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Some products on interval-valued Pythagorean fuzzy graph

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Abstract

In this paper, the complete interval-valued Pythagorean fuzzy graphs are introduced. Direct product, strong product and semi-strong product of two interval-valued Pythagorean fuzzy graphs are defined. Also we reveal that the strong product of two complete interval-valued Pythagorean fuzzy graphs is complete. Consequently, if the direct product is strong then any one of two interval-valued Pythagorean fuzzy graphs is strong. Finally, we investigated many interesting properties on strong product and semi-strong product of two interval-valued Pythagorean fuzzy graphs.

Keywords

Pythagorean fuzzy graph, Interval-valued Pythagorean fuzzy graph, Direct product, semi-strong product, strong product.

AMS Subject Classification

05C72, 03E72, 03F55.

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Contents

1. Introduction

The fuzzy set theory is a rapidly processing field of mathematics. It was first proposed by Zadeh [19] in 1965 as a general mathematical tool for dealing with uncertainty. In 1975, Rosenfeld [11] introduced the concept of fuzzy graphs. In 1986, Atanassov proposed the concept of intuitionistic fuzzy set (IFS) [4, 5] which looks more accurately to uncertainty quantification. In 1989, Atanassov and Gargov [3] defined interval-valued intuitionistic fuzzy set (IVIFS) which is generalization of intuitionistic fuzzy sets. Interval-valued fuzzy graphs were introduced by Akram and Dudec [1] in 2011. Atanassov [5] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graph. S.N.Mishra and A.Pal [8] introduced the product of interval valued intuitionistic fuzzy graph. H. Rashmanlou and Y.B. Jun [10] introduced the concept of complete interval-valued fuzzy graph. Ismayil and Ali [6, 7] studied some operations on interval-valued intuitionistic fuzzy graphs. Some works in intuitionistic fuzzy graph theory can be found in [12, 16, 17]. In 2013, Yager [18] introduced the notion of Pythagorean fuzzy set as an effective tool for handling uncertainty in real-world problems. Peng [9] defined interval-valued Pythagorean fuzzy set which is a generalization of Pythagorean fuzzy sets. Yahya Mohamed and Mohamed Ali [13–15] introduced the concept of interval-valued Pythagorean fuzzy graph(IVPFG) and they studied some operations on strong interval-valued Pythagorean fuzzy graph(SIVPFG), edge regular interval-valued Pythagorean fuzzy graphs. In this paper, the complete interval-valued Pythagorean fuzzy graphs are introduced. Direct product, Strong product and Semi-strong product of two interval-valued Pythagorean fuzzy graphs are defined. Some properties on strong product and semi-strong product of two interval-valued Pythagorean fuzzy graphs are studied.

2. Preliminaries

In this section, we recall some definitions and basic results of interval-valued Pythagorean fuzzy graphs.

Definition 2.1 ([18])**.** *Let X be a universe of discourse. An Pythagorean fuzzy set P in X is given by $P = \{< x, \mu_P(x), \gamma_P(x) > / x \in X\}$, where $\mu_P : X \to [0,1]$ denotes the degree of membership and $\gamma_P : X \to [0,1]$ denotes the degree of non-membership of the element $x \in X$ to the set P, respectively, with the condition that $0 \leq \mu_P^2(x) + \gamma_P^2(x) \leq 1$. The degree of indeterminacy $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \gamma_P^2(x)}$.*

Definition 2.2 ([9])**.** *An interval-valued Pythagorean fuzzy set A defined in a finite universe of discourse $X = \{x_1, x_2, x_3, \ldots, x_n\}$ is given by $A = \{< x, \tilde{M}(x) = [M_{AL}, M_{AU}], \tilde{N}(x) = [N_{AL}, N_{AU}] > / x \in X\}$ where $M_{AL}(x), M_{AU}(x) : X \to [0,1]$ and $N_{AL}(x), N_{AU}(x) : X \to [0,1]$ and $0 \leq M_{AU}^2 + N_{AU}^2 \leq 1$. The numbers $\tilde{M}(x)$ and $\tilde{N}(x)$ denotes the degree of membership and degree of non-membership of $x \in X$ in A. Here $D[0,1]$ denotes the set of all closed sub-intervals of the unit interval $[0,1]$. $\tilde{M} = [M_L, M_U]$, where M_L and M_U are the sup and inf of $\tilde{M}$.*

Definition 2.3 ([13])**.** *An interval valued Pythagorean fuzzy graph with underlying set V is defined to be a pair $G = (P,Q)$ where*

