Malaya Journal of Matematik, Vol. 6, No. 1, 41-48, 2018

https://doi.org/10.26637/MJM0601/0006

Oscillation theorems for certain delay difference inequalities

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Abstract

Our aim in this paper is to give some new results on the oscillatory behavior of all solutions of the delay difference inequalities

$$x(n)\left\{L_m x(n)+a(n)x(n)+(q(n)+p^j(n))x[n-m\tau]\right\}\leq 0 \quad \text{for } m \text{ odd}$$

and

$$x(n)\left\{L_m x(n)-a(n)x(n)-(q(n)+p^j(n))x[n-m\tau]\right\}\geq 0 \quad \text{for } m \text{ even}$$

under the condition $\sum^{\infty}\dfrac{1}{a_i(s)}=\infty$, $i=1,2,\cdots,m-1$. Further the result can be extended to more general equations.

Keywords

Oscillation, Delay terms, Bounded solutions, Linear and Nonlinear, Difference inequalities.

AMS Subject Classification

39A10, 39A12.

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Article History: Received **12** October **2017**; Accepted **27** December **2017**

Contents

1. Introduction

Recently there has an increasing interest in studying the oscillatory and asymptotic behavior of difference equations of various types, for examples see [2–9, 13, 15, 18, 19] Our main goal in this paper is to give first some new results on the oscillatory behavior of all solutions of the delay difference inequalities

$$x(n)\Big\{L_m x(n)+a(n)x(n)+(q(n)+p^j(n))x[n-m\tau]\Big\}\leq 0, \quad \text{for } m \text{ odd} \tag{1.1}$$

and

$$x(n)\Big\{L_m x(n)-a(n)x(n)-(q(n)+p^j(n))x[n-m\tau]\Big\}\geq 0, \quad \text{for } m \text{ even} \tag{1.2}$$

and then extent there results to equations of the form

$$L_m x(n)+(-1)^{m+1}\sum_{i=0}^{m} f_i(n,x[n-m\tau_i])=0, \tag{α}$$

and

$$L_m x(n)+(-1)^{m+1}\Big[a(n)x(n)+f\Big(n,x[n-m\tau_1],\cdots,x[n-m\tau_m]\Big)\Big]=0, \tag{β}$$

where

$$\begin{aligned} L_0 x(n) &= x(n)\\ L_k x(n) &= a_k(n)\Delta(L_{k-1}x(n)), \quad k=1,2,\cdots,m \end{aligned}$$

and $n \in \mathbb{N}_0 = \{n_0, n_0+1, \cdots\}$, $a_0(n) = a_m(n) = 1$, $\Delta x(n) = x(n+1) - x(n)$.

We assume that the following conditions without further mention

(C_1) $\{a_i(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are positive sequences for $n \geq n_0$

(C_2) $f : [n_0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ and $f_r : [n_0, \infty) \times \mathbb{R} \to \mathbb{R}$, $r = 0, 1, \cdots, m$ are continuous

(C_3)

$$\sum^{\infty} \frac{1}{a_i(s)} = \infty, \quad i = 1, 2, \cdots, m-1 \tag{1.3}$$

(C_4) τ and τ_r are positive integers $r = 1, 2, \cdots, m$

(C_5)

$$\lim_{n\to\infty} \frac{1}{\alpha_i(n)} \sum_{i=0}^{k} c_i \alpha_i(n) > 0, \quad \alpha_0(n) = 1 \tag{1.4}$$

for every choice of the constants c_i with $c_k > 0$, $k = 1, 2, \cdots, m-1$.

In what follows, we restrict our attention only to solutions $x(n)$ of (1.1) or (1.2) which are defined for $n \geq n_x$. The oscillatory character is considered in the usual sense. That is a real-valued sequence $y(n)$ defined for $n \geq n_y$ is called oscillatory if it has no last zero, otherwise it is called non-oscillatory.

For the sake of brevity, S_i will denote the set of all solution of the difference inequality (i), $i = 1, 2$ and AS_i, BS_i are subsets of S_i defined as follows:

AS_i is the set of all solutions $x(n)$ satisfying $\lim\limits_{n\to\infty} \dfrac{x(n)}{\alpha_i(n)} = 0$ where $\alpha_i(n) = \sum\limits_{c}^{n-1} \dfrac{1}{a_i(s)}$, $n \geq c > 0$.

BS_i is the set of bounded solutions, clearly $BS_i \subset AS_i \subset S_i$, $i = 1, 2$ where $\alpha_k(n) = \sum\limits_{c}^{n-1} \dfrac{1}{a_1(s_1)} \sum\limits_{c}^{s_1-1} \cdots \sum\limits_{c}^{s_{k-1}-1} \dfrac{1}{a_k(s_k)}$, $k = 1, 2, \cdots, m-1$, $c > 0$

$$\begin{aligned} \alpha_1(n) &= \sum_{c}^{n-1} \frac{1}{a_1(s_1)} \\ \alpha_2(n) &= \sum_{c}^{n-1} \frac{1}{a_1(s_1)} \sum_{c}^{s_1-1} \frac{1}{a_2(s_2)}. \end{aligned}$$

In Section 2, we establish results for (1.1) and (1.2), which extend and improve some of the results given in the literature. Extensions of results of Section 2 for equations (α) and (β) are included in Section 3.

