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Numerical solution of nonlinear fractional integro-differential equation by Collocation method

S. I. Unhale[1] and S. D. Kendre[2]*

Abstract

In this paper, we presents the Collocation Method with the help of shifted Chebyshev polynomials and shifted Legendre polynomials for the numerical solution of nonlinear fractional integro-differential equations (NFIDEs). The method introduces a promising tool for solving many NFIDEs with the help of Newton's iteration method.

Keywords

Fractional Integrodifferential Equations, Collocation method, Chebyshev Polynomials, Legendre polynomials.

AMS Subject Classification

34A08, 65L05, 65L20

[1] *Department of Mathematics, C.K. Thakur Arts, Commerce and Science College, New Panvel 410 206, India (M.S.)*

[2] *Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India (M.S.)*

***Corresponding author**: [2]sdkendre@unipune.ac.in, [2]sdkendre@yahoo.com; [1]subhashunhale@gmail.com

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1. Introduction

It is now a days well established that several real life phenomena are better described by fractional differential equations and hence the study of these type of equations is very important due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering see [1, 2, 4]. Most of the fractional differential equations do not have exact analytic solutions and therefore, approximating or numerical techniques are generally applied. Consequently, considerable attention has been given to the solutions of fractional differential equations and fractional integral equations using different numerical methods such as Predictor Corrector method, Quadrature methods, Fractional Euler, Fractional Trapezoidal method, Legendre spline interpolation method, Adomain decomposition method, Taylor series method, Picard's iterative method, Variational principle method, Iterative methods, Laplace transform, Mellin transform, Collocation method, Galerkin method and many others. Many recent papers have dealt with the solutions of fractional differential equations by using above methods, see [5–7, 9, 14, 20, 21] and some of the references cited therein.

As Chebyshev polynomials and Legendre polynomials are well known family of orthogonal polynomials on the interval $[-1,1]$ that have many applications and widely used because of their good properties in the approximation of functions. This motivated to find a numerical solution of nonlinear fractional integrodifferential equations using Collocation method with the help of Chebyshev polynomials and Legendre polynomials to reduce to system of nonlinear equations and which can be solved by Newtons Iterative method.

The aim of the present paper is to determine the numerical solution of the nonlinear fractional integrodifferential equation of the type

$$D^{\alpha}y(x) = g(x) + \int_0^1 K(x,t)f(y(t))dt, \tag{1.1}$$

$$y^{(i)}(0) = \delta_{(i)},\ i = 0,1,\dots, \tag{1.2}$$

for $0 \le x,t \le 1$ and $n-1 < \alpha \le n \in N$ where $D^{\alpha}y(x)$ indicates the α-th Caputo fractional derivatives of $y(x), g(x)$ and $K(x,t)$ are given functions, x and t are real variables varying in the interval $[0,1]$ and $y(x)$ is the unknown function to be determined.

The paper is organized as follows: Section 2, presents the preliminaries and definitions. Section 3, dedicates the solution of nonlinear fractional integrodifferential equation by Collocation method with the help of shifted Chebyshev polynomials. Section 4, obtains the solution of nonlinear fractional integrodifferential equation by Collocation method with the help of shifted Legendre polynomials. Section 5, focuses on some examples to illustrate the theory.

2. Preliminaries and Definitions

In this section we recall some definitions and properties of fractional derivatives and fractional integrals [13, 16].

Definition 1. *A real function $f(x), x>0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,1)$.*

Definition 2. *A real function $f(x), x>0$, is said to be in the space $C_\mu^m, m \in NU\{0\}$ if and only if $f^{(m)} \in C_\mu$.*

Definition 3. *The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad x>0,\ \alpha>0, \tag{2.1}$$

provided the right side is point-wise defined on [0,∞), where $\Gamma(\cdot)$ is the gamma function.

Definition 4. *The Riemann-Liouville derivative of order α with the lower limit zero for a function $f:[0,\infty)\to R$ can be written as*

$${}^{L}D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x \frac{f(t)}{(x-t)^{\alpha+1-n}}dt,\ x>0, \tag{2.2}$$

for $n-1<\alpha<n$.