- *the functions $\tilde{M}_P : V \to D[0,1]$ and $\tilde{N}_P : V \to D[0,1]$ denote the degree of membership and non membership of the element $x \in V$, respectively, such that $0 \leq M_{PU}^2(x) + N_{PU}^2(x) \leq 1$ for all $x \in V$.*
- *the functions $\tilde{M}_Q : E \subseteq V \times V \to D[0,1]$ and $\tilde{N}_Q : E \subseteq V \times V \to D[0,1]$ are defined by*

$$M_{QL}((x,y)) \leq (M_{PL}(x) \wedge M_{PL}(y)),$$
$$N_{QL}((x,y)) \geq (N_{PL}(x) \vee N_{PL}(y)) \text{ and}$$
$$M_{QU}((x,y)) \leq (M_{PU}(x) \wedge M_{PU}(y)),$$
$$N_{QU}((x,y)) \geq (N_{PU}(x) \vee N_{PU}(y))$$

such that $0 \leq M_{QU}^2((x,y)) + N_{QU}^2((x,y)) \leq 1, \forall (x,y) \in E$.

Definition 2.4. *[14] An interval-valued Pythagorean fuzzy graph(IVPFG) $G = (P,Q)$ is called strong if*

$$M_{QL}((x,y)) = (M_{PL}(x) \wedge M_{PL}(y))$$
$$N_{QL}((x,y)) = (N_{PL}(x) \vee N_{PL}(y)) \text{ and}$$
$$M_{QU}((x,y)) = (M_{PU}(x) \wedge M_{PU}(y))$$
$$N_{QU}((x,y)) = (N_{PU}(x) \vee N_{PU}(y)), \forall (x,y) \in E.$$

3. Main Results

Definition 3.1. *An interval-valued Pythagorean fuzzy graph (IVPFG) $G = (P,Q)$ is called complete if*

$$M_{QL}((x,y)) = (M_{PL}(x) \wedge M_{PL}(y))$$
$$N_{QL}((x,y)) = (N_{PL}(x) \vee N_{PL}(y)), \text{ and}$$
$$M_{QU}((x,y)) = (M_{PU}(x) \wedge M_{PU}(y))$$
$$N_{QU}((x,y)) = (N_{PU}(x) \vee N_{PU}(y)), \forall x,y \in V.$$

Example 3.2. *Consider the graph $G = (V,E)$ such that $V = \{x,y,z\}, E = \{xy, yz, zx\}$ given in Fig.1. Let P be an IVPFS of V and let Q be an IVPFS of $E \subseteq V \times V$ defined by $P = \{< x, [0.5,0.7], [0.1,0.6] >, < y, [0.6,0.7], [0.1,0.5] >, < z, [0.4,0.8], [0.2,0.5] >\}, Q = \{< xy, [0.5,0.7], [0.1,0.6] >, < yz, [0.4,0.6], [0.2,0.5] >, < xz, [0.4,0.6], [0.2,0.6] >\}$*

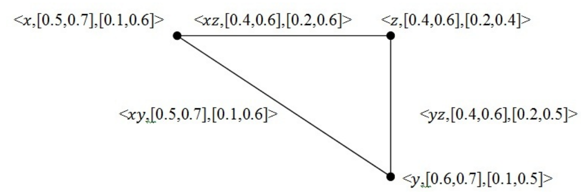

Figure 1. CIVPFG

Definition 3.3. *Let P_1 and P_2 be interval-valued Pythagorean fuzzy subsets of V_1 and V_2 and let Q_1 and Q_2 be interval-valued Pythagorean fuzzy subsets of E_1 and E_2 respectively and we assume that $V_1 \cap V_2 = \phi$. The direct product of two interval-valued Pythagorean fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \sqcap G_2 : (P_1 \sqcap P_2, Q_1 \sqcap Q_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$, where $E = \{((x_1,y_1),(x_2,y_2)) : (x_1,x_2) \in E_1, (y_1,y_2) \in E_2\}$ and is defined as follows:*

