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Results on uniqueness of meromorphic functions of differential polynomial

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Abstract

In this paper, we study the problem concerning meromorphic functions sharing a small function with weight $l \geq 0$ and present one theorem which extends a results due to Zhang and Lü [19], S.S. Bhoosnurmath and Kabbur [5], Banerjee and Majumder [3], K. S. Charak and Banarsi Lal [7].

Keywords

Uniqueness, Meromorphic function, Sharing values, Differential Polynomial, Weighted sharing.

AMS Subject Classification

Primary 30D35.

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1. Introduction and main results

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [9]. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities.

We denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \ldots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

For two positive integers n, p we define $\mu_p = min\{n, p\}$ and $\mu_p^* = p+1-\mu_p$.
Then it is clear that

$$N_p\left(r, \frac{1}{f^n}\right) \leq \mu_p N_{\mu_p^*}\left(r, \frac{1}{f}\right). \tag{1.1}$$

For notational purposes, let f and g share 1 IM. Let z_0 be a 1-point of f of order p, a 1-point of g of order q. We denote by $N_{11}\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of f and g where $p=q=1$. By $N_E^{(2}\left(r, \frac{1}{f-1}\right)$ we denote the counting function of those 1-points of f and g where $p=q\geq 2$. Also, $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ denotes the counting function of those 1-points of both f and g where $p>q$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f,g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.

We write "f,g share (a,k) " to mean that "f,g share the value a with weight k". Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0\leq p<k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

For any constant a, we define

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)},$$

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)}.$$

Clearly,

$$0\leq\delta(a,f)\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)...\leq\delta_1(a,f)=\Theta(a,f).$$

Definition. Let $n_{0j},n_{1j},...,n_{kj}$ be nonnegative integers.

The expression $M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d_{M_j}=d(M_j)=\sum_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $H[f]=\sum_{j=1}^{t}b_jM_j[f]$ is called a differential polynomial generated by f of degree

$$\overline{d}(H)=max\left\{d(M_j):1\leq j\leq t\right\}$$

and weight

$$\Gamma_H=max\left\{\Gamma_{M_j}:1\leq j\leq t\right\},$$

where $T(r,b_j)=S(r,f)$ for $j=1,2,..,t$.

The numbers $\underline{d}(H)=min\left\{d(M_j):1\leq j\leq t\right\}$ and k (the highest erder of the derivative of f in $H[f]$) are called respectively the lower degree and order of $H[f]$.

$H[f]$ is said to be homogeneous if $\overline{d}(H)=\underline{d}(H)$

$H[f]$ is called a linear differential polynomial generated by f if $\overline{d}(H)=1$. Otherwise $H[f]$ is called a non-linear differential polynomial.

We denote by $Q=max\left\{\Gamma_{M_j}-d(M_j):1\leq j\leq t\right\}=max\left\{n_{1j}+2n_{2j}+...+kn_{kj}:1\leq j\leq t\right\}$.

In 2008, Zhang and Lü ([19]) obtained the following result.

Theorem A. Let k,n be the positive integers, f be a non-constant meromorphic function, and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. If f^n and $f^{(k)}$ share a IM and

$$(2k+6)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f)>2k+12-n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k+3)\Theta(\infty,f)+2\Theta(0,f)+\delta_{2+k}(0,f)>k+6-n,$$

then $f^n=f^{(k)}$.

In the same paper, T. Zhang and W. Lü asked the following question:

Question 1. What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function $a(\not\equiv 0,\infty)$ satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$?

S.S.Bhoosnurmath and Kabbur ([5]) proved:

Theorem B. Let f be a non-constant meromorphic function and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. Let $P[f]$ be a non-constant differential polynomial in f. If f and $P[f]$ share a IM and

$$(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)>2Q+2\underline{d}(P)+\overline{d}(P)+7,$$

or if f and $P[f]$ share a CM and

$$3\Theta(\infty,f)+(\underline{d}(P)+1)\delta(0,f)>4,$$

then $f\equiv P[f]$.

