Malaya Journal of Matematik, Vol. S, No. 1, 590-594, 2019

https://doi.org/10.26637/MJM0703/0036

The zero-divisor Cayley graph of the residue class ring $(Z_n, \oplus, \odot)$

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Abstract

In this paper the notion of the zero-divisor Cayley graph $G(Z_n, D_0)$, where $(Z_n, \oplus, \odot)$ is the ring of residue classes modulo n, $n \geq 1$, an integer and D_0 is the set of nonzero zero-divisors, is introduced and it is shown that $G(Z_n, D_0)$ can be decomposed into components, if n is a power of a single prime and it is connected, if n is a product of more than one prime power.

Keywords

Zero-Divisors, Symmetric set, Cayley Graph, Zero-divisor Cayley Graph

AMS Subject Classification

05C07,05C25,05C30,05C38, 05C40,20F65.

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Article History: Received **11** March **2019**; Accepted **27** July **2019**

Contents

1. Introduction

The Cayley graph $G(X,S)$ associated with the group $(X,.)$ and its symmetric subset S (a subset S of the group $(X,.)$ is called a symmetric subset, if $s^{-1} \in S$ for every $s \in S$) is introduced to study whether given a group $(X,.)$, there is a graph Γ, whose automorphism group is isomorphic to the group $(X,.)$ [14], and Frucht established this in [12]. For details see [18]. Later independent studies on Cayley graphs have been carried out by many researches [9, 10]. Madhavi [15] introduced Cayley graphs associated with the arithmetical functions, namely, the Euler totient function $\varphi(n)$, the quadratic residues modulo a prime p and the divisor function $d(n), n \geq 1$, an integer and obtained various properties of these graphs. Later Madhavi et al. [16, 17] studied various aspects of these graphs.

The Cayley graph $G(X,S)$ associated with the group $(X,.)$ and its symmetric subset S is the graph, whose vertex set is X and the edge set $E = \{(x,y) : either \;\; xy^{-1} \in S, \; or, \; yx^{-1} \in S\}$. If $e \notin S$, where e is the identity element of X, then $G(X,S)$ is an undirected simple graph. Further $G(X,S)$ is $|S|-$ regular and contains $\frac{|X||S|}{2}$ edges [15].

Beck [8], Akbari and Mohammadian [1, 2], Anderson and Naseer [6], Anderson and Livingston [5], Livingston [19] Smith [20], Tongsuo [21], and others studied the zero-divisor graphs of commutative rings. Given a commutative ring R with identity, they defined the zero-divisor graph $\Gamma(R)$ as the graph, whose vertex set is the set $Z(R)^*$, the set of nonzero zero-divisors of R and the edge set is the set of all ordered pairs (x,y) of elements $x, y \in Z(R)^*$, such that $xy = 0$ and studied the connectedness, the diameter, the girth , the automorphisms of $\Gamma(R)$ and other prperties under conditions on the ring R. Our attempt is to associate a Cayley graph with the set of nonzero zero-divisors of a ring $(R,+,.)$ and study these graphs, with particular reference to the ring $(Z_n, \oplus, \odot)$ of residue classes modulo $n \geq 1$, an integer. The terminology and notions that are used in this paper can be found in [11] for graph theory, [13] for algebra and [7] for number theory.

2. Zero-divisor Cayley graph and its properties

In this paper we study the Cayley graph associated with the set of zero-divisors in the ring $(Z_n, \oplus, \odot)$ of residue classess modulo a positive integer n. We start with some properties of the zero-divisors of a ring that are needed in our study.

