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Existence result for neutral fractional integrodifferential equations with nonlocal integral boundary conditions

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Abstract

In this article, we study a neutral fractional integrodifferential equation supplemented with nonlocal flux type integral boundary conditions. The existence and uniqueness results are obtained by using Banach fixed point theorem and Leray-Schauder nonlinear alternative theorem. The obtained results are illustrated by examples at the end.

Keywords

Fractional differential equations, nonlocal integral boundary conditions, fixed point theorems.

AMS Subject Classification

34A08, 34B15, 34G20.

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Contents

1. Introduction

In this paper, we investigate the existence and uniqueness of solutions to the following nonlinear neutral fractional integrodifferential equations with flux type nonlocal integral boundary conditions:

$$\begin{aligned} {}^c\mathbf{D}^q[x(t)-g(t,x(t))] &= f\left(t,x(t),\int_0^t k(t,s,x(s))ds\right), \\ & \quad t\in(0,1),\, 1<q\leqslant 2 \\ x'(0) &= \alpha\int_0^\xi x'(s)ds, \\ x(1) &= \beta\phi(x'(\eta)), \\ & \quad 0\leqslant \xi,\eta\leqslant 1,\, \xi\neq\frac{1}{\alpha}, \end{aligned} \tag{1.1}$$

where ${}^c\mathbf{D}^q$ is the Caputo fractional derivative of order $q, f: [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, k:\Omega\times\mathbb{R}\to\mathbb{R}, \phi:\mathbb{R}\to\mathbb{R}$ are continuous, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable with $\Omega=\{(t,s): 0\leqslant s<t\leqslant 1\}$. In (1.1), the first of nonlocal boundary conditions can be interpreted as the flux type nonlocal integral condition which relates the flux $x'(0)$ with the continuous distribution of flux over an interval of arbitrary length $(0,\xi)$, while the second condition states that the value of unknown function $x(1)$ is proportional to nonlinear function ϕ depending on flux $x'(\eta)$.

Boundary value problems of fractional differential equations with integral boundary conditions have various applications in several applied fields such as blood flow problem, thermoelasticity, chemical engineering, underground water flow, cellular systems, heat transmission, plasma physics, population dynamics and so forth. For a detailed description of these boundary conditions, one may refer the papers [3]-[6]. Also integrodifferential equations arise in many engineering and scientific disciplines. The recent results of fractional boundary value problems with integrodifferential equations can be found in [7]-[9] and references therein.

The paper is organized as follows: in the next section we will give some basic definitions and present a lemma to establish the expression of mild solution of system (1.1). In section 3, we will study the existence and uniqueness result for mild solutions of the system (1.1) via Leray-Schauder nonlinear alternative and Banach contraction principle. Finally, in section 4, we will present some examples to illustrate our results.

2. Preliminaries

Definition 2.1. *([1]) The fractional integral of order q for a function $f \in L^1(\mathbb{R}^+)$ is defined by*

$$I^q_{0+}f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds, \quad t>0, \quad q>0.$$

Definition 2.2. *([1]) The Caputo fractional derivative of order q for a function $f \in C^{m-1}(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$ is defined by*

$${}^C\boldsymbol{D}^q_{0+}f(t) = \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}f^m(s)ds,$$

where $m-1<q<m$, $m=[q]+1$ and $[q]$ denotes the integral part of the real number q.

Lemma 2.3. *For $f \in C([0,1],\mathbb{R})$ and $g \in C^1([0,1],\mathbb{R})$, the solution of following linear fractional differential equation*

$$\begin{aligned} {}^c\boldsymbol{D}^q[x(t)-g(t)] &= f(t), \\ & \quad t\in(0,1),\, 1<q\leqslant 2 \\ x'(0) &= \alpha\int_0^\xi x'(s)ds, \\ x(1) &= \beta\phi(x'(\eta)), \\ & \quad 0\leqslant \xi,\eta\leqslant 1,\, \xi\neq\frac{1}{\alpha}, \end{aligned} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + g(t) - g(1) \\ & -\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds \\ & +\frac{t-1}{1-\alpha\xi}\Big[\alpha(g(\xi)-g(0))-g'(0) \\ & +\alpha\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau ds\Big] \\ & +\beta\phi\Big(\frac{\alpha}{(1-\alpha\xi)}(g(\xi)-g(0)) - \frac{1}{(1-\alpha\xi)}g'(0) \\ & +g'(\eta)+\frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau ds \\ & +\int_0^\eta\frac{(\eta-s)^{q-2}}{\Gamma(q-1)}f(s)ds\Big). \end{aligned} \tag{2.2}$$