Definition 5. *The Caputo derivative of order α for a function $f:[0,\infty)\to R$ can be written as*

$$D^\alpha f(x) = \begin{cases} I^{n-\alpha} f^n(x), & n-1<\alpha\le n,\ n\in N, \\ \dfrac{D^n f(x)}{Dx^n}, & \alpha = n. \end{cases} \tag{2.3}$$

3. Solution of NFIDEs by Collocation method with the help of shifted Chebyshev polynomials

The well known Chebyshev polynomials are defined on the interval $[-1,1]$ and can be determined with the use of the following recurrence formula

$$T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z),\ \ n=2,3,\ldots$$

with

$$T_0(z)=1, \qquad T_1(z)=z.$$

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z) = n\sum_{i=0}^{[\frac{n}{2}]}(-1)^i 2^{n-2i-1}\frac{(n-i-1)!}{(i)!(n-2i)!}z^{n-2i} \tag{3.1}$$

where $[\frac{n}{2}]$ denotes the integer part of $n/2$.
The orthogonality condition is

$$\int_{-1}^{1}\frac{T_i(z)T_j(z)}{\sqrt{1-z^2}}dz = \begin{cases} \pi, & for\ \ i=j=0; \\ \dfrac{\pi}{2}, & for\ \ i=j\neq 0; \\ 0, & for\ \ i\neq j. \end{cases} \tag{3.2}$$

In order to use these polynomials on the interval $[0,1]$ we define the so called shifted Chebyshev polynomials by introducing change of variables $z=2x-1$. The shifted Chebyshev polynomials $T_n(2x-1)$ be denoted by $T_n^*(x)$. Then $T_n^*(x)$can be obtained as follows

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x), \tag{3.3}$$

for $n=2,3,\ldots$ with initial conditions

$$T_0^*(x)=1, \quad T_1^*(x)=2x-1. \tag{3.4}$$

The analytic form of shifted Chebyshev polynomials $T_n^*(x)$ of degree n is given by

$$T_n^*(x) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!}x^k, \tag{3.5}$$

for $n=2,3,\ldots$
The function $y(x)$, which is square integrable functions in $[0,1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i T_i^*(x), \tag{3.6}$$

where the coefficients a_i are given by

$$a_0 = \frac{1}{\pi}\int_0^1 \frac{y(x)T_0^*(x)}{\sqrt{x-x^2}}dx,\ a_i = \frac{2}{\pi}\int_0^1 \frac{y(x)T_i^*(x)}{\sqrt{x-x^2}}dx,$$

for $i=1,2,\ldots$. In practice, only the first $(n+1)$ terms of shifted Chebyshev polynomials are considered. Then we have

$$y_n(x) \cong \sum_{i=0}^{n} a_i T_i^*(x), \quad 0\le x\le 1. \tag{3.7}$$

Theorem 3.1 (Chebyshev Truncation Theorem)**.** *[12] The error in approximating $y(x)$ by the sum of its first n terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$y_n(x) \cong \sum_{k=0}^{n} a_k T_k^*(x), \tag{3.8}$$

then

$$E_T(n) = |y(x) - y_n(x)| \leq \sum_{k=n+1}^{\infty} |a_k| \tag{3.9}$$

for all $y(x)$*, all* n *and all* $x \in [-1,1]$.

The main approximate formula of the fractional derivative of $y_n(x)$ is given in the following theorem.

