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Numerical solution of time fractional non-linear neutral delay differential equations of fourth-order

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Abstract

In this paper, we present a numerical technique for the solution of a class of time fractional nonlinear neutral delay sub-diffusion differential equation of fourth order with variable coefficients. We constructed a numerical scheme which is of second-order convergence in time and is based on L2-1σ formula for the temporal variable. The stability of the scheme is proved using discrete energy method considering several auxiliary assumptions and then we showed that our scheme is convergent in L_2 norm with convergence order $O(\tau^2+h^4)$, where τ and h are temporal and space mesh sizes respectively. In the end, we provide some numerical experiments to validate the theoretical results.

Keywords

Fractional differential equation, L2-1σ formula, Compact difference scheme, Stability, Convergence.

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Contents

1. Introduction

Differential equations involving fractional order derivatives are useful in the modeling of the various physical phenomenon in many significant scientific areas and the ability of non-integer order models in comparison with integer order models in describing the various physical phenomenon is more accurate. These days, the interest of scientists and researches has grown with Fractional Partial Differential Equations (FPDEs) in the areas of medicine, control systems, in physics, signal processing and so on [1]-[12]. Since the analytical solutions for fractional partial differential equations with delay are not so easy to obtain, therefore, it becomes necessary to look for efficient numerical schemes. Various numerical methods and existence and uniqueness theory of Fractional Differential Equations (FDEs) have been studied extensively by researchers in last few years and their study comprises of numerical methods such as Finite Difference, Finite Volume, Finite Element, Weighted residual method, Spectral methods, Hybrid methods, discontinuous Galerkin method and so on [7]-[13] and the reader can also refer to the references therein. Nevertheless, relatively less work has been done using neutral delay partial differential equations. Fractional derivative term is represented in the sense of Caputo fractional derivative $(0<\alpha<1)$, where α is the order of fractional derivative considered in the present paper. Initially, Caputo fractional derivative was approximated by using $L1-formula$ [11], which gives temporal order of convergence $O(\tau^{2-\alpha})$ for $\alpha\in(0,1)$. Recently, Alikhanov [8] introduced a new analog for Caputo fractional derivative which improves the temporal convergence order to $O(\tau^{3-\alpha})$ for $\alpha\in(0,1)$. Therefore, in the present paper, we seek to implement the $L2-1\sigma$ formula introduced by Alikhanov [8] in construction of our compact difference scheme for fourth-order neutral delay partial differential equations and compact difference operator is used to obtain the spatial convergence of order $O(h^4)$, here τ and h are temporal and space mesh sizes respectively.

The proposed difference scheme in our paper is new for the fourth-order neutral delay fractional differential equations with variable coefficients and have temporal convergence order as $O(\tau^{3-\alpha})$ which is more efficient than with using L1-

formula.

Rest of the manuscript is organized as follows: in next section definitions, various lemmas are presented to be used for the construction of the proposed compact difference scheme. Then, in the next section proposed scheme is analyzed in terms of unique solvability, stability, and convergence. In the end, numerical experimentation and conclusions are presented clearly demonstrating the validity of theoretically and numerical results.

Consider the generalized fourth order non-linear fractional sub-diffusion neutral delayed equation with variable coefficients

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + A(x)\frac{\partial^4 u(x,t)}{\partial x^4} + B(x)\frac{\partial^4 u(x,t-s)}{\partial x^4} = f(x,t,u(x,t),u(x,t-s)), 0<x<L, 0<t<T. \quad (1.1)$$

Initial and boundary conditions are as follows:

$$u(x,t) = \phi(x,t), \quad 0 \le x \le L, \quad t \in [-s,0], \quad (1.2)$$
$$u(0,t) = \alpha_1(t), \quad u(L,t) = \alpha_2(t), \quad 0 \le t \le T, \quad (1.3)$$
$$\frac{\partial^2 u(0,t)}{\partial x^2} = \beta_1(t), \quad \frac{\partial^2 u(L,t)}{\partial x^2} = \beta_2(t), 0 \le t \le T. \quad (1.4)$$

Here $s>0$ is delay, $f(x,t,u(x,t),u(x,t-s))$ stands for non-linear delayed source term, $A(x)$, $B(x)$, $\phi(x,t)$, $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ are all given and sufficiently smooth functions. Fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is considered in Caputo sense as follows:

$${}_0^C D_t^\alpha u(x,t) \equiv \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(x,\xi)}{\partial \xi} d\xi, \quad 0<\alpha<1. \quad (1.5)$$

In equation (1.1), we have two type of complexities one is delay term and second is non-linear source term. Here, we seek to obtain a linearized numerical scheme for (1.1) − (1.4) which is uniquely solvable, stable and convergent. Throughout the paper, we consider that the obtained solution $u(x,t)$ along with source function $f(x,t,\mu,\nu)$ are sufficiently smooth likewise considered in [13] in the following sense:

- Let m be the integer satisfying $ms \le T \le (m+1)s$. Define $I_r = (rs,(r+1)s)$, r = -1, 0, ..., m-1, $I_m = (ms,T)$, $I = \bigcup_{q=-1} I_q$ and assume that $u(x,t) \in C^{8,3}([0,L] \times (0,T])$,
- the partial derivatives $f_\mu(x,t,\mu,\nu)$ and $f_\nu(x,t,\mu,\nu)$ are continuous in the ε_0 neighborhood of the solution. Define
$c_1 = \sup|f_\mu(x,t,\mu,\nu)+\varepsilon_1, u(x,t-s)+\varepsilon_2|$
$c_2 = \sup|f_\nu(x,t,\mu,\nu)+\varepsilon_1, u(x,t-s)+\varepsilon_2|$

2. Notations and Preliminaries

We first divide the region $\Omega \times [-s,T]$, where $\Omega = [a,b]$. Define $\Omega_h = \{x_j = a + jh | j = 1,2,...,M\}$, where $h = \frac{b-a}{M}$ and $\Omega_t = \{t_k = (k-1+\sigma)\tau : -n \le k \le N\}$, where, $\tau = \frac{s}{n}$ and $\sigma = 1 - \frac{\alpha}{2}$. Define $\Omega_{ht} = \Omega_h \times \Omega_t$, where, $\Omega_h = \{x_j | 0 \le j \le M\}$, $\Omega_t = \{t_k | -n \le k \le N\}$ and $N = [\frac{T}{\tau}]$. $U_j^{k-1+\sigma} = u(x_j, t_{k-1+\sigma})$. $V_j^{k-1+\sigma} = \frac{\partial^2 u(x_j, t_{k-1+\sigma})}{\partial x^2}$. Consider the grid function space $\mathscr{V} = \{u_j^{k-1+\sigma} | 0 \le j \le M, -n \le k \le N\}$ defined on Ω_{ht}.