Proof. It is well known from [2], that the solution of fractional differential equation (2.1) can be written as

$$x(t) = c_1 + c_2 t + g(t) - g(0) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds, \tag{2.3}$$

for some constants $c_1, c_2 \in \mathbb{R}$. On applying the given boundary conditions in (2.1), we find that

$$\begin{aligned} c_2 = & \frac{1}{(1-\alpha\xi)}\Big(\alpha(g(\xi)-g(0))-g'(0) \\ & +\alpha\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau ds\Big) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} c_1 = & \beta\phi\Big(c_2+g'(\eta)+\int_0^\eta\frac{(\eta-s)^{q-2}}{\Gamma(q-1)}f(s)ds\Big) \\ & +g(0)-g(1)-c_2-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds, \end{aligned}$$

where c_2 is given by (2.4). Substituting the values of c_1, c_2 in (2.3), we get (2.2). □

3. Existence and Uniqueness Results

Let $\mathscr{C} := C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm $\|x\| = \sup_{t\in[0,1]}|x(t)|$, and $\mathscr{P} := C^1([0,1],\mathbb{R})$ be the Banach space of all continuously differentiable functions from $[0,1]$ to $\mathbb{R}$ equipped with norm $\|x\|_{C^1} = \sup_{t\in[0,1]}\{|x(t)|,|x'(t)|\}$.

In view of Lemma 2.3, we define an operator $F:\mathscr{P}\to\mathscr{P}$ as

$$\begin{aligned} (Fx)(t) = & \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s),Kx(s))ds \\ & +g(t,x(t))-g(1,x(1)) \\ & -\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s),Kx(s))ds \\ & +\frac{t-1}{1-\alpha\xi}\Big[\alpha(g(\xi,x(\xi))-g(0,x(0))) \\ & -g'(0,x(0)) \\ & +\alpha\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,x(\tau),Kx(\tau))d\tau ds\Big] \\ & +\beta\phi\Big(\frac{\alpha}{(1-\alpha\xi)}(g(\xi,x(\xi))-g(0,x(0))) \\ & -\frac{1}{(1-\alpha\xi)}g'(0,x(0))+g'(\eta,x(\eta)) \\ & +\int_0^\eta\frac{(\eta-s)^{q-2}}{\Gamma(q-1)}f(s,x(s),Kx(s))ds \\ & +\frac{\alpha}{1-\alpha\xi}\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \\ & f(\tau,x(\tau),Kx(\tau))d\tau ds\Big), \end{aligned} \tag{3.1}$$

where $Kx(t) = \int_0^t k(t,s,x(s))ds$, observe that the problem (1.1) has solutions if the operator F has fixed points.

Our first existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.1. *([10], Leray-Schauder nonlinear alternative) Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is continuous and compact map, then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there exists a $u \in \partial U$ (boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Theorem 3.2. *Let the functions $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \phi : \mathbb{R} \to \mathbb{R}$ be continuous and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Assume that the following hypotheses hold:*

(A1) *There exists continuous nondecreasing functions $\psi_1, \psi_2, \psi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ and $p_1, p_2, p_3 \in C([0,1], \mathbb{R}^+)$ such that ,*

(i) $|f(t,x,y)| \leqslant p_1(t)\psi_1(\|x\|), \quad \forall (t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R},$

(ii) $|g(t,x)| \leqslant p_2(t)\psi_2(\|x\|), |g'(t,x)| \leqslant p_3(t)\psi_3(\|x\|), \forall (t,x) \in [0,1] \times \mathbb{R}.$