Theorem 3.2. *[12] Let* $y(x)$*be approximated by shifted Chebyshev polynomials and also suppose* $\alpha > 0$*, then*

$$D^{\alpha}(y_n(x)) = \sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x^{k-\alpha} \tag{3.10}$$

where $w_{i,k}^{(\alpha)}$ *is given by*

$$w_{i,k}^{(\alpha)} = (-1)^{i-k} \frac{2^{2k} i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)} \tag{3.11}$$

The numerical solution of nonlinear fractional integrodifferential equation (1.1) using collocation method with the help of shifted Chebyshev polynomials is discussed below. This method is based on approximating the unknown function $y(x)$ as

$$y_n(x) \cong \sum_{i=0}^{n} a_i T_i^*(x), \quad 0 \leq x \leq 1 \tag{3.12}$$

where $T_i^*(x)$ is the shifted Chebyshev polynomial and a_i, $i = 0,1,2,\dots$ are constants.
Making use of (3.12) into (1.1), following equation is obtained

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x^{k-\alpha} = g(x) + \int_0^1 K(x,t) f(\sum_{i=0}^{n} a_i T_i^*(t)) dt \tag{3.13}$$

We now collocate equation (3.13) at $(n+1-\lceil\alpha\rceil)$ points $x_p, p = 0,1,\dots n-\lceil\alpha\rceil$ as

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} = g(x_p) + \int_0^1 K(x_p,t) f(\sum_{i=0}^{n} a_i T_i^*(t)) dt \tag{3.14}$$

for suitable collocation points we use roots of shifted Chebyshev polynomials $T_{n+1-\lceil\alpha\rceil}^*(x)$.
Also substituting equation (3.12) in the initial condition (1.1), we have

$$\sum_{i=0}^{n} (-1)^i a_i = 0. \tag{3.15}$$

From equation (3.14) and equation (3.15), we obtain $(n+1)$ system of nonlinear equations in $a_0, a_1, \dots, a_n$ given by

$$\left.\begin{array}{l} F_0(a_0,a_1,\dots,a_n) = 0 \\ F_1(a_0,a_1,\dots,a_n) = 0 \\ \qquad\qquad \vdots \\ F_n(a_0,a_1,\dots,a_n) = 0 \end{array}\right\}, \tag{3.16}$$

which can be solved by using the Newton's iteration method for system of nonlinear equation.

To develop the iterative scheme, the system of nonlinear equation (3.16) can be written in the vector form as $F(a) = 0$, where $a = (a_0, a_1, \dots, a_n)$ and $F = (F_0, F_1, \dots, F_n)$. The Taylor series expansion is

$$F(a^{k+1}) = F(a^k) + (\frac{\partial F}{\partial a})(a^{k+1} - a^k) + \cdots. \tag{3.17}$$

Truncating the Taylor's series following equation is obtained

$$F(a^k) + (\frac{\partial F(a^k)}{\partial a})(a^{k+1} - a^k) = 0, \tag{3.18}$$

which gives

$$a^{k+1} = a^k - \left(\frac{\partial F(a^k)}{\partial a}\right)^{-1} F(a^k) \tag{3.19}$$

provided that the inverse of Jacobian Matrix $\frac{\partial F(a^k)}{\partial a}$ exists. First we solve the equation

$$\frac{\partial F(a^k)}{\partial a} \triangle x = -F(a^k) \tag{3.20}$$

where

$$\triangle x = a^{k+1} - a^k. \tag{3.21}$$

Since $\frac{\partial F(a^k)}{\partial a}$ is a known matrix and $F(a^k)$ is a known vector, the equation (3.20) is just a system of linear equations, which can be solved efficiently and accurately. Once we have the solution vector $\triangle x$, we can obtain improved estimate a^{k+1} by equation (3.21).

4. Solution of NFIDEs by Collocation method with the help of shifted Legendre polynomials

The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the use of the following recurrence formula

$$L_n(z) = \frac{2n+1}{n+1} z L_{n-1}(z) - \frac{n}{n+1} L_{n-2}(z),$$

for $n = 2, 3, \ldots$
with

$$L_0(z) = 1, \quad L_1(z) = z.$$

In order to use these polynomials on the interval $[0,1]$, we define the so called shifted Legendre polynomials by introducing change of variables $z = 2x-1$. The shifted Legendre polynomials $L_n(2x-1)$ be denoted by $L_n^*(x)$. Then $L_n^*(x)$ can be obtained as follows