Definition 2.1. *The compact linear operator is given as follows:*

$$(\Re u)_j = \begin{cases} \frac{1}{12}(u_{j+1} + 10u_j + u_{j-1}), & 1 \le j \le M-1, \\ u_j, & j = 0, \quad or \quad M. \end{cases}$$

.

Definition 2.2. *Alikhanov [8] constructed a new second-order difference analog for the Caputo-fractional derivative (called $L2-1\sigma$) formula. For definition follow [8].*
$a_0 = \sigma^{1-\alpha}$, $a_l = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}$, $l \ge 1$,
$b_l = \frac{1}{2-\alpha}[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}] - \frac{1}{2}[(l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\alpha}]$, $l \ge 1$,
when k = 0, denote $c_0^{(k)} = a_0$
when $k \ge 1$

$$c_j^{(k)} = \begin{cases} a_0 + b_1, & k = 0, \\ a_k + b_{k+1} - b_k, & 1 \le j \le k-2, \\ a_k - b_k, & j = k-1. \end{cases} \quad (2.1)$$

Given grid function $u = \{u^k | -n \le k \le N\}$, *defining*

$${}_0^C D_t^\alpha u_j^{k-1+\sigma} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[c_0^{(k)} u^k - \sum_{j=1}^{k-1} (c_{k-j-1}^{(k)} - c_{k-j}^{(k)}) u^j - c_{k-1}^{(k)} u^0 \right]. \quad (2.2)$$

as the discrete fractional derivative operator, i.e., $L2-1\sigma$ formula. $\Gamma(\cdot)$ denotes the Gamma function. Alikhnov [8] estimated the error of the $L2-1\sigma$ formula to approximate the Caputo fractional derivative and provided the below Lemma.

Lemma 2.3. *[8] Suppose $u(t) \in C^3[0,t_k]$, it holds that*

$$\left| {}_0^C D_t^\alpha u(t)|_{t=t_{k-1+\sigma}} - {}_0^C D_t^\alpha u_j^{k-1+\sigma} \right| \le \frac{(4\sigma-1)\sigma^{-\alpha}}{12\Gamma(2-\alpha)} \max_{t_0 \le t \le t_k} |u^{(3)}(t)| \tau^{(3-\alpha)}. \quad (2.3)$$

Then the basic properties of the difference operator were derived.

Lemma 2.4. *[8] Suppose $\alpha \in (0,1)$, $\sigma = 1-\frac{\alpha}{2}$, and $c_j^{(k)}(0 \leq j \leq k-1)$ is defined by (2.1), it holds that*

$$c_j^{(k)} > \frac{1-\alpha}{2}(k+\sigma)^{-\alpha}, \tag{2.4}$$

$$c_0^{(k)} > c_1^{(k)} > c_2^{(k)} \ldots > c_{k-2}^{(k)} > c_{k-1}^{(k)}. \tag{2.5}$$

Furthermore, there is an important estimate for the second-order operator, which will play a key role in the analysis of the stability and convergence of our scheme.

Lemma 2.5. *[8] Suppose $u = \{u^k | -n \leq k \leq N\}$ is a grid function defined on Ω_τ, then it holds that*

$$(\sigma u^k + (1-\sigma)u^{k-1})\,{}_0^C D^\alpha_{t_{k-1+\sigma}} u \quad \geq \quad \frac{1}{2}\,{}_0^C D^\alpha_{t_{k-1+\sigma}}(u^2). \tag{2.6}$$

Lemma 2.6. *[8] Denote $\lambda(t) = (1-t)^3[5-3(1-t)^2]$ and let function $g(x) \in C^6[0,L]$. It holds that*

$$\begin{aligned}&\Re g''(x_j) = \delta_x^2 g(x_j)\\ &+ \frac{h^4}{360}\int_0^1 [g^{(6)}(x_j - th) + g^{(6)}(x_j + th)]\lambda(t)dt,\\ &1 \leq j \leq M-1.\end{aligned} \tag{2.7}$$

Now, we will begin with the construction of the compact scheme for considered equation (1.1) and boundary conditions (1.2) – (1.4) by reduced order method, that is by considering $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$ and $v(x,t-s) = \frac{\partial^2 u(x,t-s)}{\partial x^2}$, then, equations (1.1) – (1.4) are equivalent to

$$\begin{aligned}&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + A(x)\frac{\partial^2 v(x,t)}{\partial x^2} + B(x)\frac{\partial^2 v(x,t-s)}{\partial x^2}\\ &= f(x,t,u(x,t),u(x,t-s)), 0<x<L, 0<t<T,\end{aligned} \tag{2.8}$$

$$v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad v(x,t-s) = \frac{\partial^2 u(x,t-s)}{\partial x^2}, \quad 0<x<L, \quad 0<t<T, \tag{2.9}$$

$$u(x,t) = \phi(x,t), \quad 0 \leq x \leq L, \quad t \in [-s,0], \tag{2.10}$$

$$u(0,t) = \alpha_1(t), \quad u(L,t) = \alpha_2(t), \quad 0 \leq t \leq T, \tag{2.11}$$

$$v(0,t) = \beta_1(t), \quad v(L,t) = \beta_2(t), \quad 0 \leq t \leq T. \tag{2.12}$$

Define the grid functions $U_j^k = u(x_j,t_k), V_j^k = v(x_j,t_k)$, and $V_j^{k-n} = v(x_j, t_k - n)$, $0 \leq j \leq M, -n \leq k \leq N$.
Suppose $u(x,t) \in C_{x,t}^{(8,3)}([0,L]\times[-s,T])$, now, we consider the (2.8) – (2.12) at the grid point $(x_j, t_{k-1+\sigma})$, we have

$$\begin{aligned}&\frac{\partial^\alpha u(x_j,t_{k-1+\sigma})}{\partial t^\alpha} + A(x_j)\frac{\partial^2 v(x_j,t_{k-1+\sigma})}{\partial x^2}\\ &+ B(x_j)\frac{\partial^2 v(x_j,t_{k-1+\sigma-n})}{\partial x^2} = f(x_j,t_{k-1+\sigma},u(x_j,t_{k-1+\sigma}),\\ &u(x_j,t_{k-1+\sigma-n})), 0<j<M, 1 \leq k \leq N-1,\end{aligned} \tag{2.13}$$

$$v(x_j,t_{k-1+\sigma}) = \frac{\partial^2 u(x_j,t_{k-1+\sigma})}{\partial x^2}, \tag{2.14}$$

$$v(x_j,t_{k-1+\sigma-n}) = \frac{\partial^2 u(x_j,t_{k-1+\sigma}-n)}{\partial x^2}. \tag{2.15}$$

