MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 47–54.
http://doi.org/10.26637/mjm1001/004

Certain subclasses of Pseudo-type meromorphic bi-univalent functions

ADNAN GHAZY ALAMOUSH[*1]

[1] *Faculty of Science, Department of Mathematics, Taibah University, Madinah, Saudi Arabia.*

***R**eceived 26 September 2021; **A**ccepted 19 December 2021*

Abstract. In the present article, we define a new subclasses of pseudo-type meromorphic bi-univalent functions defined on $\triangle = \{z \mid : z \in \mathbb{C}\ and\ \ 1 < |z| < \infty\}$, and investigate the initial coefficient estimates $|b_0|$ and $|b_1|$. Further, several earlier results are also indicated.

AMS Subject Classifications: 30C50, 33C05.

Keywords: **Keywords:** Analytic functions, Univalent functions, Meromorphic univalent functions, Bi-Univalent functions, Meromorphic Bi-Univalent functions, Pseudo functions.

Contents

1. Introduction

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit open disk $U = \{z : z \in C, |z| < 1\}$. Also, we detote S be class of all functions in A which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in U. Let Σ' denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{1.2}$$

defined on the domain $\triangle = \{z : z \in U, 1 < |z| < \infty\}$. Since the function $g \in \Sigma'$ is univalent, then it has an inverse $g^{-1} = h$, defined by

$$g^{-1}(g(z)) = z \ \ (z \in \triangle),$$

*Corresponding author. Email address: **adnan_omoush@yahoo.com** (Adnan Ghazy ALAMOUSH)

and

$$g^{-1}(g(w)) = w \ \ (M < |w| < \infty,\ M > 0),$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_1 b_0 + b_2}{w^2} - \frac{b_1^2 + b_1 b_0^2 + 2b_0 b_2 + b_3}{w^3} + \cdots . \tag{1.3}$$

By a simple calculations, we have:

$$w = g(h(w)) = (b_0+B_0)+w+\frac{b_1 + B_1}{w}+\frac{B_2 - b_1 B_0 + b_2}{w^2}+\frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3}+\cdots . \tag{1.4}$$

Comparing the initial coefficients in (1.4), we find that

$$\begin{aligned} b_0 + B_0 = 0 \qquad & \Rightarrow \ B_0 = -b_0 \\ b_1 + B_1 = 0 \qquad & \Rightarrow \ B_1 = -b_1 \\ B_2 - b_1 B_0 + b_2 = 0 \qquad & \Rightarrow B_2 = -(b_2 + b_1 b_0) \\ B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0 \qquad & \Rightarrow B_3 = -(b_3 + 2b_0 b_1 + b_1 b_0^2 + b_1^2). \end{aligned}$$

A function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$, and the family of all meromorphic bi-univalent functions is denoted by Σ'_*. The coefficient problem was widely investigated for various interesting subclasses of the meromorphic univalent functions; for example, Schiffer [19] obtained the estimate $|b_2| < \frac{3}{2}$ for meromorphic univalent functions $f \in S$ with $b_0 = 0$. In 1983, Duren [20] obtained the inequality $|b_2| < \frac{2}{n+1}$ for $f \in S$ with $b_k = 0, 1 \leq k \leq \frac{n}{2}$.

For the coefficients of inverses of meromorphic univalent functions, Springer [16] showed that

$$|B_3| < 1 \text{ and } |B_3 + \frac{1}{2}B_1^2| < \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}\ (n = 1, 2, \cdots).$$

In 1977, Kubota [22] has proved that the Springer conjecture is true for n = 3; 4; 5. Furthermore, for $h \in \Sigma'$, Schober [15] obtained sharp bounds for $|B_{2n-1}|$ if $1 \leq n \leq 7$.

Recently, Some several researcher such as (for example [1], [2], [3], [4] [5], [6], [7], [8], [9], [10], [11], [12],[13], [14], [17], [21]) introduced new subclasses of bi-univalent functions and meromorphically bi-univalent functions and obtained estimates on the initial coefficients for functions in each of these subclasses.

