise. This space is (α,ℓ)-Hausdorff for all $\alpha \in (0,1)$ and $\ell \in (0,1]$. So the ℓ-Hausdorffness of this space is 1 for all ℓ.

We have a natural question: If the ℓ-Hausdorffness of a crisp topological space $(X,\mathscr{T})$, while considering it as smooth fuzzy topological space, is 1 for all ℓ, can we conclude that it is Hausdorff in the crisp sense? The answer is "no". Indeed, if $\mathscr{T}_1$ and $\mathscr{T}_2$ are two crisp Hausdorff topologies on a set X so that $\mathscr{T}_1 \cap \mathscr{T}_2$ is not Hausdorff, then $\mathscr{T}_1$ and $\mathscr{T}_2$ are (α,ℓ)-Hausdorff for all $\alpha \in (0,1)$ and $\ell \in (0,1]$ and hence $\mathscr{T}_1 \cap \mathscr{T}_2$ is (α,ℓ)-Hausdorff for all $\alpha \in (0,1)$ and $\ell \in (0,1]$; therefore the ℓ-Hausdorffness of $\mathscr{T}_1 \cap \mathscr{T}_2$, while considering it as smooth fuzzy topological space, is 1 for all ℓ. But $\mathscr{T}_1 \cap \mathscr{T}_2$ is not Hausdorff in the crisp sense.

5. Conclusion

In this paper, some nice results on smooth fuzzy subspaces are proved, using the context of basis. The concept of Hausdorff spaces in the context of smooth fuzzy topological spaces is developed, in a way that is entirely different the ones available in the literature. The theory can be further extended to (α,ℓ)-regular and (α,ℓ)-normal spaces, so that, a theory parallel to separation axioms in classical topological theory can be developed in a natural way.

References

[1] S. E. Abbas, On smooth fuzzy subspaces, *Int. J. Math. Math. Sci.*, 66, (2004), 3587-3602.

[2] A. K. Azad, Fuzzy hausdorff spaces and fuzzy perfect mappings, *J. Math. Anal. Appl.*, 82(2), (1981), 297-305.

[3] C. L. Chang, Fuzzy Topological Spaces, *J. Math. Anal. Appl.*, 24(1), (1968), 182-190

[4] C. Kalaivani, Fuzzy Proper Functions, *Ph. D. Thesis. Alagappa University, Karaikudi - 630 004, India*, (2012).

[5] C. K. Park, W. K. Min and M. H. Kim, Weak smooth α-structure of smooth topological spaces, *Commun. Korean Math. Soc.*, 19(1), (2004), 143-153.

[6] W. Peeters, Subspaces of smooth fuzzy topologies and initial smooth fuzzy structures, *Fuzzy sets and Systems*, 104(3), (1999), 423-433.

[7] A. A. Ramadan, M. A. Fath Alla and S. E. Abbas, Smooth Fuzzy topology on fuzzy sets, *J. Fuzzy Math.*, 10(1), (2002), 59-68.

[8] A. A. Ramadan, Smooth topological spaces, *Fuzzy sets and systems*, 48, (1992), 371-375.

[9] Rekha Srivastava, On separation axioms in a newly defined fuzzy topology, *Fuzzy Sets and Systems*, 62(3), (1994), 341-346.

[10] R. Srivastava, S. N. Lal and A. K. Srivastava, Fuzzy Hausdorff topological spaces, *J. Math. Anal. Appl.*, 81(2), (1981), 497-506.

[11] A. P. Šostak, On a fuzzy topological structure, *Rend. Circ. Matem. Palermo Ser. II*, 11, (1985), 89-103.

[12] M. Shakthiganesan and R. Vembu, On product of smooth fuzzy topological spaces, *Novi Sad Journal of Mathematics*, 46(2), (2016), 13-31.

[13] Fang Jin-ming, Yue Yue-li Base and Subbase in I-Fuzzy Topological Spaces, *Journal of Mathematical Research and Exposition*, 26(1), (2006), 89-94.

[14] Yue Yue-li, Fang Jin-ming On separation axioms in I-fuzzy topological spaces, *Fuzzy Sets and Systems*, 157, (2006), 780-793.

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