Applying the average operator $\Re$ on both sides of the (2.13) - (2.15) gives

$$\begin{aligned}&\Re\frac{\partial^\alpha u(x_j,t_{k-1+\sigma})}{\partial t^\alpha} + \Re A(x_j)\frac{\partial^2 v(x_j,t_{k-1+\sigma})}{\partial x^2}\\ &+ \Re B(x_j)\frac{\partial^2 v(x_j,t_{k-1+\sigma-n})}{\partial x^2}\\ &= \Re f(x_j,t_{k-1+\sigma},u(x_j,t_{k-1+\sigma}),u(x_j,t_{k-1+\sigma-n})),\end{aligned} \tag{2.16}$$

$$\Re v(x_j,t_{k-1+\sigma}) = \Re\frac{\partial^2 u(x_j,t_{k-1+\sigma})}{\partial x^2}, \tag{2.17}$$

$$\Re v(x_j,t_{k-1+\sigma-n}) = \Re\frac{\partial^2 u(x_j,t_{k-1+\sigma}-n)}{\partial x^2}, \tag{2.18}$$

Using Taylor's series following expressions can be easily obtained

$$\begin{aligned}&u(x_j,t_{k-1+\sigma}) = \sigma u(x_j,t_k) + (1-\sigma)u(x_j,t_{k-1})\\ &\textit{and} \quad U_j^{k-1+\sigma} = \sigma U_j^k + (1-\sigma)U_j^{k-1}\end{aligned}$$

Linearizing the non-linear source term $f(x,t,u(x,t),u(x,t-s))$ using Taylor's series gives

$$\begin{aligned}&f(x_j,t_{k-1+\sigma}) = f(x_j,t_{k-1+\sigma},2\sigma U_j^{k+1} + (2-3\sigma)U_j^k\\ &-(1-\sigma)U_j^{k-1}, \sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1})\end{aligned}$$

Using Lemma [2.6], we have

$$\begin{aligned}&\Re\frac{\partial^2 u(x_j,t_{k-1+\sigma})}{\partial x^2} = \sigma\Re\frac{\partial^2 u(x_j,t_k)}{\partial x^2}\\ &+(1-\sigma)\Re\frac{\partial^2 v(x_j,t_{k-1})}{\partial x^2}\\ &+O(\tau^2+h^4)\\ &= \delta_x^2 U_j^{k-1+\sigma} + O(\tau^2+h^4)\end{aligned} \tag{2.19}$$

Similarly, we have

$$v(x_j, t_{k-1+\sigma}) = V_j^{k-1+\sigma} + O(\tau^2) \quad (2.20)$$

$$v(x_j, t_{k-1+\sigma} - n) = V_j^{k-1+\sigma-n} + O(\tau^2) \quad (2.21)$$

$$\Re \frac{\partial^2 v(x_j, t_{k-1+\sigma})}{\partial x^2} = \delta_x^2 V_j^{k-1+\sigma} + O(\tau^2 + h^4) \quad (2.22)$$

$$\Re \frac{\partial^2 v(x_j, t_{k-1+\sigma} - n)}{\partial x^2} = \delta_x^2 V_j^{k-1+\sigma-n} + O(\tau^2 + h^4) \quad (2.23)$$

We approximate the time fractional derivative by $L2-1\sigma$ formula, (2.2), applying Lemma [2.3], and substitute equations (2.19) – (2.23) into (2.16) – (2.18); then, we obtain

$$\begin{aligned} &\Re {}_0^C D_{t_{k-1+\sigma}}^{\alpha} U_j + A_j \delta_x^2 V_j^{k-1+\sigma} + B_j \delta_x^2 V_j^{k-1+\sigma-n} \\ &= \Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma) U_j^k \\ &- (1-\sigma) U_j^{k-1}, \sigma U_j^{k-n} + (1-\sigma) U_j^{k-n-1}) + (R_1)_j^k, \end{aligned} \quad (2.24)$$

$$\Re V_j^{k-1+\sigma} = \delta_x^2 U_j^{k-1+\sigma} + (R_2)_j^k, \quad (2.25)$$

$$\Re V_j^{k-1+\sigma-n} = \delta_x^2 U_j^{k-1+\sigma-n} + (R_3)_j^k, \quad (2.26)$$

$$1 \le j \le M-1, \, 1 \le k \le N-1,$$

where

$$|(R_1)_j^k| + |(R_2)_j^k| + |(R_3)_j^k| \le \hat{c}(\tau^{3-\alpha} + h^4), \quad (2.27)$$

$$0 \le j \le M, \quad -n \le k \le N.$$

where $\hat{c}$ is a positive constant independent of τ and h.

Omitting the small terms R_1^k, R_2^k and R_3^k in (2.24) – (2.26), respectively, and taking notice of initial and boundary conditions

$$U_j^k = \phi(x_j, t_k), \quad 0 \le j \le M, \quad -n \le k \le 0, \quad (2.28)$$

$$U_0^k = \alpha_1(t_k), \quad U_M^k = \alpha_2(t_k), \quad 1 \le k \le N, \quad (2.29)$$

$$V_0^k = \beta_1(t_k), \quad V_M^k = \beta_2(t_k) \quad 1 \le k \le N. \quad (2.30)$$

We derive the finite difference scheme below for the problem (1.1) – (1.4), by replacing U_j^k with u_j^k and V_j^k with v_j^k as follows:

$$\begin{aligned} &\Re {}_0^C D_{t_{k-1+\sigma}}^{\alpha} u_j + A_j \delta_x^2 v_j^{k-1+\sigma} + B_j \delta_x^2 v_j^{k-1+\sigma-n} \\ &= \Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma) U_j^k - (1-\sigma) U_j^{k-1}, \\ &\sigma U_j^{k-n} + (1-\sigma) U_j^{k-n-1}), \end{aligned} \quad (2.31)$$

$$\Re v_j^{k-1+\sigma} = \delta_x^2 u_j^{k-1+\sigma}, \quad \Re v_j^{k-1+\sigma-n} = \delta_x^2 u_j^{k-1+\sigma-n},, \quad (2.32)$$

$$1 \le j \le M-1, 1 \le k \le N-1,$$

$$u_j^k = \phi(x_j, t_k), \quad 0 \le j \le M, \quad -n \le k \le 0, \quad (2.33)$$

$$u_0^k = \alpha_1(t_k), \quad u_M^k = \alpha_2(t_k), \quad 1 \le k \le N, \quad (2.34)$$

$$v_0^k = \beta_1(t_k), \quad v_M^k = \beta_2(t_k), \quad 1 \le k \le N. \quad (2.35)$$