(A2) $|\phi(v)| \leqslant |v|, \quad \forall v \in \mathbb{R}.$

(A3) *There exists a constant $M > 0$ such that $\frac{M}{\|p\|\psi(M)\Lambda_1} > 1$, where $\|p\| = max\{\|p_i\| : i = 1,2,3\}$,*

$$\begin{aligned}\Lambda_1 &= \frac{1}{\Gamma(q+1)}\left[2 + |\beta| q \eta^{q-1} + (1+|\beta|)\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right] \\ &\quad + 2 + |\beta| + \frac{1+|\beta|}{|1-\alpha\xi|} + 2(1+|\beta|)\frac{|\alpha|}{|1-\alpha\xi|}\end{aligned}$$

and $\psi(r) = \max\{\psi_1(r), \psi_2(r), \psi_3(r)\}$.

Then the boundary value problem (1.1) has atleast one solution in $[0,1]$.

Proof. It is easy to see that the operator $F : \mathscr{P} \to \mathscr{P}$ defined by (3.1) is continuous. Next, we show that F maps bounded set into bounded set in $\mathscr{P}$. For a positive number r, let $B_r = \{x \in \mathscr{P} : \|x\|_{C^1} \leqslant r\}$ be a bounded set in $\mathscr{P}$. Then for each $x \in B_r$,

$$\begin{aligned}|(Fx)(t)| &\leqslant \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s))|ds \\ &\quad + |g(t,x(t))| + |g(1,x(1))| \\ &\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s))|ds \\ &\quad + \frac{|t-1|}{|1-\alpha\xi|}\Big[|\alpha|(|g(\xi,x(\xi))| + |g(0,x(0))|) \\ &\quad + |g'(0,x(0))| \\ &\quad + |\alpha|\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,x(\tau),Kx(\tau))|d\tau ds\Big] \\ &\quad + |\beta|\Big(\frac{|\alpha|}{|1-\alpha\xi|}(|g(\xi,x(\xi))| + |g(0,x(0))|) \\ &\quad + \frac{1}{|1-\alpha\xi|}|g'(0,x(0))| + |g'(\eta,x(\eta))| \\ &\quad + \int_0^\eta \frac{(\eta-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s),Kx(s))|ds \\ &\quad + \frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \\ &\quad |f(\tau,x(\tau),Kx(\tau))|d\tau ds\Big) \\ &\leqslant 2\|p_1\|\psi_1(r)\frac{1}{\Gamma(q+1)} + 2\|p_2\|\psi_2(r) \\ &\quad + \frac{1}{|1-\alpha\xi|}\Big(2|\alpha|\|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) \\ &\quad + |\alpha|\|p_1\|\psi_1(r)\frac{\xi^q}{\Gamma(q+1)}\Big) \\ &\quad + |\beta|\Big(2\frac{|\alpha|}{|1-\alpha\xi|}\|p_2\|\psi_2(r) \\ &\quad + \frac{1}{|1-\alpha\xi|}\|p_3\|\psi_3(r) \\ &\quad + \|p_3\|\psi_3(r) + \|p_1\|\psi_1(r)\frac{\eta^{q-1}}{\Gamma(q)} \\ &\quad + \|p_1\|\psi_1(r)\frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\Big).\end{aligned}$$

Choosing $\psi(r) = \max\{\psi_1(r), \psi_2(r), \psi_3(r)\}$, we have

$$|(Fx)(t)| \leqslant \|p\|\psi(r)\Lambda_1. \tag{3.2}$$

On differentiating equation (3.1) with respect to t, we get

$$\begin{aligned}(Fx)'(t) &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}f(s,x(s),Kx(s))ds + g'(t,x(t)) \\ &\quad + \frac{1}{1-\alpha\xi}\Big[\alpha(g(\xi,x(\xi)) - g(0,x(0))) \\ &\quad - g'(0,x(0)) + \alpha\int_0^\xi\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \\ &\quad f(\tau,x(\tau),Kx(\tau))d\tau ds\Big],\end{aligned} \tag{3.3}$$