In 2013, Babalola [18] introduced a new subclass λ-pseudo starlike function of order $0 \leq \beta < 1$ satisfying the analytic condition

$$\Re\left\{\frac{z(f(z)')^{\lambda}}{f(z)}\right\} > \beta\ (\lambda \geq 1,\ z \in U). \tag{1.5}$$

In particular, Babalola [18] proved that all λ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in open unit disk U.

In the present paper, we introduce two new subclasses of pseudo-type of meromorphically bi-univalent functions and obtained the estimates for the initial coefficients $|b_0|$ and $|b_1|$ of functions in these subclasses. Several some consequences of the new results are also pointed out.

2. Coefficient Bounds for the Function Class $\Sigma'_*(h,p,\lambda)$

We begin by introducing the function class $\Sigma'(h,p,\lambda)$ by means of the following definition.

Definition 2.1. *Let the functions $h;p:\triangle\to C$ be analytic functions and*

$$h(z)=1+\frac{h_1}{z}+\frac{h_2}{z^2}+\frac{h_3}{z^3}+\cdots,\ \ p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^3}+\cdots,$$

such that

$$\min\{\Re(h(z)),\Re(p(z))\}>0, z\in\triangle.$$

A function $g(z)\in\Sigma'$ given by (1.2) is said to be in the class $\Sigma'_*(h,p,\lambda)$ if the following conditions are satisfied:

$$g\in\Sigma' \text{ and } \frac{z(g(z)')^{\lambda}}{g(z)}\in h(\triangle),(\ \lambda\geq 1,\ z\in\triangle),\tag{2.1}$$

and

$$\frac{w(h(w)')^{\lambda}}{h(w)}\in p(\triangle),(\ \lambda\geq 1,\ w\in\triangle),\tag{2.2}$$

where the function h is given by (1.3).

Theorem 2.2. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(h,p,\lambda)$. Then*

$$|b_0|\leq\min\left\{\sqrt{\frac{|h_1|^2+|p_1|^2}{2}},\sqrt{\frac{|h_2|+|p_2|}{2}}\right\}\tag{2.3}$$

and

$$|b_1|\leq\min\left\{\frac{|h_2|+|p_2|}{2|\lambda+1|},\frac{1}{\lambda+1}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}}+\frac{(|h_1|^2+|p_1|^2)^2}{4}\right)\right\}.\tag{2.4}$$

Proof. Let $g\in\Sigma'_*(h,p,\lambda)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $\Sigma'_*(h,p,\lambda)$, the conditions (2.1) and (2.2) can be rewritten as follows:

$$\frac{z(g(z)')^{\lambda}}{g(z)}=h(z)\tag{2.5}$$

and

$$\frac{w(h(w)')^{\lambda}}{h(w)}=p(w),\tag{2.6}$$

respectively. Here, and in what follows,the functions $h(z)\in P$ and $p(w)\in P$ have the following forms:

$$p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\frac{p_3}{z^3}+...\quad(z\in\triangle)\tag{2.7}$$

and

$$q(w)=1+\frac{q_1}{w}+\frac{q_2}{w^2}+\frac{q_3}{w^3}+...\quad(w\in\triangle).\tag{2.8}$$

Clearly, we have

$$\frac{z(g(z)')^{\lambda}}{g(z)}=1-\frac{b_0}{z}+\frac{b_0^2-(1+\lambda)b_1}{z^2}+\frac{b_0^3-(2\lambda)b_0b_1+(1+2\lambda)b_2}{z^3}+\cdots\tag{2.9}$$

and

$$\frac{w(h(w)')^{\lambda}}{h(w)} = 1 + \frac{b_0}{w} + \frac{b_0^2 + (1+\lambda))b_1}{w^2} + \frac{b_0^3 + (3(1+\lambda)b_0 b_1 + (1+2\lambda))b_2}{w^3} + \cdots . \tag{2.10}$$

Now, equating the Coefficients in (2.5) and (2.6), we get

$$-b_0 = h_1 \tag{2.11}$$

$$b_0^2 - (1+\lambda)b_1 = h_2 \tag{2.12}$$

$$b_0 = p_1 \tag{2.13}$$

$$b_0^2 + (1+\lambda)b_1 = p_2. \tag{2.14}$$

From (2.11) and (2.13), we find that

$$p_1 = -q_1 \tag{2.15}$$

and

$$2b_0^2 = h_1^2 + p_1^2 \tag{2.16}$$

that is,

$$|b_0|^2 \leq \frac{|h_1|^2 + |p_1|^2}{2}. \tag{2.17}$$

Adding (2.12) and (2.14), we get

$$2b_0^2 = h_2 + p_2 \tag{2.18}$$

that is,

$$|b_0|^2 \leq \frac{|h_2| + |p_2|}{2}. \tag{2.19}$$

From (2.18) and (2.19) we get the desired estimate on the coefficient $|b_0|$ as asserted in (2.3).