3. Analysis of the Compact Difference Scheme

Lemma 3.1. *[14], [15] For any grid function $u \in \mathscr{V}_h$, it holds that*

$$||u||^2 \le \frac{L^2}{6} ||\delta_x u||^2, \quad (3.1)$$

$$||\delta_x u||^2 \le \frac{L^2}{6} ||\delta_x^2 u||^2. \quad (3.2)$$

Lemma 3.2. *[16] For any grid function $u \in \mathscr{V}_h$, it holds that*

$$\frac{1}{3} ||u||^2 \le ||\Re u||^2 \le ||u||^2. \quad (3.3)$$

Lemma 3.3. *[16] For any grid function $u \in \mathscr{V}_h$, it holds that*

$$\frac{1}{3} ||u||^2 \le ||\Re u||^2 \le ||u||^2. \quad (3.4)$$

Proof For the proof of equation (3.4), reader can follow [16]. Inequality (3.4) can be obtained using discrete Green formula and inverse estimates given below

$$\begin{aligned} &||\delta_x^2 u||^2 \le \frac{4}{h^2} ||\delta_x u||^2, \\ &||\delta_x u||^2 \le \frac{4}{h^2} ||u||^2. \end{aligned} \quad (3.5)$$

Lemma 3.4. *Consider that u_j^k and v_j^k with $0 \le j \le M, -n \le k \le N$ are solutions of the constructed difference scheme given in (2.31)-(2.35)*

$$\Re{}_0^C D^{\alpha}_{t_{k-1+\sigma}} u_j + A_j \delta_x^2 v_j^{k-1+\sigma} + B_j \delta_x^2 v_j^{k-1+\sigma-n}$$
$$= \Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k$$
$$-(1-\sigma)U_j^{k-1}, \sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}), \quad (3.6)$$
$$\Re v_j^{k-1+\sigma} = \delta_x^2 u_j^{k-1+\sigma} + H_1^{k-1+\sigma}, \quad (3.7)$$
$$\Re v_j^{k-1+\sigma-n} = \delta_x^2 u_j^{k-1+\sigma-n} + H_2^{k-1+\sigma-n}, \quad (3.8)$$
$$1 \le j \le M-1, 1 \le k \le N-1,$$
$$u_j^k = 0, \quad 0 \le j \le M, \quad -n \le k \le 0, \quad (3.9)$$
$$u_0^k = 0, \quad u_M^k = 0, \quad (3.10)$$
$$v_0^k = 0, \quad v_M^k = 0 \quad 1 \le k \le N. \quad (3.11)$$

Then, following inequality holds

$$||\Re u^k||^2 \le ||\Re u^0||^2 + \mu \left(\frac{L^4}{18} \max_{-n \le k \le N} ||g^{k-1+\sigma}||^2 \right.$$
$$+ \left(\frac{L^4}{18} - 2 \right) \max_{-n \le k \le N} ||H_1^{k-1+\sigma}||^2$$
$$\left. + \left(\frac{L^4}{18} - 2 \right) \max_{-n \le k \le N} ||H_2^{k-1+\sigma}||^2 \right). \quad (3.12)$$

Here, function g(x,t) is linear part of source function f(x,t,u(x,t),u(x,t-s)).

Proof: Taking inner product of (3.6)-(3.8) by $\Re u_j^{k-1+\sigma}$, $\Re v_j^{k-1+\sigma}$ and $\Re v_j^{k-1+\sigma-n}$ respectively, then we obtain

$$(\Re{}_0^C D^{\alpha}_{t_{k-1+\sigma}} u_j, \Re u_j^{k-1+\sigma}) + (A_j \delta_x^2 v_j^{k-1+\sigma}, \Re u_j^{k-1+\sigma})$$
$$+ (B_j \delta_x^2 v_j^{k-1+\sigma-n}, \Re u_j^{k-1+\sigma}) = (\Re f(x_j, t_{k-1+\sigma},$$
$$2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1}, \sigma U_j^{k-n}$$
$$+ (1-\sigma)U_j^{k-n-1}, \Re u_j^{k-1+\sigma}), \quad (3.13)$$
$$(\Re v_j^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) = (\delta_x^2 u_j^{k-1+\sigma}, \Re v_j^{k-1+\sigma})$$
$$+ (H_1^{k-1+\sigma}, \Re v_j^{k-1+\sigma}), \quad (3.14)$$
$$(\Re v_j^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}) = (\delta_x^2 u_j^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n})$$
$$+ (H_2^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}). \quad (3.15)$$

Below given equations (3.17) and (3.18) can be obtained easily using the fact that u^k, $v^k \in \mathscr{V}_h$

$$(\delta_x^2 v_j^{k-1+\sigma}, \Re u_j^{k-1+\sigma}) = \left(\delta_x^2 v_j^{k-1+\sigma}, u_j^{k-1+\sigma} \right.$$
$$\left. + \frac{h^2}{12} \delta_x^2 u_j^{k-1+\sigma} \right)$$
$$= (\delta_x v_j^{k-1+\sigma}, \delta_x u_j^{k-1+\sigma}) + \frac{h^2}{12} \left(\delta_x^2 v_j^{k-1+\sigma}, \delta_x^2 u_j^{k-1+\sigma} \right)$$
$$= (\delta_x^2 u_j^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) \quad (3.16)$$

$$(\delta_x^2 v_j^{k-1+\sigma}, \Re u_j^{k-1+\sigma}) = (\delta_x^2 u_j^{k-1+\sigma}, \Re v_j^{k-1+\sigma}), \quad (3.17)$$
$$(\delta_x^2 v_j^{k-1+\sigma-n}, \Re u_j^{k-1+\sigma-n}) = (\delta_x^2 u_j^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}). \quad (3.18)$$

On adding, equations (3.13)-(3.16), and using above two identities (3.17) and (3.18), we get

$$({}_0^C D^{\alpha}_{t_{k-1+\sigma}} \Re u, \Re u_j^{k-1+\sigma}) + ||\Re v_j^{k-1+\sigma}||^2 + ||\Re v_j^{k-1+\sigma-n}||^2$$
$$= (\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1},$$
$$\sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}, \Re u_j^{k-1+\sigma}), \Re u_j^{k-1+\sigma})$$
$$+ (H_1^{k-1+\sigma}, \Re v^{k-1+\sigma}) + (H_2^{k-1+\sigma-n}, \Re v^{k-1+\sigma-n}) \quad (3.19)$$

Using equations (3.7) and (3.8) we can write the following inequality

$$\frac{1}{2}||\Re v^{k-1+\sigma}||^2 = \frac{1}{2}||\delta_x^2 u_j^{k-1+\sigma}||^2 + \frac{1}{2}||H_1^{k-1+\sigma}||^2$$
$$+ (\delta_x^2 u_j^{k-1+\sigma}, H_1^{k-1+\sigma}), \quad (3.20)$$
$$\frac{1}{2}||\Re v^{k-1+\sigma-n}||^2 = \frac{1}{2}||\delta_x^2 u_j^{k-1+\sigma-n}||^2 + \frac{1}{2}||H_1^{k-1+\sigma-n}||^2$$
$$+ (\delta_x^2 u_j^{k-1+\sigma-n}, H_1^{k-1+\sigma-n}). \quad (3.21)$$