In 1995, Thandapani E and Pandian S. [16] obtained sufficient conditions for the oscillatory and asymptotic behavior of solutions of the higher order nonlinear difference equations of the form

$$(D_m y)(n) + q(n) f(y(n - \tau_n)) = 0, \quad n = 0, 1, \cdots \tag{E_1}$$

where m is an arbitrary positive integer, $(D_0 y)(n) = y(n)$ and $(D_i y)(n) = a_i(n)\Delta(D_{i-1})(n)$, $i = 1, 2, \cdots, m$ and under the condition $\sum\limits^{\infty} \dfrac{1}{a_i(n)} = \infty$, $i = 1, 2, \cdots, m-1$.

Z.C.Wang and R.Y. Zhang in 2000 [17] consider the first order difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} \leq 0. \tag{E_2}$$

They obtained a sufficient condition generality the non existence of eventually positive solutions for the equation (E_2) with the help of the new method.

Pon. Sundaram and E. Thandapani in 2000 [10] studied the oscillatory behavior of the solutions of the second order neutral difference equation

$$\Delta(L_1^{\alpha} x(n)) + f(n, x(g(n))) = 0, \tag{E_3}$$

where $L_1 x(n) = a(n) |\Delta L_0(x(n))|^{\alpha}$ and under the condition $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{(a(n))^{1/\alpha}} = \infty$. They obtained the necessary and sufficient conditions for oscillation of almost all solutions.

Pon. Sundar and B. Kishokkumar, in 2013 [12] studied of (E_3) the extra neutral delay difference equation

$$\begin{aligned} \Delta \left[r(n) \left(\Delta^{m-1} (x(n) + p(n) x(\tau(n))) \right) \right] & \\ + q(n) f(x(\sigma(n))) = 0, & \end{aligned} \tag{E_4}$$

under the condition $\sum \dfrac{1}{r(n)} = \infty$ and $\sum \dfrac{1}{r(n)} < \infty$ and obtained sufficient condition for the oscillation of both bounded and unbounded solution of equation (E_4).

Pon.Sundar and K. Revathi in 2017 [14] consider the functional difference inequality of the form

$$(-1)^z \Delta^m x(n) sgn x(n) \geq p(n) \prod_{i=1}^{k} |x(g_i(n))|^{\alpha_i} \tag{E_5}$$

and studied the oscillations of solution of inequality (E_5) generated by general derivating arguments g_i.

In the sequel we need the following lemmas:

Lemma 1.1. *[1] Let condition (1.3) holds of $x(n)$ is a solution of equation (1.1) which is of constant sign for $n \geq n_0$ then there exists an even integer $l, 0 \leq l \leq m-1$ and an integer $n_1, n_1 \geq n_0$ such that for $n \geq n_1$,*

$$\left.\begin{aligned} x(n) L_j x(n) > 0, &\quad \text{for } 0 \leq j \leq l \\ &\text{and} \\ (-1)^{j-l} x(n) L_j x(n) > 0, &\quad \text{for } l+1 \leq j \leq n. \end{aligned}\right\} \tag{1.5}$$

Remark 1.2. *The above lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.*

Lemma 1.3. *Let conditions (i)-(v) hold. If $x(n)$ is a nontrivial solution of (1.1) or (1.2) such that*

$$x(n) \geq 0 \quad \text{and} \quad \frac{x(n)}{\alpha_i(n)} \to 0 \ \text{as} \ n \to \infty,$$

then for $n \geq n_0$

$$(-1)^k L_k x(n) > 0, \quad k = 2,3\cdots,m-1$$

and

$$L_k x(n) \to 0 \text{ montonically as } n \to \infty, k = 2,3,\cdots m-1. \quad (1.6)$$

Proof. Our proof is an adaption of the similar argument developed by Kim in discrete sequence. Put $\Delta^k y(n) = L_{k-1}x(n)$ that is

$$\begin{aligned}
\Delta y(n) &= x(n), \\
\Delta(\Delta y(n)) &= \frac{\Delta^2 y(n)}{a_1(n)}, \\
&\vdots \\
\Delta(\Delta^{m-1} y(n)) &= \frac{\Delta^m y(n)}{a_{m-1}}
\end{aligned}$$

and let b be an arbitrary point $n \geq n_0$. Then $x(n)$ satisfies the following system

$$\begin{aligned}
\Delta y(n) &= \Delta y(n) + \sum_b^{n-1} \frac{\Delta^2 y(s)}{a_1(s)} \\
\Delta^2 y(n) &= \Delta^2 y(n) + \sum_b^{n-1} \frac{\Delta^3 y(s)}{a_2(s)} \\
\Delta^m y(n) &= \Delta^{m-1} y(n) + \sum_b^{n-1} \frac{\Delta^m y(s)}{a_{m-1}(s)} \\
\Delta^m y(n) &= \Delta^m y(n) \\
&\quad - \sum_b^{n-1} \left\{a(n)x(n) + (q(n) + p^j(n))x[n-m\tau]\right\}
\end{aligned}$$