Banerjee and Majumder ([3]) considered the weighted sharing of f^n and $(f^m)^{(k)}$ and proved the following result:

Theorem C. Let f be a non-constant meromorphic function, $k,n,m\in N$ and l be a non negative integer. Suppose $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$ such that f^n and $(f^m)^{(k)}$ share (a,l). If $l\geq 2$ and

$$(k+3)\Theta(\infty,f)+(k+4)\Theta(0,f)>2k+7-n,$$

or $l=1$ and

$$(k+\frac{7}{2})\Theta(\infty,f)+(k+\frac{9}{2})\Theta(0,f)>2k+8-n,$$

or $l=0$ and

$$(2k+6)\Theta(\infty,f)+(2k+7)\Theta(0,f)>4k+13-n,$$

then $f\equiv(f^m)^{(k)}$.

In 2015, Kuldeep S. Charak and Banarasi Lal ([7]) proved the following result:

Theorem D. Let f be a non-constant meromorphic function, n be a positive integer and $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. Let $P[f]$ be a non-constant differential polynomial in f. Suppose f^n and $P[f]$ share (a,l) such that any one of the following holds:
(i) when $l\geq 2$ and

$$(Q+3)\Theta(\infty,f)+2\Theta(0,f)+\overline{d}(P)\delta(0,f)>Q+5+2\overline{d}(P)-\underline{d}(P)-n,$$

(ii) when $l=1$ and

$$(Q+\frac{7}{2})\Theta(\infty,f)+\frac{5}{2}\Theta(0,f)+\overline{d}(P)\delta(0,f)>Q+6 +2\overline{d}(P)-\underline{d}(P)-n,$$

(iii) when $l=0$ and

$$(2Q+6)\Theta(\infty,f)+4\Theta(0,f)+2\overline{d}(P)\delta(0,f)>2Q +10+4\overline{d}(P)-2\underline{d}(P)-n.$$

Then $f^n\equiv P[f]$.

Through the paper we shall assume the following notations. Let

$$\mathscr{P}(\omega)=a_{m+n}\omega^{m+n}+...+a_n\omega^n+...+a_0 =a_{n+m}\prod_{i=1}^{s}(\omega-\omega_{p_i})^{p_i}$$

where $a_j(j=0,1,2,...,n+m-1), a_{n+m}\neq 0$ and $\omega_{p_i}(i=1,2,...,s)$ are distinct finite complex numbers and $2\leq s\leq n+m$ and $p_1,p_2,...,p_s,s\geq 2,n,m$ and k are all positive integers with $\sum_{i=1}^{s}p_i=n+m$. Also let $p>max_{p\neq p_i,i=1,...,r}\{p_i\}, r=s-1$, where s and r are two positive integers.

Let

$$P(\omega_1)=a_{n+m}\prod_{i=1}^{s-1}(\omega_1+\omega_p-\omega_{p_i})^{p_i} =b_q\omega_1^q+b_{q-1}\omega_1^{q-1}+...+b_0,$$

where $a_{n+m}=b_q, \omega_1=\omega-\omega_p, q=n+m-p$. Therefore, $\mathscr{P}(\omega)=\omega_1^pP(\omega_1)$.

Next we assume

$$P(\omega_1)=b_q\prod_{i=1}^{r}(\omega_1-\alpha_i)^{p_i},$$

where $\alpha_i=\omega_{p_i}-\omega_p,(i=1,2,...,r)$, be distinct zeros of $P(\omega_1)$.

In this paper, we extend the above mentioned theorems$(A-D)$ by investigating the uniqueness of meromorphic functions of the form $f_1^pP(f_1)-a$ and $H[f]-a$ and obtain the following result.