Substituting equations (3.20) & (3.21) into (3.19), we get

$$({}_0^C D^{\alpha}_{t_{k-1+\sigma}} \Re u, \Re u^{k-1+\sigma}) + ||\delta_x^2 u_j^{k-1+\sigma}||^2 + ||H_1^{k-1+\sigma}||^2$$
$$+ 2(\delta_x^2 u_j^{k-1+\sigma}, H_1^{k-1+\sigma}) + ||\delta_x^2 u_j^{k-1+\sigma-n}||^2$$
$$+ ||H_2^{k-1+\sigma-n}||^2 + 2(\delta_x^2 u_j^{k-1+\sigma-n}, H_2^{k-1+\sigma-n})$$
$$= (\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1},$$
$$\sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}, \Re u_j^{k-1+\sigma}), \Re u_j^{k-1+\sigma})$$
$$+ (H_1^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) + (H_2^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}). \quad (3.22)$$

Using Lemma [2.5], equation (3.22) becomes

$$\frac{1}{2} {}_0^C D^{\alpha}_{t_{k-1+\sigma}} ||\Re u||^2 + ||\delta_x^2 u_j^{k-1+\sigma}||^2 + ||H_1^{k-1+\sigma}||^2$$
$$+ 2(\delta_x^2 u_j^{k-1+\sigma}, H_1^{k-1+\sigma}) + ||\delta_x^2 u_j^{k-1+\sigma-n}||^2$$
$$+ ||H_2^{k-1+\sigma-n}||^2 + 2(\delta_x^2 u_j^{k-1+\sigma-n}, H_2^{k-1+\sigma-n})$$
$$= (\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1},$$
$$\sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}, \Re u_j^{k-1+\sigma}), \Re u_j^{k-1+\sigma})$$
$$+ (H_1^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) + (H_2^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}). \quad (3.23)$$

Using ε*-Cauchy inequality* $PQ \le \frac{\varepsilon}{2}P^2 + \frac{1}{2\varepsilon}Q^2$*, then equation (3.23) becomes*

$$\begin{aligned}&\frac{1}{2}{}_0^C D^{\alpha}_{t_{k-1+\sigma}}||\Re u||^2 + 2||\delta_x^2 u_j^{k-1+\sigma}||^2 + 2||H_1^{k-1+\sigma}||^2\\&+2||\delta_x^2 u_j^{k-1+\sigma-n}||^2 + 2||H_2^{k-1+\sigma-n}||^2\\&\le (\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1},\\&\sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}, \Re u_j^{k-1+\sigma}), \Re u_j^{k-1+\sigma})\\&+(H_1^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) + (H_2^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}).\end{aligned} \tag{3.24}$$

Using Lemma 3.1 and 3.2, we have

$$||\Re u_j^{k-1+\sigma}||^2 \le \frac{L^4}{36}||\delta_x^2 u_j^{k-1+\sigma}||^2, \tag{3.25}$$

$$\begin{aligned}&(\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1} + (2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1},\\&\sigma U_j^{k-n} + (1-\sigma)U_j^{k-n-1}), \Re u_j^{k-1+\sigma})\\&\le \frac{18}{L^4}||\Re u^{k-1+\sigma}||^2 + \frac{L^4}{18}||\Re f(x_j, t_{k-1+\sigma}, 2\sigma U_j^{k+1}\\&+(2-3\sigma)U_j^k - (1-\sigma)U_j^{k-1}, \sigma U_j^{k-n}\\&+(1-\sigma)U_j^{k-n-1})||^2\\&\le \frac{1}{2}||\delta_x^2 u_j^{k-1+\sigma}||^2 + \frac{L^4}{18}(||g_j^{k-1+\sigma}||^2\\&+3(1-\sigma)||u_j^k||^2 + ||u_j^{k-n+\sigma-1}||^2),\end{aligned} \tag{3.26}$$

$$\begin{aligned}&(H_1^{k-1+\sigma}, \Re v_j^{k-1+\sigma}) \le \frac{18}{L^4}||\Re v^{k-1+\sigma}||^2\\&+\frac{L^4}{18}||H_1^{k-1+\sigma}||^2\\&\le \frac{1}{2}||\delta_x^2 u_j^{k-1+\sigma}||^2 + \frac{L^4}{18}||H_1^{k-1+\sigma}||^2,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&(H_2^{k-1+\sigma-n}, \Re v_j^{k-1+\sigma-n}) \le \frac{18}{L^4}||\Re v^{k-1+\sigma-n}||^2\\&+\frac{L^4}{18}||H_2^{k-1+\sigma-n}||^2\\&\le \frac{1}{2}||\delta_x^2 u_j^{k-1+\sigma-n}||^2\\&+\frac{L^4}{18}||H_2^{k-1+\sigma-n}||^2.\end{aligned} \tag{3.28}$$

Using equations (3.25)-(3.28), we get

$$\begin{aligned}&\frac{1}{2}{}_0^C D^{\alpha}_{t_{k-1+\sigma}}||\Re u||^2 + ||\delta_x^2 u_j^{k-1+\sigma}||^2 + ||\delta_x^2 u_j^{k-1+\sigma-n}||^2\\&\le \left(\frac{L^4}{18}-2\right)||H_1^{k-1+\sigma}||^2 + \left(\frac{L^4}{18}-2\right)||H_2^{k-1+\sigma-n}||^2\\&+\frac{L^4}{18}(||g_j^{k-1+\sigma}||^2 + 3(1-\sigma)||u_j^k||^2 + ||u_j^{k-n+\sigma-1}||^2).\end{aligned} \tag{3.29}$$

Equation (3.30) can be rewritten as follows

$$\begin{aligned}C_0^k||\Re u^k||^2 &\le \sum_{j=1}^{k-1}(C_{k-j-1}^k - C_{k-j}^k)||\Re u^j||^2 + c_{k-1}^k||\Re u^k||^2\\&+\mu\left(-\frac{16}{h^4}||u_j^{k-1+\sigma}||^2 - \frac{24}{h^4}||u_j^{k-1+\sigma-n}||^2\right.\\&+\frac{L^4}{18}(||g_j^{k-1+\sigma}||^2 + 3(1-\sigma)||u_j^k||^2 + ||u_j^{k-n+\sigma-1}||^2)\\&\left.+\left(\frac{L^4}{18}-2\right)||H_1^{k-1+\sigma}||^2 + \left(\frac{L^4}{18}-2\right)||H_2^{k-1+\sigma-n}||^2\right).\\&+\frac{L^4}{18}(||g_j^{k-1+\sigma}||^2 + 3(1-\sigma)||u_j^k||^2 + ||u_j^{k-n+\sigma-1}||^2),\end{aligned} \tag{3.30}$$

where,

$$\begin{aligned}\mu &= \tau^{\alpha}\Gamma(2-\alpha) = T^{\alpha}\Gamma(1-\alpha)(1-\alpha)N^{-\alpha}\\&< T^{\alpha}\Gamma(1-\alpha)(1-\alpha)\left(k-\frac{\alpha}{2}\right)^{-\alpha}\\&< 2C_{k-1}^k T^{\alpha}\Gamma(1-\alpha).\end{aligned} \tag{3.31}$$