1. $(M_{P_1L} \sqcap M_{P_2L})(x,y) = (M_{P_2L}(x) \wedge M_{P_2L}(y))$
   $(M_{P_2L} \sqcap M_{P_2L})(x,y) = (M_{P_2L}(x) \wedge M_{P_2L}(y))$
   $(N_{P_2L} \sqcap N_{P_2L})(x,y) = (N_{P_1L}(x) \vee N_{P_2L}(y))$
   $(N_{P_1U} \sqcap N_{P_2L})(x,y) = (N_{P_1U}(x) \vee N_{P_2L}(y))$,
   $\forall (x,y) \in V_1 \times V_2$.
2. $(M_{Q_1L} \sqcap M_{Q_2L})((x_1,y_1)(x_2,y_2)) =$
   $(M_{Q_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2))$
   $(M_{Q_1U} \sqcap M_{Q_2U})((x_1,y_1)(x_2,y_2)) =$
   $(M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
   $(N_{Q_1L} \sqcap N_{Q_2L})((x_1,y_1)(x_2,y_2)) =$
   $(N_{Q_1L}(x_1x_2) \vee N_{Q_2L}(y_1y_2))$
   $(N_{Q_1U} \sqcap N_{Q_2U})((x_1,y_1)(x_2,y_2)) =$
   $(N_{Q_1U}(x_1x_2) \vee N_{Q_2U}(y_1y_2))$
   $\forall (x_1,x_2) \in E_1, (y_1,y_2) \in E_2$.

Example 3.4. *Consider the two IVPFGs $G_1 = (P_1,Q_1)$ and $G_2 = (P_2,Q_2)$ on $V_1 = \{a,b\}$ and $V_2 = \{c,d\}$ given in Fig.2. The direct product of G_1 and G_2 given in Fig.4 is also IVPFG.*

Proposition 3.5. *If G_1 and G_2 are SIVPFGs then $G_1 \sqcap G_2$ is SIVPFG.*

Proof. If $((x_1,y_1),(x_2,y_2)) \in E$, then since G_1 and G_2 are strong we have,
$(M_{Q_1L} \sqcap M_{Q_2L})((x_1,y_1),(x_2,y_2))$
$= (M_{Q_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2))$
$= ((M_{P_1L}(x_1) \wedge M_{P_1L}(x_2)) \wedge (M_{P_2L}(y_1) \wedge M_{P_2L}(y_2)))$
$= \wedge(M_{P_1L}(x_1), M_{P_1L}(x_2), M_{P_2L}(y_1), M_{P_2L}(y_2))$
$= ((M_{P_1L}(x_1) \wedge M_{P_2L}(y_1)) \wedge (M_{P_1L}(x_2) \wedge M_{P_2L}(y_2)))$

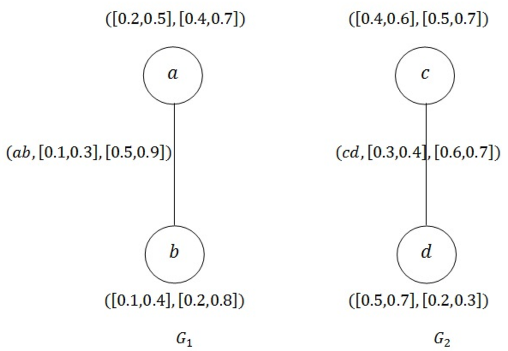

Figure 2. Interval-valued Pythagorean fuzzy graphs

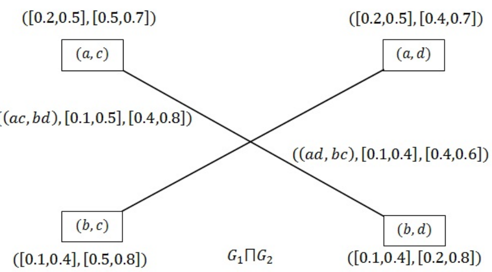

Figure 3. Direct Product

$= ((M_{P_1L} \sqcap M_{P_2L})(x_1,y_1) \wedge (M_{P_1L} \sqcap M_{P_2L})(x_2,y_2))$
$(M_{Q_1U} \sqcap M_{Q_2U})((x_1,y_1),(x_2,y_2))$
$= (M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
$= ((M_{P_1U}(x_1) \wedge M_{P_1U}(x_2)) \wedge (M_{P_2U}(y_1) \wedge M_{P_2U}(y_2)))$
$= \wedge(M_{P_1U}(x_1), M_{P_1U}(x_2), M_{P_2U}(y_1), M_{P_2U}(y_2)),$
$= ((M_{P_1U}(x_1) \wedge M_{P_2U}(y_1)) \wedge (M_{P_1U}(x_2) \wedge M_{P_2U}(y_2)))$
$= ((M_{P_1U} \sqcap M_{P_2U})(x_1,y_1) \wedge (M_{P_1U} \sqcap M_{P_2U})(x_2,y_2))$
Similarly, we can easily show that,
$(N_{Q_1L} \sqcap N_{Q_2L})((x_1,y_1),(x_2,y_2)) =$
$((N_{P_1L} \sqcap N_{P_2L})(x_1,y_1) \vee (N_{P_1L} \sqcap N_{P_2L})(x_2,y_2))$
$(N_{Q_1U} \sqcap N_{Q_2U})((x_1,y_1),(x_2,y_2)) =$
$((N_{P_1U} \sqcap N_{P_2U})(x_1,y_1) \vee (N_{P_1U} \sqcap N_{P_2U})(x_2,y_2)).$
Hence, $G_1 \sqcap G_2$ is SIVPFG. □