$$L_n^*(x) = \frac{(2n+1)(2x-1)}{n+1} L_{n-1}^*(x) - \frac{n}{(n+1)} L_{n-2}^*(x), \tag{4.1}$$

for $n = 2, 3, \ldots$
with initial conditions

$$L_0^*(x) = 1, \quad L_1^*(x) = 2x-1. \tag{4.2}$$

The analytic form of shifted Legendre polynomials $L_n^*(x)$ of degree n is given by

$$L_n^*(x) = \sum_{i=0}^{n} (-1)^{n+i} \frac{(n+i)!}{(n-i)((i)!)^2} x^i, \; n = 2, 3, \ldots \tag{4.3}$$

The orthogonality condition is

$$\int_0^1 L_i^*(x) L_j^*(x) dx = \begin{cases} \dfrac{1}{2i+1}, & for \quad i = j; \\ 0, & for \quad i \neq j. \end{cases} \tag{4.4}$$

The function $y(x)$, which is square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i L_i^*(x), \tag{4.5}$$

where the coefficients a_i are given by

$$a_i = (2i+1) \int_0^1 y(x) L_i^*(x) dx, \quad i = 0, 1, 2, \ldots$$

In practice, only the first $(n+1)$ terms of shifted Legendre polynomials are considered. Then we have

$$y_n(x) \cong \sum_{i=0}^{n} a_i L_i^*(x). \tag{4.6}$$

The main approximate formula of the fractional derivative of $y_n(x)$ is given in the following theorem.

Theorem 4.1. *[3] Let $y(x)$ be approximated by shifted Legendre polynomials and suppose $\alpha > 0$*

$$D^\alpha(y_n(x)) = \sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x^{k-\alpha} \tag{4.7}$$

where $w_{i,k}^{(\alpha)}$ *is given by*

$$w_{i,k}^{(\alpha)} = (-1)^{i+k} \frac{(i+k)!\Gamma(k+1)}{(i-k)!(k!)^2\Gamma(k+1-\alpha)} \tag{4.8}$$

The numerical solution of nonlinear fractional integrodifferential equation (1.1) using collocation method with the help of shifted Legendre polynomials is discussed below. This method is based on approximating the unknown function $y(x)$ as

$$y_n(x) \cong \sum_{i=0}^{n} a_i L_i^*(x), \quad 0 \leq x \leq 1 \tag{4.9}$$

where $L_i^*(x)$ is the shifted Legendre polynomial.
Using (4.9) in (1.1), we obtain

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x^{k-\alpha} = g(x) + \int_0^1 K(x,t) f(\sum_{i=0}^{n} a_i L_i^*(t)) dt \tag{4.10}$$

Now we collocate equation (4.10) at $(n+1-\lceil\alpha\rceil)$ points $x_p, \; p = 0, 1, \cdots n - \lceil\alpha\rceil$ as

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} = g(x_p) + \int_0^1 K(x_p,t) f(\sum_{i=0}^{n} a_i L_i^*(t)) dt. \tag{4.11}$$

For suitable collocation points we use roots of shifted Legendre polynomials $L_{n+1-\lceil\alpha\rceil}^*(x)$ and the initial condition (1.1), we obtain $(n+1)$ system of nonlinear equations in $a_0, a_1, \ldots, a_n$. These system of nonlinear equations can be solved by using the Newton's iteration method discussed above section.

5. Applications

In this section, we give some numerical examples of nonlinear fractional integrodifferential equations to illustrate the above results.

Example 5.1. *Consider the following nonlinear fractional integrodifferential equation*

$$D^\alpha y(x) = 1 - \frac{1}{4}x + \int_0^1 xt(y(t))^2 dt, \tag{5.1}$$

$$y(0) = 0. \tag{5.2}$$

where $0 \leq x < 1, \; \alpha = \frac{1}{2}$. *The differential equation (5.1)-(5.2) has the exact solution* $y(x) = x$, *if* $\alpha = 1$.