Definition 2.1. *Let $(R,+,.)$ be a commutative ring. An element $x \in R, x \neq 0$, is called a zero-divisor in $(R,+,:)$, if there exists $y \in R, y \neq 0$, such that $xy = 0$. The set of all zero-divisors of the ring $(R,+,.)$ is denoted by D_0.*

Lemma 2.2. *Let $(R,+,.)$ be a commutative ring. The set D_0 of the zero-divisors in $(R,+,.)$ is a symmetric subset of the group $(R,+)$.*

Proof. Let $x \in R$ be a zero-divisor in the ring $(R, +, .)$. Then $x \neq 0$ and there exists $y \in R, y \neq 0$, such that $xy = 0 = yx$. Consider the inverse $-x$ of x in the group $(R, +)$. Then $-x \neq 0$ and from $xy = 0$, one gets $(-x)y = -(xy) = 0$. So $-x$ is also a zero-divisor $(R, +, .)$. Hence D_0 is a symmetric subset of the group $(R, +)$. □

Lemma 2.3. *Let $n \geq 1$, be an integer. A positive integer r is not relatively prime to n if, and only if, r is a zero-divisor in the ring $(Z_n, \oplus, \odot)$.*

Proof. Let $n \geq 1$, be an integer and let $r > 1$, be a positive integer less than are equal to n such that $(n, r) \neq 1$. Then there exists an integer $s > 1$, such that $(r, n) = s$, so that s/r and s/n. So, $n = st$ and $r = sl$, for some integers $l > 1$, $t > 1$. That is, $r = sl = (n/t)l$, or, $rt = nl$. This shows that $\overline{rt} = \overline{nl}$, or, $\overline{r} \odot \overline{t} = \overline{n} \odot \overline{l} = \overline{0} \odot \overline{l} = \overline{0}$. That is, there is $\overline{t} \in Z_n$, $\overline{t} \neq 0$ such that $\overline{r} \odot \overline{t} = \overline{0}$, so that $\overline{r}$ is a zero-divisor in the ring $(Z_n, \oplus, \odot)$.

Conversely, let $\overline{r}$ be a zero-divisor in the ring $(Z_n, \oplus, \odot)$. Then there exists $\overline{s} \neq \overline{0}$ such that $\overline{r} \odot \overline{s} = \overline{0} = \overline{s} \odot \overline{r}$.

Suppose $(r, n) = 1$. Then there exists integers x and y such that $rx + ny = 1$. This gives $srx + sny = s$, or, $\overline{srx + sny} = \overline{s}$, or, $(\overline{s} \odot \overline{r}) \odot \overline{x} \oplus (\overline{s} \odot \overline{n}) \odot \overline{y} = \overline{s}$. Since $\overline{s} \odot \overline{r} = \overline{0}$ and $\overline{n} = \overline{0}$, we get $\overline{s} \neq \overline{0}$ and this leads to a contradiction to fact that $\overline{s} \neq \overline{0}$. So $(r, n) \neq 1$. □

Theorem 2.4. *For $n \geq 1$, an integer, the number of zero-divisors of the ring $(Z_n, \oplus, \odot)$ is $n - \varphi(n) - 1$.*

Proof. By the Lemma 2.3, for any positive integer $n \geq 1$, the integer r, $1 < r < n$, is not a relatively prime to n if, and only if, r is a zero-divisor in the ring $(Z_n, \oplus, \odot)$.

For any positive integer $n \geq 1$, there are $\varphi(n)$ number of integers less than n and relatively prime to n. So, the number of numbers which are less than n and not relatively prime to n is equal to $n - \varphi(n)$. Also by the definition of the zero-divisor, $\overline{0}$ is not a zero-divisor in the ring $(Z_n, \oplus, \odot)$, so that the number of zero-divisors in the ring $(Z_n, \oplus, \odot)$ is $n - \varphi(n) - 1$. □

By the Lemma 2.2, the set D_0 of zero-divisors of the ring $(Z_n, \oplus, \odot)$ is a symmetric subset of the group $(Z_n, \oplus)$. So one can think of Cayley graph associated with the group $(Z_n, \oplus)$ and its symmetric subset D_0 and this is defined as follows.