Suppose $x(n) = \Delta y(n)$ is a solution of (1.1). Then

$$\sum_b^{n-1} \left\{a(n)x(n) + (q(n) + p^j(n))x[n-m\tau]\right\}$$

is nondecreasing nonnegative sequence of n and clearly is positive on $n \geq C$ for some $C > b$. We claim that $\Delta^m y(n) > 0$. To prove this, assume the contrary, that $\Delta^m y(n) \leq 0$. Then $\Delta^m y(n)$ is non-positive, nonincreasing on $n \geq b$ and

$$\begin{aligned}
&\Delta^m y(n) = \Delta^m y(n) \\
&\qquad - \sum_b^{n-1} \left\{a(n)x(n) + (q(n) + p^j(n))x[n-m\tau]\right\} < 0 \\
&\Delta^m y(n) \leq \Delta^m y(c) < 0, \quad \text{for } n > C \quad (or) \\
&\Delta(\Delta^{m-1} y(n)) \leq \frac{1}{a_{m-1}} \Delta^m y(c).
\end{aligned}$$

Summing the above inequality from b to $n-1$, we obtain

$$\begin{aligned}
&\Delta^{m-1} y(n) \leq \Delta^{m-1} y(b) \\
&\qquad + \Delta^m y(c) \sum_b^{n-1} \frac{1}{a_{m-1}(s)} \to -\infty \ \ as \ \ n \to \infty.
\end{aligned}$$

This is turn implies that $\Delta^{m-1} y(n) \to -\infty$ as $n \to \infty$ and successively $\Delta^k y(n) \to -\infty$ as $n \to \infty$ regardless of the values $\Delta^k y(n)$, $k = 1,2,\cdots,m-1$. In particular, $\Delta y(n) = x(n) \to -\infty$ as $n \to \infty$, contrary to the hypothesis that $x(n) \geq 0$ for $n \geq n_0$. This contradiction proves that $\Delta^m y(n) > 0$.
Since b is arbitrary we conclude that $\Delta^m y(n) > 0$ for $n \geq n_0$. It is now easy to see that $\Delta^m y(n) \to 0$ as $n \to \infty$ for $m > 2$. If this were not the case, there would exist a constant $C > 0$ such that

$$\Delta^m y(n) > C \quad \text{for } \ n \geq C_1 \ \text{ for some } \ C_1 \geq 0$$

this implies, however, that

$$x(n) = \Delta y(n) > \sum_{i=0}^{m-2} \Delta^{i+1} y(c) \alpha_{i+1}(n) + C\alpha_{m-1}(n),$$

$\alpha_0(n) = 1$. If we divide the above inequality by $\alpha_2(n)$ and take the least as $n \to \infty$, we get, in view of (1.4) with $k = m-1$, a contraction to the fact that $\dfrac{x(n)}{\alpha_2(n)} \to 0$ as $n \to \infty$.

Next we shall prove that $\Delta^{m-1} y(n) < 0$ if $m > 2$ of $\Delta^{m-1} y(n) \geq 0$ then $\Delta^{m-1} y(n) \geq 0$ for $n \geq b$ and there would exist constants $C_1 > 0$ and $d > b$ such that

$$\Delta^{m-1} y(n) > C_1 \quad \text{for } \ n \geq d.$$

This would imply

$$x(n) = \Delta y(n) > \sum_{i=0}^{m-2} \Delta^{i+1} y(d) \alpha_{i+1}(n) + C\alpha_{m-2}(n),$$

which would again lead to a contradiction. Thus $\Delta^{m-1} y(n) < 0$ and hence $\Delta^{m-1} y(n) < 0$, since b is arbitrary. Moreover, we must have $\Delta^{m-1} y(n) \to 0$ as $n \to \infty$, for otherwise we would again be led to the contradiction that $x(n) \to -\infty$ as $n \to \infty$.

In this way, we can successively establish the inequalities, $\Delta^m y(n) > 0$, $\Delta^{m-1} y(n) < 0$, $\cdots$, $\Delta^4 y(n) > 0$, $\Delta^3 y(n) < 0$, for $n \geq n_0$ with the property that $\Delta^k y(n) \to 0$ as $n \to \infty$, $k = 3,4,\cdots,m$. Continuing this process, we deduce $\Delta^2 y(n) > 0$ and $\Delta y(n) \geq 0$ for $n \geq n_0$. This process the theorem for equation (1.1). The proof for equation (1.2) is similar. In this case we first prove that $\Delta^m y(n) < 0$ and $\Delta^m y(n) \to 0$ as $n \to \infty$, and continue as in the case of equation (1.1). □