Method I: Collocation method with the help of shifted Chebyshev polynomials

The suggested method is implemented for $n = 3$ and approximate the solution as follows

$$y_2(x) \cong \sum_{i=0}^{3} a_i T_i^*(x), \quad 0 \leq x \leq 1. \tag{5.3}$$

where $T_i^*(x)$ is the shifted Chebyshev polynomial and $a_i, \; i = 0, 1, 2, 3$ are constants.

An application of Collocation method with the help of shifted Chebyshev polynomial to (5.1)-(5.2), following nonlinear system of equations is obtained,

$$\begin{aligned}
F_0(a_0,a_1,a_2,a_3) = &-0.25a_0^2 - 0.166667a_1a_0\\
&+0.166667a_2a_0 + 0.1a_3a_0\\
&-0.0833333a_1^2 - 0.116667a_2^2\\
&-0.121429a_3^2 + 1.59577a_1\\
&-0.0333333a_1a_2 - 2.12769a_2\\
&+0.1a_1a_3 - 0.0714286a_2a_3\\
&-0.957461a_3 - 0.875\\
F_1(a_0,a_1,a_2,a_3) = &-0.0334936a_0^2 - 0.0223291a_1a_0\\
&+0.0223291a_2a_0 + 0.0133975a_3a_0\\
&-0.0111645a_1^2 - 0.0156304a_2^2\\
&-0.0162683a_3^2 + 0.584092a_1\\
&-0.00446582a_1a_2 - 2.12769a_2\\
&+0.0133975a_1a_3 - 0.00956961a_2a_3\\
&+4.07187a_3 - 0.983253\\
F_2(a_0,a_1,a_2,a_3) = &-0.466506a_0^2 - 0.311004a_1a_0\\
&+0.311004a_2a_0 + 0.186603a_3a_0\\
&-0.155502a_1^2 - 0.217703a_2^2\\
&-0.226589a_3^2 + 2.17986a_1\\
&-0.0622008a_1a_2 + 2.12769a_2\\
&+0.186603a_1a_3 - 0.133288a_2a_3\\
&+3.11441a_3 - 0.766747\\
F_3(a_0,a_1,a_2,a_3) = &\ a_0 - a_1 + a_2 - a_3.
\end{aligned}$$

Using Newton's iteration method for nonlinear system of equations, we obtain

$$\left.\begin{aligned}
a_0 &= 0.832197311166694\\
a_1 &= 0.5989063759298973\\
a_2 &= -0.14392391710534502\\
a_3 &= 0.08936701813145172,
\end{aligned}\right\} \tag{5.4}$$

making use of the (5.4) into (5.3), following solution is obtained

$$\begin{aligned}
y(x) = &\ 0.83219731 + 0.5989063759298973(2x-1)\\
&-0.14392391(8x^2-8x+1)\\
&+0.089367(4(2x-1)^3 - 3(2x-1)),
\end{aligned} \tag{5.5}$$

which is approximate solution of (5.1)-(5.2).
In Figure 1, we plot the approximate solution obtained by Method I and the exact solution for Example 5.1.

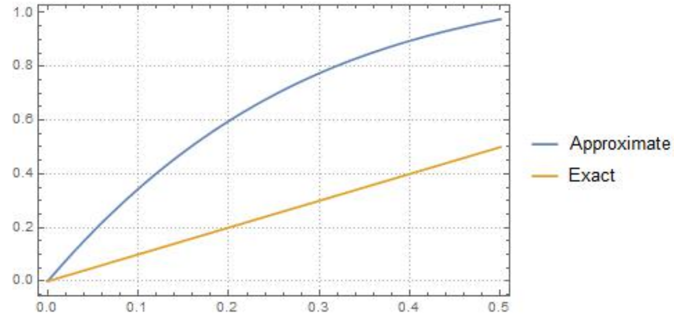

Figure 1. Approximate and Exact solution of Example 5.1 by Method I

Method II: Collocation method with the help of shifted Legendre polynomials

The suggested method is implemented for $n = 3$ and approximate the solution as follows