Proposition 3.6. *If $G_1 \sqcap G_2$ is SIVPFG, then at least G_1 or G_2 must be SIVPFG.*

Proof. Suppose that G_1 and G_2 are not SIVPFGs, there exist $x_i, y_i \in E_i, i = 1,2$ such that

$$M_{Q_iL}((x_i,y_i)) < (M_{P_iL}(x_i) \wedge MP_iL(y_i))$$
$$M_{Q_iU}((x_i,y_i)) < (M_{P_iU}(x_i) \wedge M_{P_iU}(y_i))$$
$$N_{Q_iL}((x_i,y_i)) > (N_{P_iL}(x_i) \vee N_{P_iL}(y_i))$$
$$N_{Q_iU}((x_i,y_i)) > (N_{P_iU}(x_i) \vee N_{P_iU}(y_i))$$

Consider $((x_1,y_1)(x_2,y_2)) \in E$, we have
$(M_{Q_1L} \sqcap M_{Q_2L})((x_1,y_1),(x_2,y_2))$
$= (M_{P_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2))$
$< \wedge(M_{P_1L}(x_1), M_{P_2L}(x_2), M_{P_2L}(y_1), M_{P_2L}(y_2))$
Similarly,
$(M_{Q_1U} \sqcap M_{Q_2U})((x_1,y_1),(x_2,y_2))$
$= (M_{P_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
$< \wedge(M_{P_1U}(x_1), M_{P_2U}(x_2), M_{P_2U}(y_1), M_{P_2U}(y_2))$
Since, $(M_{P_1U} \sqcap M_{P_2U})(x_1,y_1) = (M_{P_1U}(x_1) \wedge M_{P_2U}(x_2))$ and
$(M_{P_1U} \sqcap M_{P_2U})(x_1,y_1) \wedge (M_{P_1U} \sqcap M_{P_2U})(x_2,y_2))$
$= ((M_{P_1U}(x_1) \wedge M_{P_2U}(y_1)) \wedge (M_{P_1U}(x_2) \wedge M_{P_2U}(y_2)))$
$= \wedge(M_{P_1U}(x_1), M_{P_2U}(x_2), M_{P_2U}(y_1), M_{P_2U}(y_2))$
Hence,
$(M_{Q_1L} \sqcap M_{Q_2L})((x_1,y_1),(x_2,y_2))$
$< ((M_{P_1L} \sqcap M_{P_2L})(x_1,y_1) \wedge (M_{P_1L} \sqcap M_{P_2L})(x_2,y_2))$
$(M_{Q_1U} \sqcap M_{Q_2U})((x_1,y_1),(x_2,y_2))$
$< ((M_{P_1U} \sqcap M_{P_2U})(x_1,y_1) \wedge (M_{P_1U} \sqcap M_{P_2U})(x_2,y_2))$
Similarly, we can show that
$(N_{Q_1L} \sqcap N_{Q_2L})((x_1,y_1),(x_2,y_2))$
$> ((N_{P_1L} \sqcap N_{P_2L})(x_1,y_1) \vee (N_{P_1L} \sqcap N_{P_2L})(x_2,y_2))$
$(N_{Q_1U} \sqcap N_{Q_2U})((x_1,y_1),(x_2,y_2))$
$> ((N_{P_1U} \sqcap N_{P_2U})(x_1,y_1) \vee (N_{P_1U} \sqcap N_{P_2U})(x_2,y_2))$
Hence, $G_1 \sqcap G_2$ is not SIVPFG, which is a contradiction.
Hence the proof. □