Theorem 1. Let $k(\geq 1)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers and f and $f_1=f-\omega_p$ be two nonconstant meromorphic functions and $H[f]$ be a nonconstant differential polynomial generated by f. Let $\mathscr{P}(z)=a_{m+n}z^{m+n}+...+a_nz^n+...+a_0$, $a_{m+n}\neq 0$, be a polynomial in z of degree m+n such that $\mathscr{P}(f)=f_1^pP(f_1)$. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $\mathscr{P}(f)-a$ and $H[f]-a$ share $(0,l)$. If $l\geq 2$ and

$$(Q+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}(0,f)> Q+3+\mu_2+\overline{d}(H)-p, \tag{1.2}$$

or $l=1$ and

$$\left(Q+\frac{7}{2}\right)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}\left(r,\frac{1}{f}\right) +\frac{1}{2}\Theta(w_p,f)>Q+4+\mu_2+\overline{d}(H)+\frac{m+n-3p}{2}, \tag{1.3}$$

or $l=0$ and

$$(2Q+6)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+ \overline{d}(H)\delta_{k+2}(0,f)+\overline{d}(H)\delta_{k+1}(0,f) >2Q+8+\mu_2+2\overline{d}(H)+2(m+n)-3p, \tag{1.4}$$

then $\mathscr{P}(f)\equiv H[f]$.

Following example shows that the conditions in (1.2) - (1.4) in Theorem 1 can not be removed .

Example 1. Let $f(z)=cos(\alpha z)+a-\frac{a}{\alpha^{8d}}, d\in N$; where $\alpha\neq 0,\alpha^{8d}\neq 1$ and $a\in\mathbb{C}-\{0\}$. Let $\mathscr{P}(f)=f$ and $H[f]=f^{(iv)}$ share $(1,l)(l\geq 0)$ but none of the inequalities (1.2), (1.3) and (1.4) of Theorem 1 is satisfied, and $\mathscr{P}(f)\not\equiv H[f]$.

Remark 1. Theorem 1 extends Theorem $A-D$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by ψ the function as follows:

$$\psi=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. *[11] Let f be a nonconstant meromorphic function, and p,k be positive integers. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Lemma 2.2. *[4] For any two nonconstant meromorphic functions f_1 and f_2,*

$$N_p(r,f_1f_2)\leq N_p(r,f_1)+N_p(r,f_2).$$

Lemma 2.3. *[5] Let f be a nonconstant meromorphic function and $H[f]$ be a differential polynomial of f. Then*

$$m\left(r,\frac{H[f]}{f^{\overline{d}(H)}}\right)\leq(\overline{d}(H)-\underline{d}(H))m\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.2}$$

$$N\left(r,\frac{H[f]}{f^{\overline{d}(H)}}\right)\leq(\overline{d}(H)-\underline{d}(H))N\left(r,\frac{1}{f}\right) +Q\left[\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})\right]+S(r,f), \tag{2.3}$$

$$N\left(r,\frac{1}{H[f]}\right)\le Q\overline{N}(r,f)+(\overline{d}(H)-\underline{d}(H))m\left(r,\frac{1}{f}\right)$$
$$+N\left(r,\frac{1}{f^{\overline{d}(H)}}\right)+S(r,f), \tag{2.4}$$

where $Q=max_{1\le i\le m}\{n_{i0}+n_{i1}+2n_{i2}+...+kn_{ik}\}$.

Lemma 2.4. *[13] Let* ψ *be defined as in* (2.1). *If* F *and* G *share 1 IM and* $\psi\not\equiv 0$, *then*

$$N_{11}\left(r,\frac{1}{F-1}\right)\le N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.5. *[2] Let* F *and* G *share* $(1,l)$ *and* $\overline{N}(r,F)=\overline{N}(r,G)$ *and* $\psi\not\equiv 0$, *then*

$$N(r,\psi)\le\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)$$
$$+\overline{N}_0\left(r,\frac{1}{G'}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)$$
$$+S(r,f).$$

Lemma 2.6. *[4] Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f. *Let* $F=\frac{\mathscr{P}(f)}{a}=\frac{f_1^pP(f_1)}{a}$ *and* $G=\frac{H[f]}{a}$ *such that* F *and* G *shares* $(1,\infty)$. *Then one of the following cases hold:*

$$1.T(r)\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)$$
$$+\overline{N}_L(r,F)+\overline{N}_L(r,G)+S(r),$$
$$2.F\equiv G$$
$$3.FG\equiv 1$$

where $T(r)=max\{T(r,F),T(r,G)\}$ *and* $S(r)=o(T(r))$, $r\in I$, *I is a set of infinite linear measure of* $r\in\{0,\infty\}$.