by (3.3), for each $x \in B_r$ we have

$$\begin{aligned} |(Fx)'(t)| &\leqslant \|p_1\|\psi_1(r)\frac{1}{\Gamma(q)} + \|p_3\|\psi_3(r) \\ &\quad + \frac{1}{|1-\alpha\xi|}\Big(2|\alpha|\|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) \\ &\quad + |\alpha|\|p_1\|\psi_1(r)\frac{\xi^q}{\Gamma(q+1)}\Big) \\ &\leqslant \|p\|\psi(r)\Big(1 + \frac{1}{\Gamma(q)} + \frac{1+2|\alpha|}{|1-\alpha\xi|} \\ &\quad + \frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\Big), \end{aligned}$$

by denoting $\Lambda_2 = \left(1 + \frac{1}{\Gamma(q)} + \frac{1+2|\alpha|}{|1-\alpha\xi|} + \frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\right)$, we have

$$|(Fx)'(t)| \leqslant \|p\|\psi(r)\Lambda_2, \tag{3.4}$$

observe that $\Lambda_2 \leqslant \Lambda_1$. Thus by (3.2) and (3.4), we have

$$\|Fx\|_{C^1} \leqslant \|p\|\psi(r)\Lambda_1. \tag{3.5}$$

Next, we will show that F maps bounded sets into equicontinuous sets of $\mathscr{P}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$. Then we have

$$\begin{aligned} |(Fx)(t_2) - (Fx)(t_1)| &\leqslant \Big| \int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s,x(s),Kx(s))ds \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f(s,x(s),Kx(s))ds \Big| \\ &\quad + |g(t_2,x(t_2)) - g(t_1,x(t_1))| \\ &\quad + \frac{|t_2-t_1|}{|1-\alpha\xi|}\Big|\alpha(g(\xi,x(\xi)) - g(0,x(0))) - g'(0,x(0)) \\ &\quad + \alpha\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,x(\tau),Kx(\tau))d\tau ds\Big| \\ &\leqslant \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}] |f(s,x(s),Kx(s))|ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1} |f(s,x(s),Kx(s))|ds \\ &\quad + |g(t_2,x(t_2)) - g(t_1,x(t_1))| \\ &\quad + \frac{|t_2-t_1|}{|1-\alpha\xi|}\Big|\alpha(g(\xi,x(\xi)) - g(0,x(0))) - g'(0,x(0)) \\ &\quad + \alpha\int_0^{\xi}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,x(\tau),Kx(\tau))d\tau ds\Big|, \end{aligned} \tag{3.6}$$

as $t_2 \to t_1$, the right hand side of above inequality tends to zero independently of $x \in B_r$, similarly for the derivative term we have

$$\begin{aligned} |(Fx)'(t_2) - (Fx)'(t_1)| &\leqslant \Big| \int_0^{t_2} \frac{(t_2-s)^{q-2}}{\Gamma(q-1)} f(s,x(s),Kx(s))ds \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s),Kx(s))ds \Big| \\ &\quad + |g'(t_2,x(t_2) - g'(t_1,x(t_1)))| \\ &\leqslant \frac{1}{\Gamma(q-1)}\int_0^{t_1}[(t_2-s)^{q-2} - (t_1-s)^{q-2}]|f(s,x(s),Kx(s))|ds \\ &\quad + \frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}(t_2-s)^{q-2} |f(s,x(s),Kx(s))|ds + |g'(t_2,x(t_2) - g'(t_1,x(t_1)))| \end{aligned} \tag{3.7}$$

as $t_2 \to t_1$, the right hand side of above inequality tends to zero independently of $x \in B_r$, therefore by (3.6), (3.7) and Arzela-Ascoli's theorem it follows that $F : \mathscr{P} \to \mathscr{P}$ is completely continuous.

Now let $x = \lambda Fx$ where $\lambda \in (0,1)$, then $\|x\|_{C^1} < \|Fx\|_{C^1}$. Using (3.5), we have $\|x\|_{C_1} \leqslant \|p\|\psi(\|x\|)\Lambda_1$. Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Lambda_1} \leqslant 1.$$

In view of $(A3)$, there exists positive constant M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathscr{P} : \|x\|_{C^1} < M\}.$$

Note that the operator $F : \overline{U} \to \mathscr{P}$ is continuous and compact. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Consequently, by Theorem 3.1, we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). ☐

Now we will prove existence and uniqueness result based on Banach contraction principle.