Lemma 1.4. *Consider the delay difference inequalities*

$$\Delta y(n) + a(n)y(n) + p(n)y(n-\tau) \leq 0 \qquad (\beta_1)$$

$$\Delta y(n) + a(n)y(n) + p(n)y(n-\tau) \geq 0 \qquad (\beta_2)$$

and the delay difference equation

$$\Delta y(n) + a(n)y(n) + p(n)y(n-\tau) = 0, \qquad (\beta_3)$$

where τ is a positive integer, $\{a(n)\}$ and $\{p(n)\}$ are positive sequences. Assume that

$$\liminf_{n\to\infty} \sum_{n-\tau}^{n-1} p(s) > \frac{1}{e} e^{-\liminf_{n\to\infty} \sum_{n-2\tau}^{n-1} a(s)} \qquad (1.7)$$

and

$$\liminf_{n\to\infty}\sum_{n-\tau}^{n-1} p(s) > 0. \tag{1.8}$$

Then

(b_1) *Inequality* (β_1) *has no eventually positive solutions.*

(b_2) *Inequality* (β_2) *has no eventually negative solutions and*

(b_3) *Equation* (β_3) *has only oscillatory solutions.*

Proof. We will prove that the existence of eventually positive solution leads to a contradiction.

To this end suppose that $y(n)$ is a solution of (β_1) such that for no sufficiently large

$$y(n) > 0 \quad \text{for } \; n > n_0$$

Then $y(n-2\tau) > 0$ for $n \geq n_0 + 2\tau$ and from (β_1), $\Delta y(n) < 0$ for $n \geq n_1 \geq n_0$. Hence $y(n) < y(n-2\tau)$ for $n > n_2 \geq n_1$. Set

$$w(n) = \frac{y(n-2\tau)}{y(n)}, \quad \text{for } \; n > n_2. \tag{1.9}$$

Then $w(n) \geq 1$ and dividing both sides of (β_1) by $y(n)$ for $n \geq n_2$. We obtain

$$\frac{\Delta y(n)}{y(n)} + a(n) + p(n)w(n) \leq 0 \quad \text{for } \; n \geq n_2 \tag{1.10}$$

We can easily show that

$$\log w(n) \geq \left(\sum_{n-2\tau}^{n-1} p(s)w(s)\right) + \sum_{n-2\tau}^{n-1} a(s),\; n > n_3 \tag{1.11}$$

Now, summing (β_1) from $n-2\tau$ to $n-1$ and using the fact that $y(n)$ is decreasing, we find

$$y(n) - y(n-2\tau) + y(n)\sum_{n-2\tau}^{n-1} a(s) + y(n-\tau)\sum_{n-2\tau}^{n-1} p(s) \leq 0, \quad n \geq n_4.$$

Dividing the last inequality first by $y(n)$ and then by $y(n-\tau)$, we obtain respectively

$$1 - \frac{y(n-2\tau)}{y(n)} + \sum_{n-2\tau}^{n-1} a(s) + \frac{y(n-2\tau)}{y(n)}\sum_{n-\tau}^{n-1} p(s) \leq 0 \tag{1.12}$$

and

$$\frac{y(n)}{y(n-\tau)} - 1 + \frac{y(n)}{y(n-\tau)}\sum_{n-\tau}^{n-1} a(s) + \frac{y(n-2\tau)}{y(n-\tau)}\sum_{n-2\tau}^{n-1} p(s) \leq 0. \tag{1.13}$$

Let $\liminf_{n\to\infty} W(n) = l$. Then $l \geq 1$ and is finite or infinite. We consider the producing two possible cases.

Case 1: l is finite. Taking limit inf on both sides of (1.10), we obtain

$$\log l \geq \liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} a(s) + (l-s)\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s)$$

where ε is sufficiently small and so

$$\log l - l\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s) \geq \liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} a(s).$$

Using the fact that

$$\max_{l\geq 1}\left\{\log l - l\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s)\right\} = -\log\left(\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s)\right) - 1.$$

The last inequality implies

$$\begin{aligned}\log\left(\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s)\right) &\leq -1 - \liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} a(s)\\ \liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} p(s) &\leq \frac{1}{e}\exp\left(\liminf_{n\to\infty}\sum_{n-2\tau}^{n-1} a(s)\right)\end{aligned}$$

which contradicts the hypotheses (1.7).

Case 2: l is infinite. That is,

$$\lim_{n\to\infty}\frac{y(n-2\tau)}{y(n)} = +\infty.$$

In view of (1.8) and the fact that $a(n) \geq 0$, inequality (1.12) implies

$$\lim_{n\to\infty}\frac{y(n-\tau)}{y(n)} = +\infty$$

and therefore $\lim_{n\to\infty}\frac{y(n-2\tau)}{y(n-\tau)} = +\infty$ which contradicts (1.13).

Since in both cases, we are lead to a contradiction. Then the proof of part (b_1) is complete.