Lemma 2.7. *For the differential polynomial* $H[f]$,

$$N_p\left(r,\frac{1}{H[f]}\right)\le\overline{d}(H)N_{p+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)+S(r,f).$$

Proof. *Clearly for any non-constant meromorphic function* f, $N_p(r,f)\le N_q(r,f)$ *if* $p\le q$ *and* $b_1=b_2=...=b_t=1$. *Now by using the above fact and Lemma* 2.1, *Lemma* 2.2, *we get*

$$N_p\left(r,\frac{1}{H[f]}\right)\le\sum_{j=1}^{t}N_p\left(r,\frac{1}{M_j[f]}\right)+S(r,f)$$
$$=N_p\left(r,\frac{1}{M_1[f]}\right)+N_p\left(r,\frac{1}{M_2[f]}\right)+...$$
$$+N_p\left(r,\frac{1}{M_t[f]}\right)+S(r,f)$$
$$=N_p\left(r,\frac{1}{(f)^{n_{01}}(f^{(1)})^{n_{11}}...(f^{(k)})^{n_{k1}}}\right)$$
$$+N_p\left(r,\frac{1}{(f)^{n_{02}}(f^{(1)})^{n_{12}}...(f^{(k)})^{n_{k2}}}\right)+...$$
$$+N_p\left(r,\frac{1}{(f)^{n_{0t}}(f^{(1)})^{n_{1t}}...(f^{(k)})^{n_{kt}}}\right)+S(r,f)$$
$$=N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{(i)})^{n_{i1}}}\right)+N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{(i)})^{n_{i2}}}\right)$$
$$+...+N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{(i)})^{n_{it}}}\right)+S(r,f)$$
$$=\sum_{i=0}^{k}n_{i1}N_p\left(r,\frac{1}{f^{(i)}}\right)+\sum_{i=0}^{k}n_{i2}N_p\left(r,\frac{1}{f^{(i)}}\right)$$
$$+...+\sum_{i=0}^{k}n_{it}N_p\left(r,\frac{1}{f^{(i)}}\right)+S(r,f)$$
$$=\sum_{i=0}^{k}\left[(n_{i1}+n_{i2}+n_{i3}+...+n_{it})N_p\left(r,\frac{1}{f^{(i)}}\right)\right]+S(r,f)$$
$$\le\sum_{i=0}^{k}\left[(n_{i1}+n_{i2}+n_{i3}+...+n_{it})\left\{N_{p+i}\left(r,\frac{1}{f}\right)+i\overline{N}(r,f)\right\}\right]$$
$$+S(r,f)$$
$$\le max_{1\le j\le t}\left\{\sum_{i=0}^{k}n_{ij}N_{p+k}\left(r,\frac{1}{f}\right)\right\}$$
$$+max_{1\le j\le t}\left\{\sum_{i=0}^{k}(n_{i1}+n_{i2}+n_{i3}+...+n_{it})\,i\overline{N}(r,f)\right\}$$
$$+S(r,f)$$
$$\le\overline{d}(H)N_{p+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)+S(r,f).$$

Hence the proof.

Lemma 2.8. *Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f. *Let us define* $F=\frac{\mathscr{P}(f)}{a}=\frac{f_1^pP(f_1)}{a}$ *and* $G=\frac{H[f]}{a}$. *Then* $FG\not\equiv 1$.