Theorem 3.3. *Assume that* $(A2)$ *and following hypotheses hold:*

(B1) *The function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists positive constant* L_1 *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leqslant L_1\{|x_1-x_2| + |y_1-y_2|\}, \quad t \in [0,1], x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

(B2) *The function* $g : [0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuously differentiable and there exist positive constants* L_2, L_3 *such that*

$$|g(t,x_1)-g(t,x_2)|\leqslant L_2|x_1-x_2|,$$
$$|g'(t,x_1)-g'(t,x_2)|\leqslant L_3|x_1-x_2|,\quad t\in[0,1], x_1,x_2\in\mathbb{R}.$$

(B3) *The function* $k : [0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous and there exists positive constant* L_4 *such that*

$$|k(t,s,x_1)-k(t,s,x_2)|\leqslant L_4|x_1-x_2|,\quad t,s\in[0,1], x_1,x_2\in\mathbb{R}.$$

(B4) *Let* $\rho=\max\{\rho_1,\rho_2\}<1$ *where,*

$$\rho_1 = \left(2+(1+|\beta|)\frac{|\alpha|\xi^q}{|1-\alpha\xi|}+|\beta|q\eta^{q-1}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)} +\left(2+2(1+|\beta|)\frac{|\alpha|}{|1-\alpha\xi|}\right)L_2 +\left(|\beta|+(1+|\beta|)\frac{1}{|1-\alpha\xi|}\right)L_3,$$

and

$$\rho_2 = \left(q+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)} +2\frac{|\alpha|}{|1-\alpha\xi|}L_2+\left(1+\frac{1}{|1-\alpha\xi|}\right)L_3.$$

Then the boundary value problem (1.1) has a unique solution in $[0,1]$.

Proof. For $x,y\in\mathscr{P}$ and $t\in[0,1]$, we obtain

$$\begin{aligned}|(Fx)(t)-(Fy)(t)| \leqslant{}& \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s)) -f(s,y(s),Ky(s))|ds\\
&+|g(t,x(t))-g(t,y(t))| +|g(1,x(1))-g(1,y(1))|\\
&+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s)) -f(s,y(s),Ky(s))|ds\\
&+\frac{1}{|1-\alpha\xi|}\Big[|\alpha|(|g(\xi,x(\xi)) -g(\xi,y(\xi))|) +|\alpha|(|g(0,x(0))-g(0,y(0))|)\\
&+|g'(0,x(0))-g'(0,y(0))|\\
&+|\alpha|\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)} |f(\tau,x(\tau),Kx(\tau)) -f(\tau,y(\tau),Ky(\tau))|d\tau ds\Big]\\
&+|\beta|\Big(\frac{|\alpha|}{|1-\alpha\xi|}(|g(\xi,x(\xi)) -g(\xi,y(\xi))| +|g(0,x(0))-g(0,y(0))|)\\
&+\frac{1}{|1-\alpha\xi|}|g'(0,x(0))-g'(0,y(0))| +|g'(\eta,x(\eta))-g'(\eta,y(\eta))|\\
&+\int_0^\eta\frac{(\eta-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s),Kx(s)) -f(s,y(s),Ky(s))|ds\\
&+\frac{|\alpha|}{|1-\alpha\xi|}\int_0^\xi\int_0^s\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,x(\tau),Kx(\tau)) -f(\tau,y(\tau),Ky(\tau))|d\tau ds\Big)\\
\leqslant{}& \Big[\frac{2}{\Gamma(q+1)}L_1(1+L_4)+2L_2 +\frac{1}{|1-\alpha\xi|}\Big(2|\alpha|L_2+L_3 +|\alpha|L_1(1+L_4)\frac{\xi^q}{\Gamma(q+1)}\Big)\\
&+|\beta|\Big(2\frac{|\alpha|}{|1-\alpha\xi|}L_2+\frac{1}{|1-\alpha\xi|}L_3+L_3 +L_1(1+L_4)\frac{\eta^{q-1}}{\Gamma(q}+L_1(1+L_4) \frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\Big)\Big]\|x-y\|\\
\leqslant{}& \rho_1\|x-y\|,\end{aligned}\tag{3.8}$$

where,

$$\rho_1 = \left(2+(1+|\beta|)\frac{|\alpha|\xi^q}{|1-\alpha\xi|}+|\beta|q\eta^{q-1}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)} +\left(2+2(1+|\beta|)\frac{|\alpha|}{|1-\alpha\xi|}\right)L_2 +\left(|\beta|+(1+|\beta|)\frac{1}{|1-\alpha\xi|}\right)L_3. \tag{3.9}$$