The result (b_2) follows immediately from the observation that of $y(n)$ is a solution of (β_2) then $-y(n)$ is a solution of (β_1).

From the above results, if follows that the delay difference equation (β_3) has no eventually positive or eventually negative solutions and therefore we are lead to the conclusion that (β_3) has oscillatory solutions only. □

2. Main Results

In this section, we begin with the following theorem.

Theorem 2.1. *Let conditions* (1.3), (1.7) *and* (1.8) *hold, and in addition, suppose*

$$\frac{1}{a_{m-l}(n-i\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)^{(i-1)} \geq (i-1)p(n) \geq 0, \quad (2.1)$$

$i = 1, 2, \cdots, m$, *then every solution of* (1.1) *is oscillatory.*

Proof. Assume that there exists a solution $x(n)$ of (1.1) such that for n_0 sufficiently large

$$x(n) > 0 \quad \text{for} \quad n \geq n_0.$$

Then $x(n-m\tau) > 0$ for $n > n_0 + m\tau$. Hence (1.1) becomes

$$L_m x(n) + a(n)x(n) + (q(n) + p^j(n))x(n-m\tau) \leq 0, \quad (1.1')$$

n is odd. By Lemma 1.1, there exists an even integer l, $0 \leq l \leq m$ such that for $n > n_0 + m\tau$

$$L_j x(n) > 0 \quad 0 \leq j \leq l \quad (2.2)$$

and

$$(-1)^{j-l} L_j x(n) > 0 \quad l+1 \leq j \leq m.$$

We claim that $l = 0$, that is, for $n > n_0 + m\tau$

$$(-1)^j L_j x(n) > 0 \quad \text{for} \quad j = 0, 1, \cdots, m. \quad (2.3)$$

To prove it, assume that $l > 0$. Then by the generalized discrete Taylor's formula

$$\begin{aligned}
L_l x(n) &= L_l x(n_1) + \left(\sum_{n_2}^{n-1} \frac{1}{a_{l+1}(s_{l+1})}\right) L_{l+1} x(n) \\
&+ \left(\sum_{n_1}^{n-1} \frac{1}{a_{l+2}(s_{l+2})} \sum_{n_1}^{s_{l-2}} \frac{1}{a_{l+1}(s_{l+1})}\right) L_{l+2} x(n) \\
&+ (-1)^{m-l+1} \sum_{n_1}^{n-1} \left[\sum_{n_1}^{s_{n-1}} \frac{1}{(a_{m-1} s_{m-1})} \sum_{n_1}^{s_{n-2}} \cdots \sum_{n_1}^{s_{l+2}} \frac{1}{a_{l+1}}(s_{l+1})\right] \\
&\qquad \times L_m x(s_{m-\tau})
\end{aligned}$$

for every $n \geq n_1$ with n_1 sufficiently large. Using (1.1), (2.2) and the fact that the integer $m+l$ is odd and $x(n)$ and $p(n)$ are nondecreasing sequence we have

$$\begin{aligned}
L_l x(n) &\leq L_l x(n_1) - p^j(n_1) x(n_1 - m\tau) \\
&\times \sum_{n_1}^{n-1} \left[\sum_{n_1}^{s_{m-1}} \frac{1}{a_{m-1}(s_{m-1})} \sum_{n_1}^{s_{m-2}} \cdots \sum_{n_1}^{s_{l+2}} \frac{1}{a_{l+1}(s_{l+1})}\right].
\end{aligned}$$

By (1.3), $L_l x(n) \to -\infty$ as $n \to \infty$, that is, $L_l x(n) < 0$ for all large n, which contradicts (2.2) and proves (2.3). Set

$$\begin{aligned}
y(n) \quad = \quad & L_{m-1} x(n) - p(n) L_{m-2} x(n-\tau) \\
& + p^{(2)}(n) L_{m-3} x(n-2\tau) \\
& + \cdots + p^{(j-1)}(n) x(n-(m-1)\tau).
\end{aligned}$$

Then in view of (2.3),

$$y(n) > 0. \quad (2.4)$$

Observe that

$$\begin{aligned}
&\Delta y(n) + p(n) y(n-\tau) = \\
&L_m x(n) + p^{(j-1)}(n-\tau) p(n) x(n-m\tau) \\
&+ (j-1) p^{(j-2)}(n) x(n-(m-1)\tau) \\
&+ p^{(m-1)}(n) \left[\left[\frac{1}{a_1(n-(m-1)\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)^{(j-2)}\right] \right. \\
&\qquad \times L_1 x(n-(m-1)\tau) \\
&\left. - \frac{(j-2)}{p^2(n)} L_1 x(n-(m-2)\tau)\right] \\
&\qquad \vdots \\
&- p^{(3)}(n) \left[\left[\frac{1}{a_{m-3}(n-3\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)^{(2)}\right] \right. \\
&\qquad \times L_{m-3} x(n-3\tau) \\
&\left. - \frac{2}{p^2(n)} L_{m-3} x(n-4\tau)\right] \\
&+ p^{(2)}(n) \left[\left[\frac{1}{a_{m-2}(n-2\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)\right] \right. \\
&\qquad \times L_{m-2} x(n-2\tau) \\
&\left. - \frac{1}{p^2(n)} L_{m-2} x(n-3\tau)\right] \\
&- p(n) \left[\frac{1}{a_{m-1}(n-\tau)} - 1\right] L_{m-1} x(n-\tau).
\end{aligned}$$