Definition 2.5. *Consider the group $(Z_n, \oplus)$ and its symmetric subset D_0 of zero-divisors in the ring $(Z_n, \oplus, \odot)$. The graph G whose vertex set $V = Z_n = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\}$ and whose edge set $E = \{(x, y) : either\ x - y \in D_0, or, y - x \in D_0\}$ is defined as the* ***zero-divisor Cayley graph*** *and it is denoted by $G(Z_n, D_0)$.*

Lemma 2.6. *The graph $G(Z_n, D_0)$ is $(n - \varphi(n) - 1)-$regular. Moreover the number of edges in $G(Z_n, D_0)$ is given by $\frac{n}{2}(n - \varphi(n) - 1)$.*

Proof. By the Theorem 1.4.5, [15]. The graph $G(Z_n, D_0)$ is $(n - \varphi(n) - 1)-$ regular and the total number of edges in $G(Z_n, D_0)$ is $\frac{|Z_n|(n-\varphi(n)-1)}{2}$. That is, $G(Z_n, D_0)$ is $(n - \varphi(n) - 1)-$ regular and its size is $\frac{n(n-\varphi(n)-1)}{2}$. □

Example 2.7. *In the ring $(Z_7, \oplus, \odot)$, the set D_0 of zero-divisors is the empty set and the graph contains only vertices . The graph of $G(Z_7, D_0)$ is given in Figure 1.*

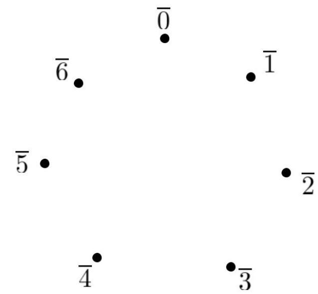

Figure 1: $G(Z_7, D_0)$

Example 2.8. *In the ring $(Z_8, \oplus, \odot)$, the set D_0 of zero-divisors is the $\{\overline{2}, \overline{4}, \overline{6}\}$. Since $\overline{7} - \overline{3} = \overline{4} \in D_0$, there is an edge between $\overline{3}$ and $\overline{7}$. Also, $\overline{5} - \overline{1} = \overline{4} \in D_0$ and there is an edge between $\overline{1}$ and $\overline{5}$. Similarly other edges can be found and the graph of $G(Z_8, D_0)$ is given in Figure 2.*

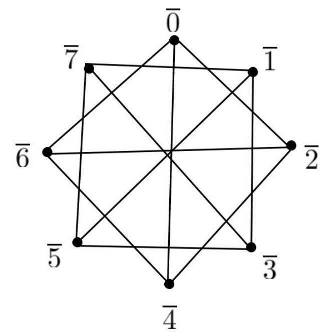

Figure 2: $G(Z_8, D_0)$

Example 2.9. *In the ring $(Z_{10}, \oplus, \odot)$, the set D_0 of zero-divisors is the $\{\overline{2}, \overline{4}, \overline{5}, \overline{6}, \overline{8}\}$ and the graph of $G(Z_{10}, D_0)$ is given in Figure 3.*

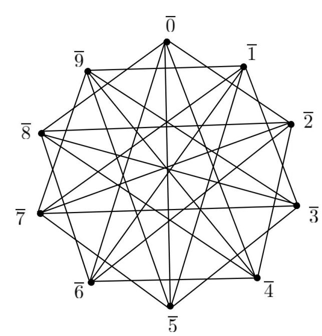

Figure 3: $G(Z_{10}, D_0)$

Lemma 2.10. *For a prime p, the graph $G(Z_p, D_0)$ contains only isolated vertices.*

Proof. Let p be a prime. Then for every $r, 1 \leq r \leq p - 1$, $(r, p) = 1$ and the ring $(Z_p, \oplus, \odot)$ has no zero-divisors, so that the edge set is empty and the graph has only isolated vertices. □

3. The disconnected property of the zero-divisor Cayley graph $G(Z_n, D_0)$, where n is a power of a single prime

When n is a power of a single prime say, $n = p^r$, p be a prime and $r > 1$, the zero-divisor Cayley graph $G(Z_{p^r}, D_0)$ has an interesting property, that, it is decomposed into disjoint union of p components.