$$y_2(x) \cong \sum_{i=0}^{3} a_i L_i^*(x), \quad 0 \le x \le 1. \tag{5.6}$$

where $L_i^*(x)$ is the shifted Legendre polynomial and $a_i,\ i = 0,1,2,3$ are constants.
An application of Collocation method with the help of shifted Legendre polynomials to (5.1)-(5.2), following nonlinear system of equations is obtained,

$$\begin{aligned}
F_0(a_0,a_1,a_2,a_3) = &-0.25a_0^2 - 0.166667a_1a_0\\
&-0.0833333a_1^2 - 0.05a_2^2\\
&-0.0357143a_3^2 + 1.59577a_1\\
&-0.0666667a_1a_2 - 1.59577a_2\\
&-0.0428571a_2a_3 + 1.88411095 * 10^{-15}a_3\\
&-0.875\\
F_1(a_0,a_1,a_2,a_3) = &-0.0563508a_0^2 - 0.0375672a_1a_0\\
&-0.0187836a_1^2 - 0.0112702a_2^2\\
&-0.00805012a_3^2 + 0.757618a_1\\
&-0.0150269a_1a_2 - 1.93131a_2\\
&-0.00966a_2a_3 + 2.99198a_3 - 0.971825\\
F_2(a_0,a_1,a_2,a_3) = &-0.443649a_0^2 - 0.295766a_1a_0\\
&-0.147883a_1^2 - 0.0887298a_2^2\\
&-0.0633785a_3^2 + 2.12579a_1\\
&-0.118306a_1a_2 + 1.16747a_2\\
&-0.076054a_2a_3 + 1.8086a_3 - 0.778175\\
F_3(a_0,a_1,a_2,a_3) = &\ a_0 - a_1 + a_2 - a_3
\end{aligned}$$

Using Newton's iteration method for nonlinear system of equations, we obtain

$$\left.\begin{aligned}
a_0 &= 0.7765706927568846\\
a_1 &= 0.5681834659041213\\
a_2 &= -0.12918879599732455\\
a_3 &= 0.07919843085543878
\end{aligned}\right\} \tag{5.7}$$

making use of (5.7) into (5.6), following solution is obtained

$$\begin{aligned} y(x) &= 0.776571 + 0.568183(2x-1) \\ &\quad - 0.129189(6x^2 - 6x + 1) \\ &\quad + 0.0791984(20x^3 - 30x^2 + 12x - 1) \end{aligned} \tag{5.8}$$

which is approximate solution of (5.1)-(5.2).
In Figure 2, we plot the approximate solution obtained by Method II and the exact solution for Example 5.1.

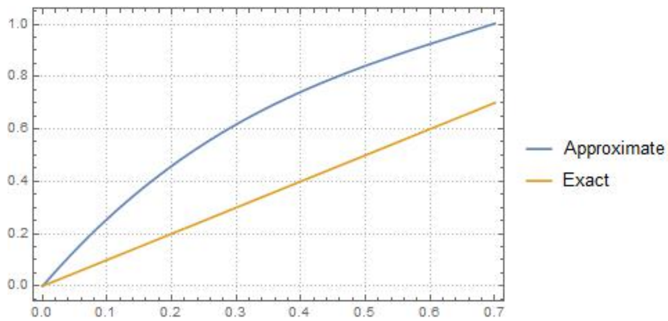

Figure 2. Approximate and Exact solution of Example 5.1 by Method II

Example 5.2. *Consider the following nonlinear fractional integrodifferential equation*

$$D^{\alpha} y(x) = \frac{\frac{8x^{3/2}}{3} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{1260}x + \int_0^1 xt(y(t))^4 dt, \tag{5.9}$$

$$y(0) = 0. \tag{5.10}$$

where $0 \le x < 1,\ \alpha = \frac{1}{2}$. *The differential equation (5.9)-(5.10) has the exact solution* $y(x) = x^2 - x$, *if* $\alpha = 1$.