Using the monotonicity of L_k, $k = 0, 1, \cdots, m$ and the condition (2.1) we have

$$\begin{aligned}
&\Delta y(n) + p(n) y(n-\tau) \leq -a(n) x(n) - q(n) x(n-m\tau) \\
&- p^{(j)}(n) \left[1 - \left(\frac{p(n-\tau)}{p(n)}\right)^{(j-1)} - (m-1)\frac{1}{p^2(n)}\right] \\
&\qquad \times x(n-m\tau) \\
&+ p^{(j-1)}(n) \left[\frac{1}{a_1(n-(m-1)\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)^{(j-2)} \right. \\
&\qquad \left. - (m-2)\frac{1}{p^2(n)}\right] L_1 x(n-(m-1)\tau) + \\
&\ldots \\
&+ p^{(2)}(n) \left[\frac{1}{a_{m-2}(n-2\tau)} - \frac{p(n-\tau)}{p(n)} - \frac{1}{p^2(n)}\right] \\
&\qquad \times L_{m-2} x(n-2\tau) \\
&- p(n) \left[\frac{1}{a_{m-1}(n-\tau)} - 1\right] L_{m-1} x(n-\tau).
\end{aligned}$$

Since by Lemma 1.4(b_1), the above inequality has no eventually positive solutions, we get a contradiction to (2.4). The case $x(n) < 0$ for $n \geq n_0$ is similar and thus the proof is omitted. □

Theorem 2.2. *Let conditions* (1.3)*,* (1.4)*,* (1.7)*,* (1.8) *and* (2.1) *hold, then every solution $x(n)$ of* (1.2) *with the property that $\frac{x(n)}{\alpha_i(n)} \to 0$ as $n \to \infty$ is oscillatory.*

Proof. Assume that there exists a solution of equation (1.2) such that

$$x(n) > 0, \ \lim_{n\to\infty} \frac{x(n)}{\alpha_i(n)} = 0, \ \ n \geq n_0.$$

Then $x(n - m\tau) > 0$ for $n > n_1 + m\tau$ and from (1.2), $L_m x(n) > 0$ for $n > n_0 + m\tau$. By Lemma 1.3,

$$(-1)^k L_k x(n) > 0, \quad k = 0, 1, \cdots, m. \tag{2.5}$$

Set

$$\begin{aligned} y(n) \ &= \ L_{m-1}x(n) - p(n)L_{m-2}x(n-\tau) \\ &\quad +p^{(2)}(n)L_{m-3}x(n-2\tau) \\ &\quad \cdots - p^{(j-1)}(n)x(n-(m-1)\tau) \end{aligned} \tag{2.6}$$

which for sufficiently large n such that $y(n) < 0$. Taking differences on both sides of (2.6), we obtain

$$\begin{aligned} &\Delta y(n) + p(n)y(n-\tau) \\ &= L_m x(n) - p^{(j-1)}(n-\tau)p(n)x(n-m\tau) \\ &-(n-1)p^{(j-2)}x(n-(m-1)\tau) \\ &-p^{(j-1)}(n)\Bigg[\Bigg[\frac{1}{a_1(n-(m-1)\tau)} \\ &\qquad -\left(\frac{p(n-\tau)}{p(n)}\right)^{(j-2)}\Bigg]L_1x(n-(m-1)\tau) \\ &\qquad +(j-2)\frac{1}{p^2(n)}L_1x(n-(m-2)\tau)\Bigg] \\ &\qquad\qquad \times L_1x(n-(m-1)\tau)+ \\ &\ \ldots \\ &+p^{(2)}(n)\left[\left[\frac{1}{a_{m-2}(n-2\tau)} - \frac{p(n-\tau)}{p(n)}\right]L_{m-2}x(n-2\tau)\right. \\ &\qquad \left. -\frac{1}{p^2(n)}L_{m-2}x(n-3\tau)\right] \\ &-p(n)\left[\frac{1}{a_{m-1}(n-\tau)} - 1\right]L_{m-1}x(n-\tau). \end{aligned}$$