Similarly for the derivative term we have,

$$\begin{aligned}|(Fx)'(t)-(Fy)'(t)| \leqslant{}& \int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s),Kx(s)) -f(s,y(s),Ky(s))|ds\\
&+|g'(t,x(t))-g'(t,y(t))|\\
&+\frac{1}{|1-\alpha\xi|}\Big[|\alpha||g(\xi,x(\xi)) -g(\xi,y(\xi))| +|\alpha||g(0,x(0))-g(0,y(0))| +|g'(0,x(0))-g'(0,y(0))|\end{aligned}$$

$$
\begin{aligned}
&+|\alpha|\int_0^{\xi}\int_0^{s}\frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\\
&|f(\tau,x(\tau),Kx(\tau))\\
&-f(\tau,y(\tau),Ky(\tau))|d\tau ds\Big]\\
\leqslant\;& \Big[\frac{1}{\Gamma(q)}L_1(1+L_4)+L_3\\
&+\frac{1}{|1-\alpha\xi|}\Big(2|\alpha|L_2+L_3\\
&+|\alpha|L_1(1+L_4)\frac{\xi^q}{\Gamma(q+1)}\Big)\Big]\|x-y\|\\
\leqslant\;& \rho_2\|x-y\|, \qquad (3.10)
\end{aligned}
$$

where,

$$
\begin{aligned}
\rho_2 \;=\;& \left(q+\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)}\\
&+2\frac{|\alpha|}{|1-\alpha\xi|}L_2+\left(1+\frac{1}{|1-\alpha\xi|}\right)L_3. \qquad (3.11)
\end{aligned}
$$

By (3.8) and (3.10) we have

$$\|Fx-Fy\|_{C^1}<\rho\|x-y\|_{C^1}, \quad x,y\in\mathscr{P},$$

since $\rho<1$ by assumption $(B4)$, consequently F is a contraction. Hence by Banach contraction principle, the problem (1.1) has a unique solution. □

4. Examples

Example(1): Consider the following fractional boundary value problem

$$
\begin{cases}
{}^c\mathbf{D}^{\frac{3}{2}}[x(t)-\frac{e^{-t}}{35(1+11e^t)}x(t)] = & \frac{1}{(t+4)^2}|x(t)| \\
& +\frac{1}{16}\int_0^t\frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds, \qquad (4.1)\\
x'(0)=\frac{1}{2}\int_0^{\frac{1}{3}}x'(s)ds,\;\; x(1)= & \frac{1}{3}\phi(x'(\frac{3}{4})).
\end{cases}
$$

Here, $t\in(0,1)$, $q=\frac{3}{2}$, $\alpha=\frac{1}{2}$, $\xi=\frac{1}{3}$, $\beta=\frac{1}{3}$, $\eta=\frac{3}{4}$ and

$$
\phi(v)=\begin{cases}\sqrt{|v|}, & |v|\geqslant 1;\\ v^2, & |v|<1.\end{cases} \qquad (4.2)
$$

With the given values, we find that $\Lambda_1\approx 9.45$. Clearly,

$$f(t,x,Kx)=\frac{1}{(t+4)^2}|x(t)|+\frac{1}{16}Kx(t),$$

where $Kx(t)=\int_0^t\frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds$ and $g(t,x)=\frac{e^{-t}}{35(1+11e^t)}x(t)$.