Proof. *On contrary suppose* $FG\equiv 1$, *i.e.,*

$$f_1^pP(f_1)H[f]=a^2.$$

From above it is clear that the function f *can't have any zero and poles. Therefore* $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)=\overline{N}(r,f)$. *So by the First Fundamental Theorem and Lemma* 2.3, *we have*

$$(m+n+\overline{d}(H))T(r,f)=T\left(r,\frac{a^2}{f_1^pP(f_1)f^{\overline{d}(H)}}\right)+S(r,f)$$
$$\le T\left(r,\frac{H[f]}{f^{\overline{d}(H)}}\right)+S(r,f)$$
$$\le m\left(r,\frac{H[f]}{f^{\overline{d}(H)}}\right)+N\left(r,\frac{H[f]}{f^{\overline{d}(H)}}\right)$$
$$+S(r,f)$$
$$\le(\overline{d}(H)-\underline{d}(H))T(r,f)$$
$$+Q\left[\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})\right]+S(r,f)$$
$$\le(\overline{d}(H)-\underline{d}(H))T(r,f)+S(r,f).$$

Thus

$$(m+n+\underline{d}(H)))T(r,f)\le S(r,f),$$

which is a contradiction.

3. Proof of the Theorem

Proof of Theorem 1.

Let $F=\dfrac{\mathscr{P}(f)}{a}=\dfrac{f_1^pP(f_1)}{a}$ and $G=\dfrac{H[f]}{a}$.
Since $\mathscr{P}(f)-a$ and $H[f]-a$ share $(0,l)$, F, G share $(1,l)$ except the zeros and poles of $a(z)$. Also note that $\overline{N}(r,F)=\overline{N}(r,f)+S(r,f)$ and $\overline{N}(r,G)=\overline{N}(r,f)+S(r,f)$. Let ψ be defined as in (2.1).
We consider the following cases.

Case 1. Suppose $\psi\not\equiv 0$. By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}&T(r,F)+T(r,G)\\&\le \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\&+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-\overline{N}_0\left(r,\frac{1}{F'}\right)\\&-\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G),\end{aligned}\tag{3.1}$$

where $\overline{N}_0\left(r,\frac{1}{F'}\right)$ denotes the reduced counting function of the zeros of F' which are not the zeros of $F(F-1)$.
Since F and G share 1 IM, it is easy to verify that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)&=N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&=\overline{N}\left(r,\frac{1}{G-1}\right).\end{aligned}\tag{3.2}$$

Using Lemmas 2.4, 2.5, and (3.1), (3.2), we get

$$\begin{aligned}T(r,F)+T(r,G)&\le 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\&+N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&+S(r,F)+S(r,G).\end{aligned}\tag{3.3}$$

Subcase 1.1. Let $l\ge 2$.
Obviously,

$$\begin{aligned}&N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+3\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\le N\left(r,\frac{1}{G-1}\right)+S(r,F)\\&\le T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{3.4}$$

Using (3.3) and (3.4), we get

$$T(r,F)\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,F)+S(r,F).\tag{3.5}$$

Using Lemmas 2.1, 2.7, and (1.1), (3.5), we get

$$\begin{aligned}(n+m)T(r,f)&\le 3\overline{N}(r,f)+N_2\left(r,\frac{1}{f_1^pP(f_1)}\right)\\&+N_2\left(r,\frac{1}{H[f]}\right)+S(r,f)\\&\le 3\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)\\&+(n+m-p)T(r,f)+\overline{d}(H)N_{k+2}\left(r,\frac{1}{f}\right)\\&+Q\overline{N}(r,f)+S(r,f)\\&\le (Q+3)\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)\\&+(n+m-p)T(r,f)+\overline{d}(H)N_{k+2}\left(r,\frac{1}{f}\right)\\&+S(r,f).\end{aligned}$$

So, $$\begin{aligned}&(Q+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}(0,f)\\&\le Q+3+\mu_2+\overline{d}(H)-p,\end{aligned}$$

which violates (1.2).