Now using the monotonicity of L_k, $k = 0, 1, \cdots m$ and the condition (2.1), we have

$$\begin{aligned} &\Delta y(n) + p(n)y(n-\tau) \\ &\geq a(n)x(n) - q(n)x(n-m\tau) \\ &+p^{(j)}(n)\left[1 - \left(\frac{p(n-\tau)}{p(n)}\right)^{(j-1)} - (j-1)\frac{1}{p^2(n)}\right] \\ &\qquad\qquad \times x(n-m\tau) \\ &+p^{(j-1)}(n)\Bigg[\frac{1}{a_1(n-(m-1)\tau)} - \left(\frac{p(n-\tau)}{p(n)}\right)^{(j-2)} \\ &\qquad -(j-2)\frac{1}{p^2(n)}\Bigg]L_1x(n-(m-1)\tau)+ \\ &\ldots \\ &+p^{(2)}(n)\left[\frac{1}{a_{m-2}(n-2\tau)} - \frac{p(n-\tau)}{p(n)} - \frac{1}{p^2(n)}\right] \\ &\qquad\qquad \times L_{m-2}x(n-2\tau) \\ &-p(n)\left[\frac{1}{a_{m-1}(n-\tau)} - 1\right]L_{m-1}x(n-\tau) \geq 0. \end{aligned}$$

The above inequality has no eventually negative solutions in view of Lemma 1.4(b_2). This contradicts the fact that $y(n)$ is negative.

The proof of theorem $x(n) < 0$ is similar and hence is omitted. □

Example 2.3. *The third order inequality*

$$x(n)\left\{\Delta\left(\frac{e}{1+e}\left(\Delta\left(\frac{e}{e+1}\Delta x(n)\right)\right)\right)\right\} \\ +x(n) + \frac{e^2}{e^5}x(n-3) \leq 0, \ \ n > 3, \quad (E_6)$$

has an oscillatory solution $x(n) = (-1)^n e^{-n}$ and the second order inequality

$$x(n)\left\{\Delta\left(\frac{1}{e+1}\Delta x(n)\right)\right\} - x(n) - ce^{(2)}x(n-2) \geq 0, \ \ n > 2$$

has an oscillatory solution $x(n) = (-1)^n e^n$. Only condition (2.1) *is not satisfied.*

In the following theorem we discuss the case when $p(n)$ in (1.1) and (1.2) is not a monotone nondecreasing sequence. We replace the sequence $q(n) + p^{(m)}(n)$ by $p(n)$ and assume that $p(n)$ satisfies (1.7) and (1.8).

Theorem 2.4. *Let condition* (2.1) *in Theorems 2.1 and 2.2 be replaced by*

$$\frac{1}{a_{m-1}(n-i\tau)} \geq p(n), \quad i = 1, 2, \cdots m, \tag{2.1$'$}$$

then the conclusions of Theorems 2.1 and 2.2 hold.

Proof. The proof of Theorem 2.4 is similar to the proofs of Theorem 2.1 and 2.2 except that in Theorem 2.1 we replace $y(n)$ by

$$y(n) = L_{m-1}x(n) - L_{m-2}x(n-\tau) + \cdots + x(n-(m-1)\tau)$$

and in Theorem 2.2 we replace $y(n)$ by

$y(n) = L_{m-1}x(n) - L_{m-2}x(n-\tau) - \cdots - x(n-(m-1)\tau)$.

$\square$

Example 2.5. *Consider the inequality*

$$x(n)\left\{\Delta^2 x(n) + 4x(n-3)\right\} \le 0, \quad n > 3. \qquad (E_7)$$

It is easy to check that the hypothesis of Theorem 2.4 are satisfied while Theorems 2.1 and 2.2 are not applicable. This equation (E_7) has an oscillatory solution $x(n) = (-1)^n e^n$.

Remark 2.6. *In (1.1) and (1.2) if we let $a(n) = 0$ and $c(n) = q(n) + p^{(j)}(n)$, then the equation $L_m x(n) + (-1)^{n+1}c(n)x(n-\tau) = 0$ is positive for the conclusions of Theorems 2.1 and 2.2. For the sake of completeness we state the result.*

Theorem 2.7. *If*

$$\sum_{n-\tau}^{n-1} \frac{1}{a_1(s_{n-1})} \sum_{s_{n-1}}^{n-1} \cdots \sum_{s_1}^{n-1} c(s) > 1$$

then the conclusions of Theorems 2.1 and 2.2 hold for the inequalities

$$x(n)\left\{L_m x(n) + (-1)^{m+1}c(n)x(n-\tau)\right\} \le 0, \;\; n \text{ is odd}$$

and

$$x(n)\left\{L_m x(n) + (-1)^{m+1}c(n)x(n-\tau)\right\} \ge 0, \;\; n \text{ is even}$$

respectively.