Remark 3.1. *In the study of disconnected property of $G(Z_{p^r}, D_0)$, where p is a prime and $r > 1$, is an integer, the following decomposition of the vertex set Z_{p^r} of $G(Z_{p^r}, D_0)$, as $C_0, C_1, C_2, ..., C_{p-1}$ play a key role.*
$C_0 = \{\overline{0}, \overline{p}, ..., i\overline{p}, ..., j\overline{p}, ..., (p^{r-1}-1)\overline{p}\}$,
$C_1 = \{\overline{1}, \overline{p}+\overline{1}, ..., i\overline{p}+\overline{1}, ..., j\overline{p}+\overline{1}, ..., (p^{r-1}-1)\overline{p}+\overline{1}\}$,
$C_2 = \{\overline{2}, \overline{p}+\overline{2}, ..., i\overline{p}+\overline{2}, ..., j\overline{p}+\overline{2}, ..., (p^{r-1}-1)\overline{p}+\overline{2}\}$,

$$\vdots$$

$C_{p-1} = \{\overline{p-1}, ..., i\overline{p}+\overline{p-1}, ..., j\overline{p}+\overline{2}, ..., (p^{r-1}-1)\overline{p}+\overline{p-1}\}$.

Lemma 3.2. *For a prime p and an integer $r > 1$, the set D_0 of zero-divisors in the ring $(Z_{p^r}, \oplus, \odot)$ is given by $D_0 = \{\overline{p}, 2\overline{p}, ..., i\overline{p}, ..., j\overline{p}, ..., (p^{r-1}-1)\overline{p}\}$ and the number of zero-divisors of the ring $(Z_{p^r}, \oplus, \odot)$ is $p^{r-1}-1$.*

Proof. For each integer i, $0 \le i \le p^{r-1}-1$, $i\overline{p}$ is a zero-divisor of the ring $(Z_{p^r}, \oplus, \odot)$, since $(i\overline{p})(\overline{p^{r-1}}) = \overline{ip^r} = \overline{0}$. So, every element in the set

$$D_0 = \{\overline{p}, 2\overline{p}, ..., i\overline{p}, ..., j\overline{p}, ..., (p^{r-1}-1)\overline{p}\},$$

is a zero-divisors of ring $(Z_{p^r}, \oplus, \odot)$ and it contains $p^{r-1}-1$ elements. By the Theorem 2.4, the number of elements in the set D_0 of zero-divisor of $(Z_{p^r}, \oplus, \odot)$ is equal to $p^r - \varphi(p^r) - 1$, or, $p^r - (p^r - p^{r-1}) - 1 = p^{r-1} - 1$, since $\varphi(p^r) = p^r - p^{r-1}$. This shows that the set

$$D_0 = \{\overline{p}, 2\overline{p}, ..., i\overline{p}, ..., j\overline{p}, ..., (p^{r-1}-1)\overline{p}\}$$

is the set of zero-divisors of $(Z_{p^r}, \oplus, \odot)$, and the number of zero-divisors of $(Z_{p^r}, \oplus, \odot)$, is $p^{r-1}-1$. □

Lemma 3.3. *For $0 \le k \le p-1$, , each C_k contains p^{r-1} distinct vertices of $G(Z_{p^r}, D_0)$.*

Proof. For $0 \le k \le p-1$, consider the subset C_k of vertices of $G(Z_{p^r}, D_0)$ is given by

$$C_k = \{\overline{k}, \overline{p}+\overline{k}, 2\overline{p}+\overline{k}, ..., i\overline{p}+\overline{k}, ..., j\overline{p}+\overline{k}, ..., (p^{r-1}-1)\overline{p}+\overline{k}\}.$$

If possible, let $\overline{k} + i\overline{p} = \overline{k} + j\overline{p}$. For $i \ne j, 0 \le i < j \le p^{r-1}-1$. Then $(j-i)\overline{p} = \overline{0}$. Since, $i \ne j, 0 \le i < j \le p^{r-1}-1$, we have $0 \le j-i \le p^{r-1}-1$. But $o(p)$ in $(Z_{p^r}, \oplus)$ is p^{r-1}. So, for any positive integer $t \le p^{r-1}, t\overline{p} \ne \overline{0}$ and thus, $(j-i)\overline{p} = \overline{0}$ with $j-i < p^{r-1}$ leads to a contradiction. Hence our assumption that $\overline{k} + i\overline{p} = \overline{k} + j\overline{p}$, for $i \ne j, 0 \le i < j \le p^{r-1}-1$, is wrong and C_k contains p^{r-1} distinct elements. □