Subcase 1.2. Let $l=1$.
It is easy to verify that

$$\begin{aligned}&N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\le N\left(r,\frac{1}{G-1}\right)+S(r,F)\\&\le T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{3.6}$$

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right)&\le\frac{1}{2}N\left(r,\frac{F}{F'}\right)\\&\le\frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,F)\\&\le\frac{1}{2}\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right)+S(r,F).\end{aligned}\tag{3.7}$$

Using (3.3), (3.6) and (3.7), we get

$$\begin{aligned}T(r,F)&\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{7}{2}\overline{N}(r,F)\\&+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F).\end{aligned}\tag{3.8}$$

Using Lemmas 2.1, 2.7 and (1.1), (3.8), we get

$$\begin{aligned}&(n+m)T(r,f)\\&\leq\left(Q+\frac{7}{2}\right)\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)\\&+\overline{d}(H)\delta_{k+2}\left(r,\frac{1}{f}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{f-w_p}\right)\\&+\frac{3}{2}(n+m-p)T(r,f)+S(r,f).\end{aligned}$$

$$\begin{aligned}\text{So, }&\left(Q+\frac{7}{2}\right)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)\\&+\overline{d}(H)\delta_{k+2}\left(r,\frac{1}{f}\right)+\frac{1}{2}\Theta(w_p,f)\\&\leq Q+4+\mu_2+\overline{d}(H)+\frac{m+n-3p}{2},\end{aligned}$$

which violates (1.3).

Subcase 1.3. Let $l=0$.
It is easy to verify that

$$\begin{aligned}N_{11}&\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+2\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\leq N\left(r,\frac{1}{G-1}\right)+S(r,F)\\&\leq T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{3.9}$$

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right)&\leq N\left(r,\frac{1}{F-1}\right)-\overline{N}\left(r,\frac{1}{F-1}\right)\\&\leq N\left(r,\frac{F}{F'}\right)\leq N\left(r,\frac{F'}{F}\right)+S(r,F)\\&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+S(r,F).\end{aligned}\tag{3.10}$$

Using (3.3), (3.9) and (3.10), we get

$$\begin{aligned}T(r,F)\leq\;&N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+6\overline{N}(r,F)\\&+2\overline{N}\left(r,\frac{1}{F}\right)+N_1\left(r,\frac{1}{G}\right)+S(r,F).\end{aligned}\tag{3.11}$$

Using Lemmas 2.1, 2.7 and (1.1), (3.11), we get

$$\begin{aligned}(n+m)T(r,f)\leq\;&N_2\left(r,\frac{1}{f_1^pP(f_1)}\right)+N_2\left(r,\frac{1}{H[f]}\right)\\&+6\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f_1^pP(f_1)}\right)\\&+N_1\left(r,\frac{1}{H[f]}\right)+S(r,f).\end{aligned}$$

So,

$$\begin{aligned}&(2Q+6)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)\\&+\overline{d}(H)\delta_{k+2}(0,f)+\overline{d}(H)\delta_{k+1}(0,f)\\&\leq 2Q+8+\mu_2+2\overline{d}(H)+2(m+n)-3p,\end{aligned}$$

which violates (1.4).

Case 2. Let $\psi\equiv 0$.
On Integration we get

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B,$$

where $A(\neq 0),B$ are complex constants.
It is clear that F and G share $(1,\infty)$. Also by construction of F and G we see that F and G share $(\infty,0)$ also.

So using Lemma 2.7, (1.1) and condition (1.2), we obtain

$$\begin{aligned}&N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_L(r,F)\\&+\overline{N}_L(r,G)+S(r)\\&\leq(Q+3)\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)+(n+m-p)T(r,f)\\&+\overline{d}(H)N_{k+2}\left(r,\frac{1}{f}\right)+S(r)\\&\leq\left\{Q+3+\mu_2+n+m-p+\overline{d}(H)\right\}T(r,f)\\&-\left\{(Q+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}(0,f)\right\}\\&T(r,f)+S(r)\\&<T(r,F)+S(r).\end{aligned}$$

Hence inequality (1) of Lemma 2.6, does not hold. Again in view of Lemma 2.8, we get $FG\not\equiv 1$. Therefore $F\equiv G$ i.e., $\mathscr{P}(f)\equiv H[f]$.

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