Method I: Collocation method with the help of shifted Chebyshev polynomials

Similarly as in Example 5.1 applying the Collocation method with the help of shifted Chebyshev polynomial to (5.9)-(5.10), we obtain,

$$\left.\begin{aligned} a_0 &= -0.12500000268834607 \\ a_1 &= -3.2462632438396387 * 10^{-9} \\ a_2 &= 0.12499999951591143 \\ a_3 &= 7.382860772130038 * 10^{-11} \end{aligned}\right\} \tag{5.11}$$

Hence the approximate solution of (5.9)-(5.10) is

$$\begin{aligned} y(x) &= -0.125 - 3.24626 * 10^{-9}(2x-1) \\ &\quad + 0.125(8x^2 - 8x + 1) \\ &\quad + 7.38286 * 10^{-11}(4(2x-1)^3 - 3(2x-1)). \end{aligned} \tag{5.12}$$

In Figure 3, we plot the approximate solution obtained by Method I and the exact solution for Example 5.2.

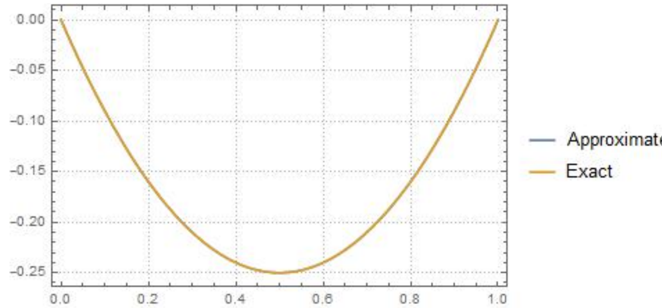

Figure 3. Approximate and Exact solution of Example 5.2 by Method I

Method II: Collocation method with the help of shifted Legendre polynomials

Similarly as in Example 5.1 applying the Collocation method with the help of shifted Legendre polynomial to (5.9)-(5.10), we obtained,

$$\left.\begin{aligned} a_0 &= -0.16667380012560998 \\ a_1 &= 3.971261224148561 * 10^{-6} \\ a_2 &= 0.16667060752789342 \\ a_3 &= -7.16385894071834 * 10^{-6} \end{aligned}\right\} \tag{5.13}$$

Hence the approximate solution of (5.9)-(5.10) is

$$\begin{aligned} y(x) &= -0.166674 + 3.97126 * 10^{-6}(2x-1) \\ &\quad + 0.166671(6x^2 - 6x + 1) \\ &\quad - 7.16386 * 10^{-6}(20x^3 - 30x^2 + 12x - 1). \end{aligned} \tag{5.14}$$

In Figure 4, we plot the approximate solution obtained by Method II and the exact solution for Example 5.2.

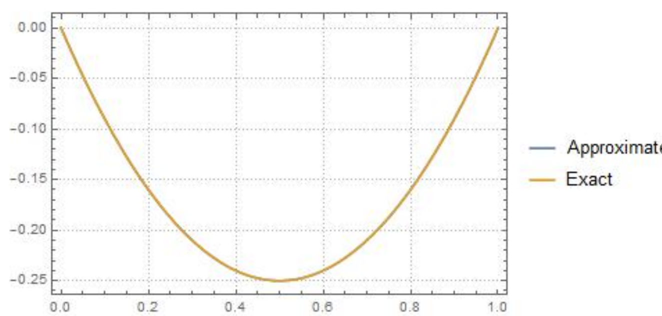

Figure 4. Approximate and Exact solution of Example 5.2 by Method II

6. Conclusion

The Collocation method is implemented with the help of shifted Chebyshev polynomials and shifted Legendre polynomials for solving nonlinear fractional integrodifferential equation. The fractional derivatives are considered in the Caputo sense. This method derives a good approximation and reliable techniques to handle nonlinear fractional integrodifferential equations. The properties of Chebyshev polynomials and Legendre polynomials are used to reduce nonlinear fractional integrodifferential equation to the solution of system of algebraic equations. The solution obtained using this method is in excellent agreement with the exact solution and show that this method is effective.

All numerical results are obtained using Mathematica 11.

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