From Lemma 2.4, we know that

$$\begin{aligned}&C_{k-1}^k > \frac{1-\alpha}{2}\left(k-1-\frac{\alpha}{2}\right)^{-\alpha} > \frac{1-\alpha}{2}\left(k-\frac{\alpha}{2}\right)^{-\alpha},\\&1 \le k \le N.\end{aligned} \tag{3.32}$$

Denote

$$\begin{aligned}D &= |\Re u^0||^2 + \mu\left(\frac{L^4}{18}\max_{-n\le k\le N}(||g_j^{k-1+\sigma}||^2\right.\\&+3(1-\sigma)||u_j^k||^2 + ||u_j^{k-n+\sigma-1}||^2)\\&+\left(\frac{L^4}{18}-2\right)\max_{-n\le k\le N}||H_1^{k-1+\sigma}||^2\\&\left.+\left(\frac{L^4}{18}-2\right)\max_{-n\le k\le N}||H_2^{k-1+\sigma-n}||^2\right).\end{aligned} \tag{3.33}$$

Equation (3.30) becomes

$$\begin{aligned}C_0^k||\Re u^k||^2 &\le \sum_{j=1}^{k-1}(C_{k-j-1}^k - C_{k-j}^k)||\Re u^j||^2 + C_{k-1}^k D\\&-\frac{40}{h^4}||u_j^k||^2.\end{aligned} \tag{3.34}$$

Below given inequality can be proved easily by induction and same can be followed in [16]

$$||\Re u^k||^2 \le D, \quad 1 \le k \le N, \tag{3.35}$$

$$C_0^k||\Re u^k||^2 \le C_0^k D. \tag{3.36}$$

Above inequality holds obviously for k = 0, and we assume that this is valid for k = 1,2,...,M-1, i.e.,

$$||\Re u^k||^2 \le D, \quad 1 \le k \le M-1. \tag{3.37}$$

Then, for $2 \le m \le N$ *from (3.34), we have*

$$C_0^k||\Re u^k||^2 \le \sum_{j=1}^{k-1}(C_{k-j-1}^k - C_{k-j}^k)||\Re u^j||^2 + C_{k-1}^k D - \frac{40}{h^4}||u_j^k||^2 \le \sum_{j=1}^{k-1}(C_{k-j-1}^k - C_{k-j}^k)D + C_{k-1}^k D = C_0^k D. \tag{3.38}$$

Which completes the proof.

3.1 Stability

Theorem 3.5. *(Stability) With the help of Lemma 3.4, following theorem can be written as follows:*
The proposed compact difference scheme (2.31)-(2.35) is unconditionally stable for given function $\phi(x,t)$ *and the source function* $f(x,t,u(x,t),u(x,t-s))$.

3.2 Solvability

Theorem 3.6. *(Solvability) The established compact difference scheme (2.31)-(2.35) is uniquely solvable. From [16], it needs only to prove uniqueness of linear homogeneous system as given below has an only trivial solution.*

$$\Re_0^C D_{t_{k-1+\sigma}}^{\alpha} u_j + A_j \delta_x^2 v_j^{k-1+\sigma} + B_j \delta_x^2 v_j^{k-1+\sigma-n} = 0, \tag{3.39}$$

$$\Re v_j^{k-1+\sigma} = \delta_x^2 u_j^{k-1+\sigma}, \Re v_j^{k-1+\sigma-n} = \delta_x^2 u_j^{k-1+\sigma-n}, \quad 1 \le j \le M-1, 1 \le k \le N-1, \tag{3.40}$$

$$u_j^k = 0, \quad 0 \le j \le M, \quad -n \le k \le 0, \tag{3.41}$$

$$u_0^k = 0, \quad u_M^k = 0, \quad v_0^k = 0, \quad v_M^k = 0, \quad 1 \le k \le N. \tag{3.42}$$

Proof Using Theorem 1 from [16] and Lemma 3.4; equations (2.31)-(2.35) possesses a unique solution.

3.3 Convergence

Theorem 3.7. *(Convergence)([16]) Suppose* $\{U_j^k|0 \le j \le M, -n \le k \le N\}$ *and* $\{u_j^k|0 \le j \le M, -n \le k \le N\}$ *are solutions of the equations (1.1)-(1.4) and the constructed difference scheme (2.31)-(2.35), then it holds that*

$$||e^k|| \le c(\tau^{3-\alpha} + h^4). \tag{3.43}$$

where $e_j^k = U_j^k - u_j^k$ *and* $\hat{e}_j^k = V_j^k - v_j^k$.
Proof Subtracting equations (1.2)-(2.1) from (2.31)-(2.35), we get the following error equations as given below

$$\Re e_{t_{k-1+\sigma}}^{\alpha} u_j + A_j \delta_x^2 \hat{e}_j^{k-1+\sigma} + B_j \delta_x^2 \hat{e}_j^{k-1+\sigma-n} = 0, \tag{3.44}$$

$$\Re \hat{e}_j^{k-1+\sigma} = \delta_x^2 e_j^{k-1+\sigma}, \tag{3.45}$$

$$\Re \hat{e}_j^{k-1+\sigma-n} = \delta_x^2 e_j^{k-1+\sigma-n}, \quad 1 \le j \le M-1, \quad 1 \le k \le N-1, \tag{3.46}$$

$$e_j^k = 0, \quad 0 \le j \le M, \quad -n \le k \le N, \tag{3.47}$$

$$e_0^k = 0, \quad e_M^k = 0, \quad \hat{e}_0^k = 0, \quad \hat{e}_M^k = 0, \quad 1 \le k \le N, \tag{3.48}$$

Using Lemma 3.4 we get the following inequality

$$||\Re e^k||^2 \le ||\Re e^0||^2 + \mu\left(\frac{L^4}{18}\max_{-n\le k\le N}||g^{k-1+\sigma}||^2 + \left(\frac{L^4}{18}-2\right)\max_{-n\le k\le N}||H_1^{k-1+\sigma}||^2 + \left(\frac{L^4}{18}-2\right)\max_{-n\le k\le N}||H_2^{k-1+\sigma}||^2\right). \tag{3.49}$$

Using Lemma 3.2, we can easily obtain claimed inequality (3.43) and equation (3.49).