$$
\begin{aligned}
|f(t,x,Kx)| \;\leqslant\;& \frac{1}{(t+4)^2}|x(t)|+\frac{1}{16}\int_0^t\frac{1}{9}e^{-s}ds\\
\leqslant\;& \frac{1}{16}(\|x\|_{C^1}+\frac{1}{9}),
\end{aligned}
$$

$$|g(t,x)|\leqslant\frac{e^{-t}}{35(1+11e^t)}\|x\|_{C^1},$$

$$g'(t,x) \;=\; \frac{1}{35}\left[\frac{e^{-t}(x'-x)-22x+11x'}{(1+11e^t)^2}\right],$$

therefore, we have

$$|g'(t,x)|\leqslant\frac{1}{(1+11e^t)^2}\|x\|_{C^1}.$$

Hence, $p_1(t)=\frac{1}{16}$, $p_2(t)=\frac{e^{-t}}{35(1+11e^t)}$, $p_3(t)=\frac{1}{(1+11e^t)^2}$, $\psi_1(r)=r+\frac{1}{9}$, $\psi_2(r)=r$, $\psi_3(r)=r$, and $\|p\|=\frac{1}{16}$, $\psi(r)=r+\frac{1}{9}$. Now using the condition in $(A3)$ that is $\frac{M}{\|p\|\psi(M)\Lambda_1}>1$, we find that $M>0.1603$. Hence, by Theorem 3.2, the boundary value problem (4.1) has atleast one solution on $[0,1]$.

Example(2): Consider the following fractional boundary value problem

$$
\begin{cases}
{}^c\mathbf{D}^{\frac{3}{2}}[x(t)-\frac{1}{9}e^{-t}x(t)] = & \frac{1}{(t+6)^2}\frac{|x(t)|}{1+|x(t)|}\\
& +\frac{1}{36}\int_0^t e^{\frac{-1}{5}x(s)}ds, \qquad (4.3)\\
x'(0)=\frac{1}{2}\int_0^{\frac{1}{3}}x'(s)ds, & x(1)=\frac{1}{3}\phi(x'(\frac{3}{4})).
\end{cases}
$$

Here, $t\in(0,1)$, $q=\frac{3}{2}$, $\alpha=\frac{1}{2}$, $\xi=\frac{1}{3}$, $\beta=\frac{1}{3}$, $\eta=\frac{3}{4}$ and ϕ is given as (4.2). Clearly, $f(t,x,Kx)=\frac{1}{(t+6)^2}\frac{|x(t)|}{1+|x(t)|}+\frac{1}{36}Kx(t)$, where $Kx(t)=\int_0^t e^{\frac{-1}{5}x(s)}ds$ and $g(t,x)=\frac{1}{9}e^{-t}x(t)$.

$$
\begin{aligned}
|k(t,s,x(s))-k(t,s,y(s))| \;=\;& |e^{\frac{-1}{5}x(s)}-e^{\frac{-1}{5}y(s)}|\\
\leqslant\;& \frac{1}{5}\|x-y\|_{C^1},
\end{aligned}
$$

$$
\begin{aligned}
|f(t,x,Kx)-f(t,y,Ky)| \;\leqslant\;& \frac{1}{(t+6)^2}\frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)}\\
&+\frac{1}{36}\|Kx-Ky\|\\
\leqslant\;& \frac{1}{36}\Big[\|x-y\|_{C^1}+\|Kx-Ky\|_{C^1}\Big],
\end{aligned}
$$

$$|g(t,x)-g(t,y)| \;\leqslant\; \frac{1}{9}\|x-y\|_{C^1},$$

$$
\begin{aligned}
g'(t,x) \;=\;& \frac{1}{9}e^{-t}(x'-x),\\
|g'(t,x)-g'(t,y)| \;\leqslant\;& \frac{1}{9}\Big[\|x'-y'\|+\|x-y\|\Big]\\
\leqslant\;& \frac{2}{9}\|x-y\|_{C^1}.
\end{aligned}
$$

Therefore $L_1=\frac{1}{36}$, $L_2=\frac{1}{9}$, $L_3=\frac{2}{9}$, $L_4=\frac{1}{5}$, and we get $\rho_1=0.893<1$ and $\rho_2=0.662<1$. Hence by Theorem 3.3, the boundary value problem (4.3) has a unique solution on $[0,1]$.

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