Lemma 3.4. *For $0 \le k \le p-1$, each C_k is a complete subgraph of $G(Z_{p^r}, D_0)$.*

Proof. For this one has to show that there is an edge between every pair of distinct vertices in C_k. To see this, let $i\overline{p}+\overline{k}, j\overline{p}+\overline{k} \in C_k$ for $0 \le i < j \le p^{r-1}-1$. Then,

$$(j\overline{p}+\overline{k}) - (i\overline{p}+\overline{k}) = (j-i)\overline{p}.$$

Since $(j-i)\overline{p}\overline{p^{r-1}} = \overline{0}$, this shows $(j-i)\overline{p}$ is a zero divisor of $(Z_{p^r}, \oplus, \odot)$ and $(j\overline{p}+\overline{k}) - (i\overline{p}+\overline{k}) \in D_0$, so that there is an edge between any pair of distinct vertices in C_k, proving that C_k is a complete subgraph of $G(Z_{p^r}, D_0)$. □

Lemma 3.5. *For $0 \le k < l \le p-1, C_k \cap C_l = \phi$.*

Proof. For $0 \le k < l \le p-1$, we have

$C_k = \{\overline{k}, \overline{p}+\overline{k}, 2\overline{p}+\overline{k}, ..., i\overline{p}+\overline{k}, ..., j\overline{p}+\overline{k}, ..., (p^{r-1}-1)\overline{p}+\overline{k}\}$,

and

$C_l = \{\overline{l}, \overline{p}+\overline{l}, 2\overline{p}+\overline{l}, .., i\overline{p}+\overline{l}, ..., j\overline{p}+\overline{l}, ..., (p^{r-1}-1)\overline{p}+\overline{l}\}$.

If possible, assume that $C_k \cap C_l \ne \phi$. Then, there exists $u \in C_k \cap C_l$. Now $u \in C_k$ implies that $u = \overline{k} + i\overline{p}$ for some $i, 0 \le i \le p^{r-1}-1$. Similarly, $u \in C_l$, implies that $u = \overline{l} + j\overline{p}$ for some $j, 0 \le j \le p^{r-1}-1$. For definiteness we may assume that $i < j$. Then we have, $\overline{k} + i\overline{p} = u = \overline{k} + i\overline{p}$, or, $\overline{l} - \overline{k} + (j-i)\overline{p} = \overline{0}$. From this one gets $(\overline{l}-\overline{k})\overline{p^{r-1}} + (j-i)\overline{p^r} = \overline{0}$, or, $(\overline{l}-\overline{k})\overline{p^{r-1}} = \overline{0}$, since $\overline{p^r} = \overline{0}$. That is, $(l-k)\overline{p^{r-1}} = \overline{0}$, since $\overline{t} = t\overline{1}$, for any $t, 1 < t < p^{r-1}$. Now $0 \le k < l \le p-1$, so that $0 \le l-k \le p-1 < p$. That is, $(l-k)\overline{p^{r-1}} = \overline{0}$ with $l-k < p$. Since, $o(\overline{p^{r-1}}) = p$, in $(Z_{p^r}, \oplus)$, this leads to a contradiction. So, our assumption that $C_k \cap C_l \ne \phi$ is wrong and hence C_k and C_l are disjoint. □

Lemma 3.6. *For $0 \le k < l \le p-1$, there is no edge between any vertex of C_k and any vertex of C_l.*