4. Numerical validation

This section discusses the theoretical and numerical results through examples. Here, we considered an example with delay term s=0.1(¿0) whose exact solution is known and is used for verification with our obtained numerical results. The efficiency of the scheme is numerically examined by taking sufficiently small spatial and temporal steps. Tables [1] and [2] provides the temporal and spatial rate of convergence orders and hence proves the validity of our theoretical results with temporal and spatial order of convergence are $O(\tau^2)$ and $O(h^4)$ respectively. A good agreement between theoretical and numerical results is obtained. Let $E_{L_2}(h,\tau) = \max_{1\le k\le N}||u^k - U^k||$, $E_\infty(h,\tau) = \max_{1\le k\le N}||u^k - U^k||_\infty$, $Order(\tau) = \log_2\left(\frac{E_{L_2}(h,2\tau)}{E_{L_2}(h,\tau)}\right)$ and $Order(h) = \log_2\left(\frac{E_{L_2}(2h,\tau)}{E_{L_2}(h,\tau)}\right)$.

Example 4.1.

$$f(x,t,u(x,t),u(x,t-s)) = u(x,t)^2 - u(x,t-0.1)$$
$$+G(x,t).$$
$$where \quad G(x,t) = \frac{1}{6}\Gamma(\alpha+4)\sin(2\pi x)t^{3+\alpha}$$
$$+16\pi^4 x\sin(2\pi x)t^{3+\alpha}$$
$$+16\pi^4(x^2+1)\sin(2\pi x)(t-0.1)^{3+\alpha}$$
$$+(\sin(2\pi x)t^{3+\alpha})^2 - \sin(2\pi x)(t-0.1)^{3+\alpha},$$
$$A(x) = x, B(x) = x^2+1,$$
$$\phi(x,t) = \sin(2\pi x)t^{3+\alpha}, 0 \le x \le 1, \quad t \in [-0.1,0]$$
$$\alpha_1(t) = 0, \alpha_2(t) = 0, \beta_1(t) = 0, \beta_2(t) = 0, 0 \le t \le 1,$$

The exact solution of the Example 4.1 is $u(x,t) = \sin(2\pi x)t^{3+\alpha}$.

First we fix $h = \frac{1}{50}$ and keep varying τ. In table 1, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_\infty(\tau,h))$. It is observed that temporal accuracy of order $O(\tau^2)$ is achieved conforming the theoretical results.
Next we fix $\tau = \frac{1}{50}$ and keep varying h. In table 2, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_\infty(\tau,h))$. It is observed that spatial accuracy of order $O(h^4)$ is achieved conforming the theoretical results.

The efficiency of the scheme is numerically examined by taking sufficiently small spatial and temporal steps. The results illustrate that our scheme has temporal accuracy of $O(h^4)$ and temporal convergence of order $O(\tau^2)$. A good agreement between theoretical and numerical results is obtained.

First we fix $h = \frac{1}{50}$ and keep varying τ. In table 1, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_\infty(\tau,h))$. It is observed that temporal accuracy of order $O(\tau^2)$ is achieved conforming the theoretical results.
Next we fix $\tau = \frac{1}{50}$ and keep varying h. In table 2, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_\infty(\tau,h))$. It is observed that spatial accuracy of order $O(h^4)$ is achieved conforming the theoretical results.
Exact and numerical solution surface plot are given in Figure 1 and 2 with $\alpha = 0.3$, $h = \tau = \frac{1}{50}$ and a good agreement between numerical and exact solution can be observed.

Table 1. Comparison of temporal convergence order with L1-formula [9],[11] and computational error using $L_1 - norm$ and $L_\infty - norm$ for example 4.1.

			Using L1-formula	Using Present Scheme		Using L1-formula	Using Present Scheme
α	τ	$E_{L_2}(h,\tau)$	$Order(\tau^{2-\alpha})$	$Order(\tau^{3-\alpha})$	$E_\infty(h,\tau)$	$Order(\tau^{2-\alpha})$	$Order(\tau^{3-\alpha})$
0.3	1/30	6.6119e-004	1.5789	1.9868	5.2005e-004	1.5919	1.9876
	1/60	2.4058e-004	1.6217	1.9977	1.7166e-004	1.6321	1.9981
	1/80	1.6322e-005	1.6901	1.9985	7.3215e-005	1.6911	1.9990
0.6	1/30	7.8624e-004	1.2886	1.9892	8.4616e-004	1.2768	1.9968
	1/60	3.0526e-005	1.3473	1.9969	3.5020e-005	1.3377	1.9991
	1/80	9.1927e-006	1.3890	1.9993	9.1822e-006	1.3701	2.0007
0.8	1/30	2.3755e-004	1.0919	1.9898	2.3604e-004	1.0818	1.9989
	1/60	1.4600e-005	1.1285	1.9985	4.9427e-005	1.1176	1.9995
	1/80	8.8152e-006	1.1789	2.0011	6.5601e-006	1.7664	2.0009

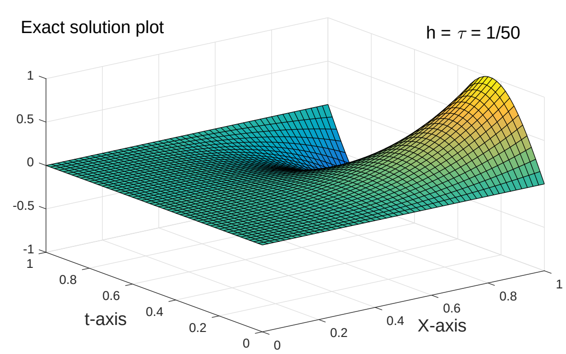

Figure 1. Exact solution surface with $h = \tau = \frac{1}{50}$

Table 2. The computational error and convergence orders in space for example 4.1.