Next, in order to find the bound on $|b_0|$, by subtracting the equation (2.12) from the equation (2.14), we get

$$2(1+\lambda)b_1 = p_2 - h_2, \tag{2.20}$$

that is,

$$|b_1| \leq \frac{|h_2| + |p_2|}{|2(1+\lambda)|}. \tag{2.21}$$

By squaring and adding (2.12) and (2.14), using (2.18) in the computation leads to

$$b_1^2 = \frac{1}{(1+\lambda)^2}\left(\frac{h_2^2 + p_2^2}{2} - \frac{[h_1^2 + p_1^2]^2}{4}\right). \tag{2.22}$$

that is,

$$|b_1| \leq \frac{1}{1+\lambda}\left(\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{(|h_1|^2 + |p_1|^2)^2}{4}}\right). \tag{2.23}$$

From (2.21) and (2.23) we get the desired estimate on the coefficient $|b_1|$ as asserted in (2.4). ■

Remark 2.3. *If we take*

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots, \ (0 < \alpha \leq 1, \ z \in \triangle),$$

and

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\mu)}{z} + \frac{2(1-\mu)}{z^2}, \ (0 < \mu \leq 1, \ z \in \triangle),$$

respectively, in the Theorem 2.2, we obtain the following results which is an improvement of estimates obtained by Srivastava el. at [17].

Corollary 2.4. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(\lambda, \alpha)$. Then*

$$|b_0| \leq 2\alpha \tag{2.24}$$

and

$$|b_1| \leq \frac{2\sqrt{5}\alpha^2}{\lambda+1}. \tag{2.25}$$

Corollary 2.5. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(\lambda, \mu, \alpha)$. Then*

$$|b_0| \leq 2(1-\mu) \tag{2.26}$$

and

$$|b_1| \leq \frac{2(1-\mu)\sqrt{4\mu^2 - 8\mu + 5\alpha^2}}{\lambda+1}. \tag{2.27}$$

3. Coefficient Bounds for the Function Class $\Sigma'_*(h, p, \lambda, \beta)$

We first introduce the function class $\Sigma'_*(h, p, \lambda, \beta)$ as follows.

Definition 3.1. *Let the functions $h; p : \triangle \to C$ be analytic functions and*

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \;\; p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,$$

such that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0, z \in \triangle.$$

A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\Sigma'_*(h, p, \lambda, \beta)$ if the following conditions are satisfied:

$$g \in \Sigma' \text{ and } (1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda} + \beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right) \in h(\triangle),\; (0 < \beta \leq 1,\; \lambda \geq 1, z \in U^*), \tag{3.1}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda} + \beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right) \in p(\triangle)(0 < \beta \leq 1,\; \lambda \geq 1,\; w \in U^*), \tag{3.2}$$

where the function h is given by (1.3).

Next, we now derive the estimates on the Coefficients $|b_0|$ and $|b_1|$ for the meromorphically bi univalent function class $\Sigma'_{\lambda,\beta}(\mu)$.

Theorem 3.2. *Let $g(z)$ be given by (1.2) be in the class $\Sigma'_*(h, p, \lambda, \beta)$. Then*

$$|b_0| \leq \min\left\{\sqrt{\frac{|h_1|^2 + |p_1|^2}{2(\lambda - \lambda\beta - \beta)^2}}, \sqrt{\frac{|h_1| + |p_1|}{|\lambda(\lambda-1)(1-\beta) + 2\beta|}}\right\} \tag{3.3}$$

and

$$|b_1| \leq \min$$

$$\left\{\frac{|p_2| + |h_2|}{|2(\beta - \lambda + 2\lambda\beta)|}, \frac{1}{|\lambda - \beta - 2\lambda\beta|}\left(\sqrt{\frac{|h_2|^2 + |p_2|^2}{2} + \frac{[(\lambda(\lambda-1)(1-\beta) + 2\beta)]^2(|h_1|^2 + |p_1|^2)^2}{16(\lambda - \lambda\beta - \beta)^4}}\right)\right\}. \tag{3.4}$$