Definition 3.7. *Let P_1 and P_2 be interval–valued Pythagorean fuzzy subsets of V_1 and V_2 and let Q_1 and Q_2 interval valued Pythagorean fuzzy subsets of E_1 and E_2 respectively and we assume that $V_1 \cap V_2 = \phi$. The semi strong product of two IVPFGs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \diamond G_2 : (P_1 \diamond P_2, Q_1 \diamond Q_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$, where $E = \{((x,y_1),(x,y_2)) : x \in V_1, (y_1,y_2) \in E_2\} \cup \{((x_1,y_1),(x_2,y_2)) : (x_1,x_2) \in E_1, (y_1,y_2) \in E_2\}$ and is defined as follows:*

1. $(M_{P_1L} \diamond M_{P_2L})(x,y) = (M_{P_1L}(x) \wedge M_{P_2L}(y)),$
   $(M_{P_1U} \diamond M_{P_2U})(x,y) = (M_{P_1U}(x) \wedge M_{P_2U}(y)),$
   $(N_{P_1L} \diamond N_{P_2L})(x,y) = (N_{P_1L}(x) \vee N_{P_2L}(y)),$
   $(N_{P_1U} \diamond N_{P_2U})(x,y) = (N_{P_1U}(x) \vee N_{P_2U}(y)),$
   $\forall (x,y) \in V_1 \times V_2$

2. $(M_{Q_1L} \diamond M_{Q_2L})((x,y_1),(x,y_2))$
   $= (M_{P_1L}(x) \wedge M_{Q_2L}(y_1y_2)),$
   $(M_{Q_1U} \diamond M_{Q_2U}((x,y_1),(x,y_2))$
   $= (M_{P_1U}(x) \wedge M_{Q_2U}(y_1y_2)),$
   $(N_{Q_1L} \diamond N_{Q_2L})((x,y_1),(x,y_2))$
   $= (N_{P_1L}(x) \vee N_{Q_2L}(y_1,y_2)),$
   $(N_{Q_1U} \diamond N_{Q_2U})((x,y_1),(x,y_2))$
   $= (N_{P_1U}(x) \vee N_{Q_2U}(y_1y_2))$

3. $(M_{Q_1L} \diamond M_{Q_2L})((x_1,y_1),(x_2,y_2))$
   $= (M_{Q_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2)),$
   $(M_{Q_1U} \diamond M_{Q_2U})((x_1,y_1),(x_2,y_2))$
   $= (M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2)),$
   $(N_{Q_1L} \diamond N_{Q_2L})((x_1,y_1),(x_2,y_2))$
   $= (N_{Q_1L}(x_1x_2) \vee N_{Q_2L}(y_1y_2)),$
   $(N_{Q_1U} \diamond N_{Q_2U})((x_1,y_1),(x_2,y_2))$
   $= (N_{Q_1U}(x_1x_2) \vee N_{Q_2U}(y_1y_2)),$
   $\forall (x_1,x_2) \in E_1, (y_1,y_2) \in E_2.$

Example 3.8. *Consider the two IVPFGs $G_1 = (P_1,Q_1)$ and $G_2 = (P_2,Q_2)$ on $V_1 = \{a,b\}$ and $V_2 = \{c,d\}$ respectively.*

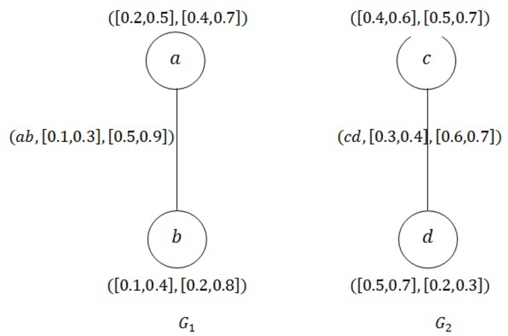

Figure 4. IVPFGs

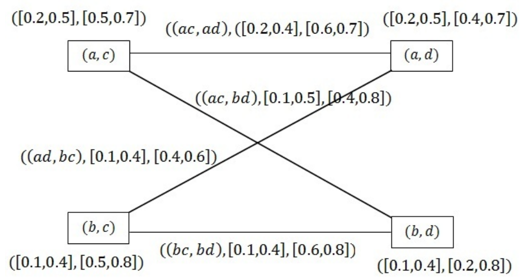

Figure 5. $G_1 \diamond G_2$

Proposition 3.9. *If G_1 and G_2 are SIVPFGs then $G_1 \diamond G_2$ is SIVPFG.*

Proof. This proof is similar to the proposition 3.5. □

Proposition 3.10. *If $G_1 \diamond G_2$ is SIVPFG then at least G_1 or G_2 must be SIVPFG.*