α	h	$E_{L_2}(h,\tau)$	$Order(\tau)$	$E_\infty(h,\tau)$	Order(τ)
0.3	1/20	2.6600e-006	3.9986	9.1840e-006	4.0008
	1/30	7.4381e-007	3.9882	4.4657e-007	3.9989
	1/50	1.9632e-008	3.9861	3.9617e-008	3.9964
0.6	1/20	5.7485e-006	3.9999	6.5893e-006	4.0012
	1/30	3.3914e-007	3.9941	1.7462e-007	3.9979
	1/50	9.0368e-008	3.9926	7.4129e-008	3.9896
0.9	1/20	4.4380e-006	3.9993	2.3051e-006	4.0015
	1/30	8.9951e-007	3.9977	8.4683e-007	3.9958
	1/50	6.7162e-008	3.9960	4.0009e-008	3.9886

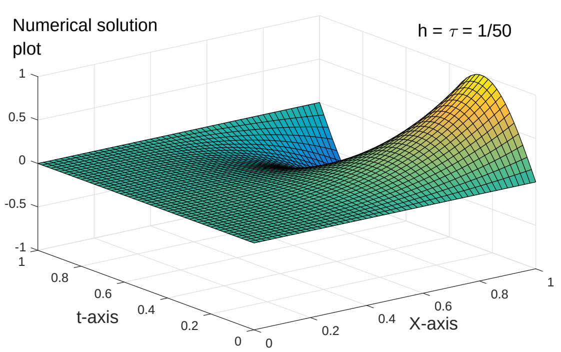

Figure 2. Numerical solution surface with $h = \tau = \frac{1}{50}$

Example 4.2. *Example 4.2 is considered with non-local initial and boundary conditions. A positive delay quantity (s=0.1) is assumed. Numerical results are calculated using the proposed compact difference scheme and obtained results are found in good agreement with theoretical results. Tables 3 and 4 shows the temporal and spatial convergence orderers and hence confirming the theoretical results.*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^4 u(x,t)}{\partial x^4} + \frac{\partial^4 u(x,t-0.1)}{\partial x^4} + G(x,t),$$
$$G(x,t) = \exp(x)\Gamma(\alpha+4)\frac{t^3}{6} - 2t\exp(x)$$
$$-\exp(x)(t^{3+\alpha} + (t-0.1)^{3+\alpha}) + \frac{1}{10}\exp(x)$$
$$u(x,t) = \exp(x)(t^{3+\alpha}+t), \quad 0 \le x \le 1, \quad t \in [-0.1, 0]$$
$$u(0,t) = t^{3+\alpha}+t, \quad u(1,t) = e(t^{3+\alpha}+t), \quad 0 \le t \le 1,$$
$$\frac{\partial^2 u(0,t)}{\partial x^2} = t^{3+\alpha}+t, \frac{\partial^2 u(1,t)}{\partial x^2} = e(t^{3+\alpha}+t), 0 \le t \le 1.$$

The exact solution of the problem is $u(x,t) = \exp(x)(t^{3+\alpha}+t)$.

The efficiency of the scheme is numerically examined by taking sufficiently small spatial and temporal steps. The results illustrate that our scheme has temporal accuracy of $O(h^4)$ and temporal convergence of order $O(\tau^2)$. A good agreement between theoretical and numerical results is obtained.

First we fix $h = \frac{1}{50}$ and keep varying τ. In table 3, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_{L_\infty}(\tau,h))$. It is observed that temporal accuracy of order $O(\tau^2)$ is achieved conforming the theoretical results.
Next we fix $\tau = \frac{1}{50}$ and keep varying h. In table 4, error is presented in L_2 norm $(E_2(\tau,h))$ and L_∞ norm $(E_{L_\infty}(\tau,h))$. It is observed that spatial accuracy of order $O(h^4)$ is achieved conforming the theoretical results.

Table 3. The computational error and convergence orders in time for example 4.2.

α	τ	$E_{L_2}(h,\tau)$	$Order(\tau)$	$E_\infty(h,\tau)$	Order(τ)
0.4	1/30	3.6370e-004	1.9946	8.4100e-003	1.9840
	1/50	1.0015e-005	1.9969	6.5628e-004	1.9965
	1/70	8.2963e-006	1.9978	9.7968e-005	1.9989
0.6	1/30	6.7639e-004	1.9985	1.3642e-004	1.9945
	1/50	7.4628e-005	1.9999	7.4687e-004	1.9969
	1/70	3.9653e-006	2.0003	8.9643e-005	2.0020
0.8	1/30	5.7684e-004	1.9986	4.8593e-004	1.9980
	1/50	4.6820e-005	1.9999	6.2849e-004	1.9996
	1/70	9.5008e-006	2.0010	9.6280e-005	2.0018

Table 4. The computational error and convergence orders in space for example 4.2.

α	h	$E_{L_2}(h,\tau)$	$Order(h)$	$E_{L_\infty}(h,\tau)$	Order(h)
0.4	1/30	4.8460e-004	3.9845	3.8001e-004	3.9879
	1/60	6.7658e-005	3.9915	5.5299e-005	3.9923
	1/90	3.3255e-007	3.9979	9.4630e-007	3.9979
0.6	1/30	1.8651e-003	3.9858	1.4688e-004	3.9941
	1/60	9.7650e-005	3.9975	4.0273e-005	3.9980
	1/90	8.8850e-007	3.9994	7.9611e-007	4.0010
0.8	1/30	2.9273e-004	3.9875	9.6370e-004	3.9980
	1/60	6.7965e-005	3.9941	4.2988e-005	3.9991
	1/90	4.4620e-007	4.0001	6.3685e-007	4.0011

Exact and numerical solution surface plot are given in Figure 3 and 4 with $\alpha = 0.3$, $h = \tau = \frac{1}{50}$ and a good agreement between numerical and exact solution can be observed.

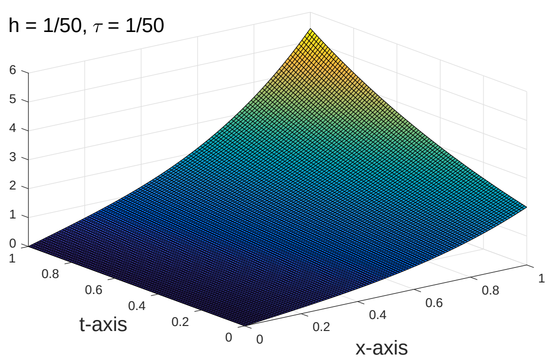

Figure 3. Exact solution surface with $h = \tau = \frac{1}{50}$

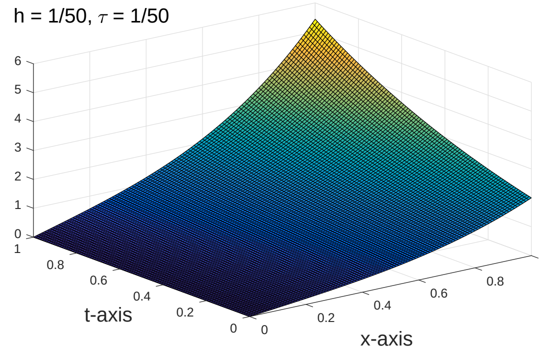

Figure 4. Numerical solution surface with $h = \tau = \frac{1}{50}$

5. Conclusion

This paper presents a second-order compact difference scheme for time variable and of fourth order for the spatial variable for fourth order non-linear neutral delay sub-diffusion wave equation with variable coefficients. Using the discrete energy method, stability and convergence of the proposed scheme in the L_2-norm are proved. Numerical calculations of the test problem confirm the reliability of the theoretical results.

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