Proof. For $0 \le k < l \le p-1$, let $i\overline{p}+\overline{k} \in C_k$ and $j\overline{p}+\overline{l} \in C_l$. Then $(j\overline{p}+\overline{l}) - (i\overline{p}+\overline{k}) = (j-i)\overline{p} + (\overline{l}-\overline{k})$. Since $0 \le k \le p-1$, and $0 \le l \le p-1$, we have $l-k \le p-1 < p$, it follows that $l-k$ is not a multiple of p. Hence $(j-i)p + (l-k)$ is not a multiple of p so that it is not a be a zero-divisor of $(Z_{p^r}, \oplus, \odot)$. This shows that there is no edge between $i\overline{p}+\overline{k} \in C_k$ and $j\overline{p}+\overline{l} \in C_l$. □

Theorem 3.7. *For a prime p and an integer $r > 1$, the graph $G(Z_{p^r}, D_0)$ contains p disjoint components of $G(Z_{p^r}, D_0)$, each of which is a complete subgraph of $G(Z_{p^r}, D_0)$.*

Proof. Let $n = p^r, r > 1$, be an integer. Consider the decomposition of the vertex set of $G(Z_{p^r}, D_0)$ as given in Remark 3.1. By the Lemma 3.4, there is no edge between any vertex of C_k and any vertex of C_l, for some $k, l, 0 \le k < l \le p-1$. Hence, the graph $G(Z_{p^r}, D_0)$ contains p number of components, and each of which is a complete subgraph of $G(Z_{p^r}, D_0)$. □

Example 3.8. *The graph $G(Z_9, D_0)$ and its disjoint components are given in Figure 4 and and Figure 5 respectively.*

Example 3.9. *The graph $G(Z_{16}, D_0)$ and its disjoint components are given in Figure 6 and Figure 7 respectively.*

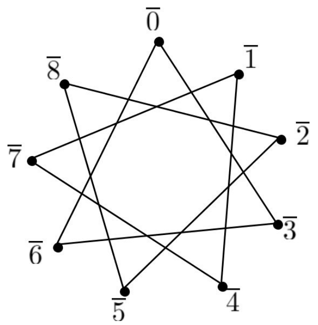

Figure 4: $G(Z_9, D_0)$

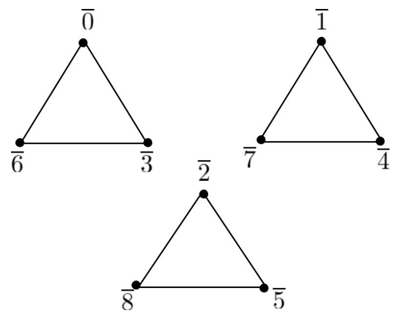

Figure 5: The disjoint components of $G(Z_9, D_0)$

4. The connected property of the zero-divisor Cayley graph $G(Z_n, D_0)$, where n is not a power of a single prime

In this section, it is shown that the graph $G(Z_n, D_0)$, where n is not a power of a single prime, is a connected graph. For this, a decomposition of vertex set V of $G(Z_n, D_0)$, similar to that given in Remark 3.1, is considered. Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < ... < p_r$ are primes, $\alpha_i \geq 1$, $1 < i \leq r$ are integers.

Remark 4.1. *Consider the following subsets of vertices $V_0, V_1, V_2, ..., V_{p_1-1}$ of the vertex set V of $G(Z_n, D_0)$.*

$V_0 = \{\overline{0}, \overline{p_1}, 2\overline{p_1}, ..., i\overline{p_1}, ..., (\frac{n-p_1}{p_1})\overline{p_1}\}$,

$V_1 = \{\overline{p_2}, \overline{p_1}+\overline{p_2}, 2\overline{p_1}+\overline{p_2}, ..., i\overline{p_1}+\overline{p_2}, ..., (\frac{n-p_1}{p_1})\overline{p_1}+\overline{p_2}\}$,

$V_2 = \{2\overline{p_2}, 2\overline{p_1}+2\overline{p_2}, ..., i\overline{p_1}+2\overline{p_2}, ..., (\frac{n-p_1}{p_1})\overline{p_1}+2\overline{p_2}\}$,

$\vdots$

$V_{p_1-1} = \{(p_1-1)\overline{p_2}, ..., i\overline{p_1}+(p_1-1)\overline{p_2}, ..., (\frac{n-p_1}{p_1})\overline{p_1}+(p_1-1)\overline{p_2}\}$.