Proof. This proof is similar to the proposition 3.6 □

Definition 3.11. *Let P_1 and P_2 be interval – valued Pythagorean fuzzy subsets of V_1 and V_2 and let Q_1 and Q_2 interval valued Pythagorean fuzzy subsets of E_1 and E_2 respectively and we assume that $V_1 \cap V_2 = \phi$. The strong product of two IVPFGs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \otimes G_2 : (P_1 \otimes P_2, Q_1 \otimes Q_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$,where $E = \{((x,y_1),(x,y_2)) : x \in V_1, (y_1,y_2) \in E_2\} \cup \{((x_1,z),(x_2,z)) : z \in V_2, (x_1,x_2) \in E_1\} \cup \{((x_1,y_1),(x_2,y_2)) : (x_1,x_2) \in E_1, (y_1,y_2) \in E_2\}$ and is defined as follows:*

1. $(M_{P_1L} \otimes M_{P_2L})(x,y) = (M_{P_1L}(x) \wedge M_{P_2L}(y))$
   $(M_{P_1U} \otimes M_{P_2U})(x,y) = (M_{P_1U}(x) \wedge M_{P_2U}(y))$
   $(N_{P_1L} \otimes N_{P_2L})(x,y) = (N_{P_1L}(x) \vee N_{P_2L}(y))$
   $(N_{P_1U} \otimes N_{P_2U})(x,y) = (N_{P_1U}(x) \vee N_{P_2U}(y))$,
   $\forall (x,y) \in V_1 \times V_2$.

2. $(M_{Q_1L} \otimes M_{Q_2L})((x,y_1),(x,y_2))$
   $= (M_{P_1L}(x) \wedge M_{Q_2L}(y_1y_2))$
   $(M_{Q_1U} \otimes M_{Q_2U})((x,y_1),(x,y_2))$
   $= (M_{P_1U}(x) \wedge M_{Q_2U}(y_1y_2))$
   $(N_{Q_1L} \otimes N_{Q_2L})((x,y_1),(x,y_2))$
   $= (N_{P_1L}(x) \vee N_{Q_2L}(y_1y_2))$
   $(N_{Q_1U} \otimes N_{Q_2U})((x,y_1),(x,y_2))$
   $= (N_{P_1U}(x) \vee N_{Q_2U}(y_1y_2))$

3. $(M_{Q_1L} \otimes M_{Q_2L})((x_1,z),(y_1,z))$
   $= (M_{Q_1L}(x_1y_1) \wedge M_{P_2L}(z))$
   $(M_{Q_1U} \otimes M_{Q_2U})((x_1,z),(y_1,z))$
   $= (M_{Q_1U}(x_1y_1) \wedge M_{P_2U}(z))$
   $(N_{Q_1L} \otimes N_{Q_2L})((x_1,z),(y_1,z))$
   $= (N_{Q_1L}(x_1y_1) \vee N_{P_2L}(z)))$
   $(N_{Q_1U} \otimes N_{Q_2U})((x_1,z),(y_1,z))$
   $= (N_{Q_1U}(x_1y_1) \vee N_{P_2U}(z))$,
   $\forall z \in V_2, x_1y_1 \in E_1$.

4. $(M_{Q_1L} \otimes M_{Q_2L})((x_1,y_1),(x_2,y_2))$
   $= (M_{Q_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2))$
   $(M_{Q_1U} \otimes M_{Q_2U})((x_1,y_1),(x_2,y_2))$
   $= (M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
   $(N_{Q_1L} \otimes N_{Q_2L})((x_1,y_1),(x_2,y_2))$
   $= (N_{Q_1L}(x_1x_2) \vee N_{Q_2L}(y_1y_2))$
   $(N_{Q_1U} \otimes N_{Q_2U})((x_1,y_1),(x_2,y_2))$
   $= (N_{Q_1U}(x_1x_2) \vee N_{Q_2U}(y_1y_2))$,
   $\forall (x_1,x_2) \in E_1, (y_1,y_2) \in E_2$.