Proof. Let $g \in \Sigma'_*(h,p,\lambda,\beta)$. Then, by Definition 3.1 of meromorphically bi-univalent function class $\Sigma'_*(h,p,\lambda,\beta)$, the conditions (3.1) and (3.2) can be rewritten as follows:

$$(1-\beta)\left(\frac{g(z)}{z}\right)^{\lambda}+\beta\left(\frac{z(g(z)')^{\lambda}}{g(z)}\right)=h(z) \tag{3.5}$$

and

$$(1-\beta)\left(\frac{h(w)}{w}\right)^{\lambda}+\beta\left(\frac{w(h(w)')^{\lambda}}{h(w)}\right)=p(w), \tag{3.6}$$

respectively. Here, just as in our proof of Theorem 2.1, with the functions $p(z) \in P$ and $q(w) \in P$ have the forms in (2.7) and (2.7), and comparing the corresponding coefficients in (3.5) and (3.6), we have

$$(\lambda-\lambda\beta-\beta)b_0=h_1 \tag{3.7}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\lambda-\beta-2\lambda\beta)b_1=h_2 \tag{3.8}$$

$$-(\lambda-\lambda\beta-\beta)b_0=p_1 \tag{3.9}$$

$$\frac{1}{2}(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2+(\beta-\lambda+2\lambda\beta)b_1=p_2. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$h_1=-p_1 \tag{3.11}$$

and

$$2(\lambda-\lambda\beta-\beta)^2b_0^2=h_1^2+p_1^2 \tag{3.12}$$

that is,

$$|b_0|^2 \le \frac{|h_1|^2+|p_1|^2}{2(\lambda-\lambda\beta-\beta)^2}. \tag{3.13}$$

From (3.8) and (3.10), we get

$$(\lambda(\lambda-1)(1-\beta)+2\beta)b_0^2=h_1+p_1, \tag{3.14}$$

that is,

$$|b_0|^2 \le \frac{|h_1|+|p_1|}{|\lambda(\lambda-1)(1-\beta)+2\beta|}. \tag{3.15}$$

From (3.13) and (3.15) we get the desired estimate on the coefficient $|b_0|$ as asserted in (3.3).
Next, in order to find the bound on $|b_1|$, by subtracting the equation (3.8) from the equation (3.10), we get

$$2(\beta-\lambda+2\lambda\beta)b_1=p_2-h_2, \tag{3.16}$$

that is,

$$|b_1| \le \frac{|p_2|+|h_2|}{|2(\beta-\lambda+2\lambda\beta)|}. \tag{3.17}$$

By squaring and adding (3.8) and (3.10), using (3.14) in the computation leads to

$$b_1^2=\frac{1}{(\lambda-\beta-2\lambda\beta)^2}\left(\frac{h_2^2+p_2^2}{2}-\frac{[(\lambda(\lambda-1)(1-\beta)+2\beta)]^2[h_1^2+p_1^2]^2}{16(\lambda-\lambda\beta-\beta)^4}\right), \tag{3.18}$$

that is,

$$|b_1| \le \frac{1}{|\lambda-\beta-2\lambda\beta|}\left(\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{([(\lambda(\lambda-1)(1-\beta)+2\beta)]^2(|h_1|^2+|p_1|^2)^2}{16(\lambda-\lambda\beta-\beta)^4}}\right). \tag{3.19}$$

From (3.17) and (3.19) we get the desired estimate on the coefficient $|b_1|$ as asserted in (3.4). ■

Future Work: For function $g \in \Sigma'_*(\lambda,\beta,\phi)$ given by (1.2) by taking $\phi = h(z) = p(z)$ as in Remark 2.3 or ($\phi = \frac{1+Az}{1+Bz}$ $-1 \le B < A \le 1$), we can obtain the initial coefficient estimates $|b_0|$ and $|b_1|$ by routine procedure (as in Theorem 2.2) and so we omit the details.

4. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] A. A. AMOURAH, A. G. AL AMOUSH AND M. AL-KASEASBEH, Gegenbauer Polynomials and Bi-univalent Functions, *Palestine Journal of Mathematics*, **10(2)**(2021) , 625–632.

[2] A. A. AMOURAH, Faber polynomial coefficient estimates for a class of analytic bi-univalent functions, *AIP Conference Proceedings*, 2096(1)(2019), 020024.

[3] A. G. ALAMOUSH, Coefficient Estimates for New Subclass of Pseudo-Type Meromorphic Bi-Univalent Functions, *Italian Journal of Pure and Applied Mathematics*, (46), accepted.

[4] A. G. ALAMOUSH, A subclass of pseudo-type meromorphic bi-univalent functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **69(2)**(2020), 1025-1032.

[5] A. G. ALAMOUSH, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials, *Malaya Journal of Matematik*, 7(2018), 618–624.

[6] A. G. ALAMOUSH, Coefficient Estimates for Certain Subclass of Bi-Bazilevic Functions Associated With Chebyshev Polynomials, *Acta Universitatis Apulensis*, **60**(2019), 53–59.

[7] A. G. ALAMOUSH, Coefficient estimates for a new subclasses of lambda-pseudo bi-univalent functions with respect to symmetrical points associated with the Horadam Polynomials, *Turkish Journal of Mathematics*, **3**, 2865–2875.

[8] A. G. ALAMOUSH, On subclass of analytic bi-close-to-convex functions, *Int. J. Open Problems Complex Analysis*, **13(1)**(2021), 10–18.

[9] A. G. ALAMOUSH, On a subclass of bi-univalent functions associated to Horadam polynomials, *International Journal of Open Problems in Complex Analysis*, **12(1)**(2020), 58–56.

[10] A. G. ALAMOUSH AND M. DARUS, Coefficient bounds for new subclasses of bi-univalent functions using Hadamard product, *Acta Universitatis Apulensis*, (2014), 153–161.

[11] A. G. ALAMOUSH AND M. DARUS, Coefficients estimates for bi-univalent of fox-wright functions, *Far East Journal of Mathematical Sciences*, (2014), 249–262.

[12] A. G. ALAMOUSH AND M. DARUS, Faber polynomial Coefficients estimates for a new subclass of meromorphic bi-univalent functions, *Advances in Inequalities and Applications*, (2016), 2016:3.

[13] A. G. ALAMOUSH AND M. DARUS, On coefficient estimates for new generalized subclasses of bi-univalent functions, *AIP Conference Proceedings*, 1614(2014), 844.

[14] G. P. KAPOOR AND A.K. MISHRA, Coefficients estimates for inverses of starlike functions of positive order, *J. Math. Anal. Appl.*, **329(2)**(2007), 922–934.

[15] G. SCHOBE, Coefficients of inverses of meromorphic univalent functions, *Proc. Amer. Math. Soc.*, **67(1)**(1977), 111–116.

[16] G. SPRINGER, The Coefficients problem for schlicht mappings of the exterior of the unit circle, *Trans. Amer. Math. Soc.*, **70**(1951), 421–450.

[17] H. M. SRIVASTAVA, B. SANTOSH JOSHI, S. SAYALI JOSHI, H. PAWAR, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Pal. Jour. Math.*, **5**(2016), 250–258.

[18] K. O. BABALOLA, On λ-pseudo-starlike functions, *Jour. Class. Anal.*, **3**(2013), 137–147.

[19] M. SCHIFFER, On an extremum problem of conformal representation, *Bull. de la Soc. Math. de Fra.*, **66**(1938), 48–55.

[20] P. L. DUREN, Coefficients of meromorphic schlicht functions, *Proc. Amer. Math. Soc.*, **28**(1971), 169–172.

[21] S. G. HAMIDI , S.A. HALIM, J.M. JAHANGIRI, Coefficients estimates for a class of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris Sér. I*, **351**(2013), 349–352.

[22] Y. KUBOTA, Coefficients of meromorphic univalent functions, *Kod. Math. Sem. Rep.*, **28(2-3)**(1977), 253–261.