Lemma 4.2. *For $0 \leq k \leq p_1 - 1$, each V_k contains distinct vertices and the number of vertices in each V_k is $\frac{n}{p_1}$.*

Proof. For $0 \leq k \leq p_1 - 1$, let

$$V_k = \{k\overline{p_2}, \overline{p_1}+k\overline{p_2}, ..., i\overline{p_1}+k\overline{p_2}, ..., (\frac{n-p_1}{p_1})\overline{p_1}+k\overline{p_2}\}.$$

If possible, let $i\overline{p_1}+k\overline{p_2} = j\overline{p_1}+k\overline{p_2}$, for some i, j where $0 \leq i < j \leq \frac{n-p_1}{p_1} < \frac{n}{p_1}$. Then $(j-i)\overline{p_1} = \overline{0}$. Since $j - i < \frac{n}{p_1}$, this implies that $(j-i)p_1 < n$, which leads to a contradiction.

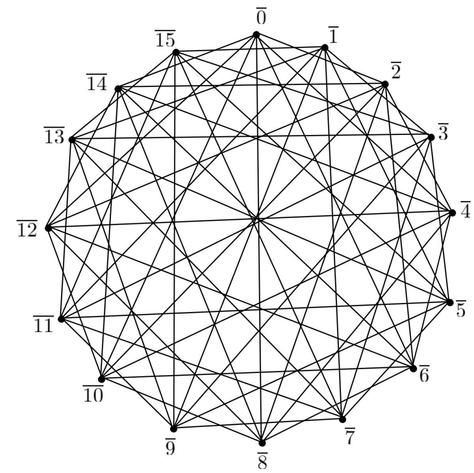

Figure 6: $G(Z_{16}, D_0)$

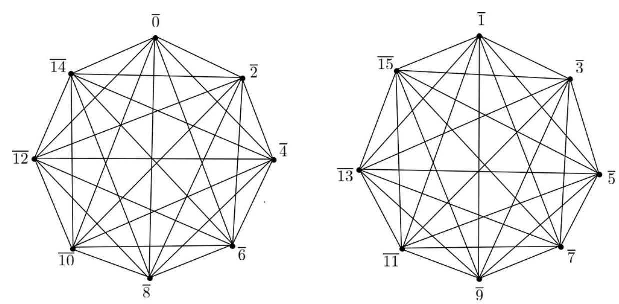

Figure 7: The components of $G(Z_{16}, D_0)$

So, our assumption that $i\overline{p_1}+k\overline{p_2} = j\overline{p_1}+k\overline{p_2}$ is wrong and $i\overline{p_1}+k\overline{p_2}$ and $j\overline{p_1}+k\overline{p_2}$ are distinct. That is, each V_k contains $\frac{n-p_1}{p_1}+1 = \frac{n}{p_1}$ distinct vertices of $G(Z_n, D_0)$. □

Lemma 4.3. *For $0 \leq k \leq p_1 - 1$, each V_k is a complete subgraph of $G(Z_n, D_0)$.*

Proof. Let $u, v \in V_k$. Then $u = i\overline{p_1}+k\overline{p_2}$ and $v = j\overline{p_1}+k\overline{p_2}$ for some $i, j, 0 \leq i < j \leq \frac{n-p_1}{p_1}$. Then,

$$u - v = (j\overline{p_1}+k\overline{p_2}) - (i\overline{p_1}+k\overline{p_2}) = (j-i)\overline{p_1}, 0 \leq i < j \leq \frac{n-p_1}{p_1}.$$

Since $\overline{p_1}$ is a zero-divisor in the ring $(Z_n, \oplus, \odot)$, $r\overline{p_1}$ is also a zero-divisor of the ring $(Z_n, \oplus, \odot)$ and this shows that u and v are adjacent, so that V_k is complete subgraph of $G(Z_n, D_0)$. □

The following theorem establishes that, if n is not a power of a single prime then $G(Z_n, D_0)$ is connected.