Example 3.12. *Consider the two IVPFGs $G_1 = (P_1,Q_1)$ and $G_2 = (P_2,Q_2)$ on $V_1 = \{a,b\}$ and $V_2 = \{c,d\}$ respectively.*

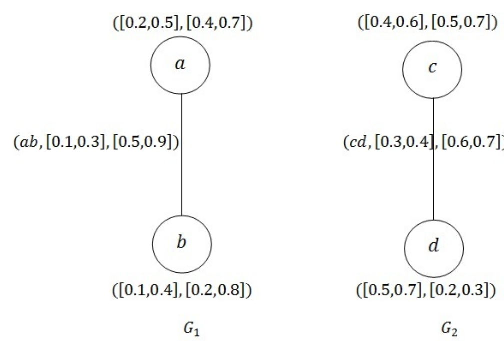

Figure 6. IVPFGs

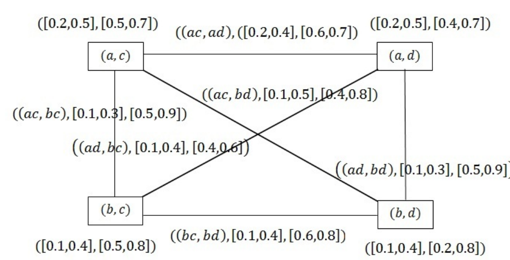

Figure 7. $G_1 \otimes G_2$

Proposition 3.13. *If G_1 and G_2 are CIVPFGs then $G_1 \otimes G_2$ is CIVPFG.*

Proof. If $((x,y_1),(x,y_2)) \in E$, then
$(M_{Q_1L} \otimes M_{Q_2L})((x,y_1),(x,y_2))$
$= (M_{P_1L}(x) \wedge M_{Q_2L}(y_1y_2))$
$= (M_{P_1L}(x) \wedge (M_{P_2L}(y_1) \wedge M_{P_2L}(y_2)))$
$= (M_{P_1L}(x) \wedge M_{P_2L}(y_1) \wedge M_{P_2L}(y_2))$,
$= ((M_{P_1L}(x) \wedge M_{P_2L}(y_1)) \wedge (M_{P_1L}(x) \wedge M_{P_2L}(y_2)))$
$= ((M_{P_1L} \otimes M_{P_2L})(x,y_1) \wedge (M_{P_1L} \otimes M_{P_2L})(x,y_2))$
$(M_{Q_1U} \otimes M_{Q_2U})((x_1,y_1),(x_2,y_2))$