Remark 2.8. *If we let $a_i = 1$, $i = 1,2,3,\cdots,m-1$, $p(n)$ is a positive constant, and $a(n) = 0$, the conditions (1.4) and (2.1) are trivially satisfied. Moreover $\alpha_1(n) = n$ for $a_1(n) = 1$. Then the cases of sequences $x(n)$ satisfying the property that $\lim_{n\to\infty} \frac{x(n)}{n} = 0$ includes all bounded sequences.*

3. Some Extensions

In this section we are interested in extending our results of section 2 to more general equations namely (α) which takes the form

$$L_m x(n) + \sum_{i=0}^{k} f_i(n, x[n-m\tau_i]) = 0, \;\; n \text{ is odd} \qquad (3.1)$$

and

$$L_m x(n) - \sum_{i=0}^{k} f_i(n, x[n-m\tau_i]) = 0, \;\; n \text{ is even} \qquad (3.2)$$

where L_m is defined as above.
We assume that for each i, $0 \le i \le k$, $f_i \in C[[n_0,\infty)\times\mathbb{R},\mathbb{R}]$,

$$\frac{f_i(n,x)}{x} \ge q_i(n) \ge 0, \;\; for\; x \ne 0 \qquad (3.3)$$

where $q_i \in C[[n_0,\infty),[0,\infty)]$, $\tau_0 = 0$, $1 \le i \le k$ are positive constants, and $\tau = \min\{\tau_1, \tau_2, \cdots, \tau_k\}$.

It follows as in the proofs of Theorems 2.1 and 2.2 that if $x(n) > 0$ for $n \ge n_1 \ge n_0$ we have $\Delta x(n) < 0$ for $n \ge n_2 \ge n_1$. Thus

$$\frac{f_i(n, x[n-m\tau_i])}{x[n-m\tau]} \ge \frac{f_i(n, x[n-m\tau_i])}{x[n-m\tau_i]} \ge q_i(n), \;\; 0 \le i \le k. \qquad (3.4)$$

The above inequality is also true for the case where $x(n) < 0$, $n \ge n_0$ and hence (3.1) and (3.2) reduce to

$$x(n)\left\{L_m x(n) + q_0(n)x(n) + \left(\sum_{i=1}^{k} q_i(n)\right)x[n-m\tau]\right\} \le 0, n \text{ is odd} \qquad (3.1')$$

and

$$x(n)\left\{L_m x(n) - q_0(n)x(n) - \left(\sum_{i=1}^{k} q_i(n)\right)x[n-m\tau]\right\} \ge 0, n \text{ is even.} \qquad (3.2')$$

Now, the required extensions follows immediately by letting

$$q_0(n) = a(n) \quad \text{and} \quad p^{(m)}(n) + q(n) = \sum_{i=0}^{k} q_i(n),$$

where $a(n)$, $p(n)$ and $q(n)$ are defined as above and satisfy the hypotheses of our theorems.

Example 3.1. *Consider the equation*

$$\Delta^m x(n) + (-1)^{(2m+1)} \frac{(1+e)^m}{e^{2m}} x(n-m) = 0. \qquad (E_8)$$

We conclude that every solution of (E_8) is oscillatory if m is odd, while m is even, every solution of (E_8) with the property that $\frac{x(n)}{n} \to 0$ as $n \to \infty$ is oscillatory.

Finally, the results presented is this paper can be extended to equation (β) which can be written as

$$L_m x(n) + a(n)x(n) + f(n, x[n-m\tau_1], \cdots x[n-m\tau_k]) = 0, \qquad (3.5)$$

n is odd and

$$L_m x(n) - a(n)x(n) + f(n, x[n-m\tau_1], \cdots x[n-m\tau_k]) = 0, \qquad (3.6)$$

n is even, where L_m is defined as above, τ_i, $i = 1,2,\cdots,k$ are positive constants and $\tau = \min\{\tau_1, \cdots, \tau_k\}$.

$$f \in C[[n_0,\infty)\times\mathbb{R}^m,\mathbb{R}]$$
$$f(n,y_1,y_2,\cdots,y_k) > 0 \;\; if\; y_i > 0\; for\; all\; i$$
$$f(n,y_1,y_2,\cdots,y_k) < 0 \;\; if\; y_i < 0\; for\; all\; i$$
$$f(n,y_1,y_2,\cdots,y_k) \text{ is nondecreasing on} Y \;\; n \ge n_0$$

where

$$Y = \left\{ (y_1, \cdots, y_k) | y_i \in \mathbb{R} \text{ and either every } y_i \geq 0 \right.$$

$$\left. \text{or every } y_i < 0 \; for \; i = 1, 2, \cdots k \right\}$$

and

$$|f(n, y_1, \cdots, y_k)| \geq \gamma(n)|y(n)| \;\; for \; y \neq 0 \;\; and \;\; n \geq n_0$$

where $\gamma \in C[[n_0, \infty), (0, \infty)]$.

If we write $\gamma(n) = q(n) + p^{(m)}(n)$ where $a(n)$, $p(n)$ and $q(n)$ are as given before and hence we obtain the desired extension.

Example 3.2. *Consider the equation*

$$\Delta^m x(n) + (-1)^{(m+1)} \frac{(1+e)^m}{e^{4m-2n}} x(n-m) \times x^2(n-m) = 0, \qquad (E_9)$$

where $m = \tau$ and $\gamma(n) = \dfrac{(1+e)^m}{e^{4m-2n}}$, one can easily see that every solution of (E_9) is oscillating if m is odd, while if m is even, every solution of (E_9) such that $\dfrac{x(n)}{n} \to 0$ as $n \to \infty$ is also oscillatory.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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