Theorem 4.4. *Let $n > 1$, be an integer, which is not a power of a single prime. Then the graph $G(Z_n, D_0)$ is a connected graph.*

Proof. Let $n > 1$, be an integer, which is not a power of a single prime and let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < ... < p_r$ are primes $\alpha_i \geq 1, 1 < i \leq r$ are integers.
Case i: Let $u, v \in V_l$, for some $l, 0 \leq l \leq p_1 - 1$. Then $u =$

$i\overline{p_1} + l\overline{p_2}$ and $v = j\overline{p_1} + l\overline{p_2}$ for some $i, j, 0 \le i < j \le \frac{n-p_1}{p_1}$. By the Lemma 3.2,

$$u = [i\overline{p_1} + l\overline{p_2}] - [(i+1)\overline{p_1} + l\overline{p_2}] - ... - [j\overline{p_1} + l\overline{p_2}] = v$$

is a path joining u and v and thus the graph $G(Z_n, D_0)$ is a connected graph.

Case ii: Let $u \in V_k$ and $v \in V_l$ for some $k, l, 0 \le k < l \le p_1 - 1$. Then $u = i\overline{p_1} + k\overline{p_2}$ and $v = j\overline{p_1} + l\overline{p_2}$ for some $i, j, 0 \le i < j \le \frac{n-p_1}{p_1}$. Consider $i\overline{p_1} + l\overline{p_2} \in V_l$. (This is possible since $i < j \le \frac{n-p_1}{p_1}$). Since V_l is a complete subgraph of $G(Z_n, D_0)$, there is an edge between $j\overline{p_1} + l\overline{p_2}$ and $i\overline{p_1} + l\overline{p_2}$. Further $(i\overline{p_1} + l\overline{p_2}) - (i\overline{p_1} + k\overline{p_2}) = (l-k)\overline{p_2}$ is also a zero-divisor of the ring $(Z_n, \oplus, \odot)$. So, there is an edge between $i\overline{p_1} + l\overline{p_2}$ and $i\overline{p_1} + k\overline{p_2}$. That is,

$$u = [i\overline{p_1} + k\overline{p_2}] - [j\overline{p_1} + l\overline{p_2}] - [i\overline{p_1} + l\overline{p_2}] = v$$

is a path joining u and v and thus the graph $G(Z_n, D_0)$ is a connected graph.

□

Example 4.5. *In the graph $G(Z_{10}, D_0)$, the set D_0 of zero-divisors is $\{\overline{2}, \overline{4}, \overline{5}, \overline{6}, \overline{8}\}$. Here $10 = 5.2$, $p_1 = 2$ and $p_2 = 5$. Now the vertex set is the union of V_0 and V_1, where $V_0 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$ and $V_1 = \{\overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{1}\}$. Consider the two vertices $\overline{4}$ and $\overline{7}$. The path $\overline{4} - \overline{9} - \overline{7}$ connects $\overline{4}$ and $\overline{7}$. Similarly the vertices $\overline{1}, \overline{5} \in V_1$ are connected by the edge $(\overline{1}, \overline{5})$. These paths are shown in Figure 8, by bold face edges.*

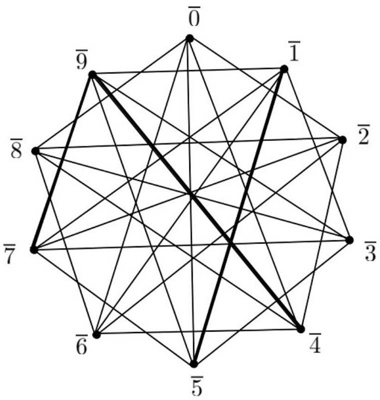

Figure 8: $G(Z_{10}, D_0)$

Acknowledgment

The authors express their thanks to Prof. L.Nagamuni Reddy for his valuable suggestions during the preparation of this paper.

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