$= (M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
$= ((M_{P_1U}(x_1) \wedge M_{P_1U}(x_2)) \wedge (M_{P_2U}(y_1) \wedge M_{P_2U}(y_2)))$
$= \wedge(M_{P_1U}(x_1), M_{P_1U}(x_2), M_{P_2U}(y_1), M_{P_2U}(y_2)),$
$= ((M_{P_1U}(x_1) \wedge M_{P_2U}(y_1)) \wedge (M_{P_1U}(x_2) \wedge M_{P_2U}(y_2)))$
$= ((M_{P_1U} \otimes M_{P_2U}(x_1, y_1) \wedge (M_{P_1U} \otimes M_{P_2U})(x_2, y_2)).$
Similarly,
$(N_{Q_1L} \otimes N_{Q_2L})((x, y_1), (x, y_2))$
$= ((N_{P_1L} \otimes N_{P_2L})(x, y_1) \vee (N_{P_1L} \otimes N_{P_2L})(x, y_2))$
$(N_{Q_1U} \otimes N_{Q_2U})((x, y_1), (x, y_2)$
$= ((N_{P_1U} \otimes N_{P_2U})(x, y_1) \vee (N_{P_1U} \otimes N_{P_2U})(x, y_2))$
If $((x_1, z), (x_2, z)) \in E$, since G_1 and G_2 are complete, then
$(M_{Q_1L} \otimes M_{Q_2L})((x_1, z), (x_2, z))$
$= (M_{P_1L}(z) \wedge M_{Q_2L}(x_1x_2))$
$= (M_{P_1L}(z), \wedge(M_{P_2L}(x_1) \wedge M_{P_2L}(x_2)))$
$= \wedge(M_{P_1L}(z), M_{P_2L}(x_1), M_{P_2L}(x_2)),$
$= ((M_{P_1L}(z) \wedge M_{P_2L}(x_1)) \wedge (M_{P_1L}(z) \wedge M_{P_2L}(x_2)))$
$= ((M_{P_1L} \otimes M_{P_2L})(z, x_1) \wedge (M_{P_1L} \otimes M_{P_2L})(z, x_2))$
$(M_{Q_1U} \otimes M_{Q_2U})((x_1, z), (x_2, z))$
$= (M_{P_1U}(z) \wedge M_{Q_2U}(x_1x_2))$
$= (M_{P_1U}(z) \wedge (M_{P_2U}(x_1) \wedge M_{P_2U}(x_2)))$
$= \wedge(M_{P_1U}(z), M_{P_2U}(x_1), M_{P_2U}(x_2))$
$= ((M_{P_1U}(z) \wedge M_{P_2U}(x_1)) \wedge (M_{P_1U}(z) \wedge M_{P_2U}(x_2)))$
$= ((M_{P_1U} \otimes M_{P_2U})(z, x_1) \wedge (M_{P_1U} \otimes M_{P_2U})(z, x_2)).$
Similarly,
$(N_{Q_1L} \otimes M_{Q_2L})((x_1, z), (x_2, z)$
$= ((N_{P_1L} \otimes N_{P_2L})(z, x_1) \vee (N_{P_1L} \otimes N_{P_2L})(z, x_2))$
$(N_{Q_1U} \otimes N_{Q_2U})((x_1, z), (x_2, z))$
$= ((N_{P_1U} \otimes N_{P_2U})(z, x_1) \vee (N_{P_1U} \otimes N_{P_2U})(z, x_2))$
If $((x_1, y_1), (x_2, y_2)) \in E$, since G_1 and G_2 are complete,
$(M_{Q_1L} \otimes M_{Q_2L})((x_1, y_1), (x_2, y_2))$
$= (M_{Q_1L}(x_1x_2) \wedge M_{Q_2L}(y_1y_2))$
$= ((M_{P_1L}(x_1) \wedge M_{P_1L}(x_2)) \wedge (M_{P_2L}(y_1) \wedge M_{P_2L}(y_2)))$
$= \wedge(M_{P_1L}(x_1), M_{P_1L}(x_2), M_{P_2L}(y_1), M_{P_2L}(y_2)),$
$= ((M_{P_1L}(x_1) \wedge M_{P_2L}(y_1)) \wedge (M_{P_1L}(x_2) \wedge M_{P_2L}(y_2)))$
$= ((M_{P_1L} \otimes M_{P_2L})(x_1, y_1) \wedge (M_{P_1L} \otimes M_{P_2L})(x_2, y_2)$ $(M_{Q_1U} \otimes M_{Q_2U})((x_1, y_1), (x_2, y_2)) = (M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1y_2))$
$= ((M_{P_1U}(x_1) \wedge M_{P_1U}(x_2)) \wedge (M_{P_2U}(y_1) \wedge M_{P_2U}(y_2)))$
$= \wedge(M_{P_1U}(x_1), M_{P_1U}(x_2), M_{P_2U}(y_1), M_{P_2U}(y_2)),$
$= ((M_{P_1U}(x_1) \wedge M_{P_2U}(y_1)) \wedge (M_{P_1U}(x_2) \wedge M_{P_2U}(y_2)))$
$= ((M_{P_1U} \otimes M_{P_2U})(x_1, y_1) \wedge (M_{P_1U} \otimes M_{P_2U})(x_2, y_2).$
Similarly,
$(N_{Q_1L} \otimes M_{Q_2L})((x_1, y_1), (x_2, y_2))$
$= ((N_{P_1L} \otimes N_{P_2L})(x_1, y_1) \vee (N_{P_1L} \otimes N_{P_2L})(x_2, y_2))$
$(N_{Q_1U} \otimes N_{Q_2U})((x_1, y_1), (x_2, y_2))$
$= ((N_{P_1U} \otimes N_{P_2U})(x_1, y_1) \vee (N_{P_1U} \otimes N_{P_2U})(x_2, y_2))$
Hence, $G_1 \otimes G_2$ is CIVPFG. □

Proposition 3.14. *If G_1 and G_2 are SIVPFGs then $G_1 \otimes G_2$ is SIVPFG.*

Proof. This proof is similar to the proposition 3.5. □

Proposition 3.15. *If $G_1 \otimes G_2$ is SIVPFG then at least G_1 or G_2 must be SIVPFG.*

Proof. This proof is similar to the proposition 3.6 □

4. Conclusion

In this paper, we proved that the direct product, semi-strong product and strong product of two IVPFGs is also an IVPFG, Consequently, if the direct product is strong, then any one of two IVPFGs is strong and also if the semi-strong product is strong, then any one of two IVPFG is strong. Also we proved that the strong product of two complete IVPFGs is complete.

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