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Blow up of the solutions for the Petrovsky equation with fractional damping terms

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Abstract
In this paper, we prove a blow up result for solutions of the Petrovsky equation with fractional damping term with negative initial energy.

Keywords
Blow up, Petrovsky equation, Fractional damping term.

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Contents

1. Introduction

In this paper, we investigate the following Petrovsky equation with fractional damping terms

$$\begin{cases} u_{tt}+\Delta^2 u+\partial_t^{1+\alpha}u=|u|^{p-1}u, & x\in\Omega,\ t>0,\\ u(x,t)=\frac{\partial u(x,t)}{\partial \nu}=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), x\in\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, ν is the outer normal. The constants $p>1$,and $-1<\alpha<1$. The notation $\partial_t^{1+\alpha}$ stands for the Caputo's fractional derivative of order $1+\alpha$ with respect to the time variable [6, 8]. It is defined as follows

$$\partial_t^{1+\alpha}w(t)=\begin{cases} I^{-\alpha}\frac{d}{dt}w(t) & \text{for } -1<\alpha<0\\ I^{1-\alpha}\frac{d^2}{dt^2}w(t) & \text{for } \ \ 0<\alpha<1\end{cases}$$

where $I^{\beta},\beta>0$ is fractional integral

$$I^{\beta}\frac{d}{dt}w(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}w(\tau)d\tau.$$

When $-1<\alpha<0$, the term $\partial_t^{1+\alpha}u$ is said to be fractional damping. Also, $\alpha=-1$ and $\alpha=0$ the term $\partial_t^{1+\alpha}u$ is said to be weak damping and strong damping term, respectively. The fractional damping term plays a dissipative role, which is stronger than weak damping and weaker than strong damping [3].

Messaoudi [5] studied the local existence and blow up of the solution to the equation

$$u_{tt}+\Delta^2 u+|u_t|^{q-1}u_t=|u|^{p-1}u. \tag{1.2}$$

Wu and Tsai [11] obtained global existence and blow up of the solution of the problem (1.2). Later, Chen and Zhou [2] studied blow up of the solution of the problem (1.2) for positive initial energy.

Li et al. [4] studied global existence and blow up of the solution to the equation

$$u_{tt}+\Delta^2 u-\Delta u+|u_t|^{q-1}u_t=|u|^{p-1}u. \tag{1.3}$$

Recently, Pişkin and Polat [7] proved decay of solution of problem (1.3).

Tatar [9] studied exponential growth of the solution to the wave equation with fractional damping

$$u_{tt}-\Delta u+\partial_t^{1+\alpha}u=|u|^{p-1}u. \tag{1.4}$$

Also, he [1, 10] studied blow up of the solution to the equation (1.4).

In this paper, we establish the blow up of the solution with negative initial energy by using the technique of [1].

2. Preliminaries

In this section, we present some materials needed for our main results. Furthermore we will consider only the case

$-1 < \alpha < 0$. The classical energy functional associated to problem (1.1) is

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p+1}\|u\|^{p+1} \tag{2.1}$$

In deriative of (2.1), we have

$$E'(t) = -\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx.$$

Is obtained. Now, if we define our modified energy function as

$$E_\varepsilon(t) = E(t) - \varepsilon\int_\Omega uu_t dx \tag{2.2}$$

where $0 < \varepsilon < 1$ and to be determined later. (2.1) replace and a differation of (2.2) with respect to t yields

$$\begin{aligned} E'_\varepsilon(t) &= -\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \\ &\quad -\varepsilon\int_\Omega |u_t|^2 dx - \varepsilon\int_\Omega |u|^{p+1}dx - \varepsilon\int_\Omega |\Delta u|^2 dx \\ &\quad +\frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau)d\tau dx \end{aligned} \tag{2.3}$$

Also, we define the following functions for use in our theorem,

$$H(t) = -\left(e^{-\sigma\varepsilon t}E_\varepsilon(t) + \mu F(t) + d\right), \tag{2.4}$$

$$F(t) = \int_0^t \int_\Omega G(t-\tau)e^{-\sigma\varepsilon\tau}u_\tau^2 dx d\tau, \tag{2.5}$$

and

$$G(t) = e^{\beta t}\int_t^\infty e^{-\beta\tau}\tau^{-(\alpha+1)}d\tau \tag{2.6}$$

where $\sigma = \frac{p+1}{2}$ and β, μ, d are positive consants.

Lemma 2.1. *If $E_\varepsilon(0) < 0$ and p is sufcifiently large, then $H(t) > 0$ and $H'(t) > 0$.*

Proof. By taking a derivative of (2.4) and (2.5), we have

$$H'(t) = \sigma\varepsilon e^{-\sigma\varepsilon t}E_\varepsilon(t) - e^{-\sigma\varepsilon t}E'_\varepsilon(t) - \mu F'(t), \tag{2.7}$$

$$\begin{aligned} F'(t) &= \beta^\alpha\Gamma(-\alpha)e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx \\ &\quad -\int_0^t \int_\Omega (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau}u_\tau^2 dx d\tau + \beta F(t) \end{aligned} \tag{2.8}$$

Substituting (2.8), (2.2) and (2.3) into (2.7), we obtain

$$\begin{aligned} H'(t) &= \left[\frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2} - \mu\beta^\alpha\Gamma(-\alpha)\right]e^{-\sigma\varepsilon t}.\int_\Omega u_t^2 dx \\ &\quad +\left(\frac{\sigma}{2}+1\right)\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx \\ &\quad -\sigma\varepsilon^2 e^{-\sigma\varepsilon t}.\int_\Omega uu_t dx \\ &\quad +\left(\varepsilon - \frac{\sigma\varepsilon}{p+1}\right)e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx \\ &\quad +\frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &\quad -\frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &\quad +\mu\int_0^t\int_\Omega (t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2 dx d\tau - \mu\beta F(t) \end{aligned} \tag{2.9}$$

For the fifth term on the right side of (2.9), using Young inequality, we get

$$\begin{aligned} & e^{-\sigma\varepsilon t}\int_\Omega u_t\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &= e^{-\sigma\varepsilon t}\int_\Omega u_t dx\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &\le \delta_1 e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx \\ &\quad +\frac{1}{4\delta_1}e^{-\sigma\varepsilon t}\int_\Omega\left[\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau\right]^2 dx. \end{aligned}$$

Writing $-(\alpha+1) = -\frac{\alpha+1}{2} - \frac{\alpha+1}{2}$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & e^{-\sigma\varepsilon t}.\int_\Omega u_t\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx \\ &\le \delta_1 e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx + \\ &\quad \frac{(\sigma\varepsilon)^\alpha\Gamma(-\alpha)}{4\delta_1}\int_\Omega\int_0^t \frac{e^{-\sigma\varepsilon\tau}u_\tau^2}{(t-\tau)^{(\alpha+1)}}d\tau dx. \end{aligned} \tag{2.10}$$

Smilarly, we have

$$\int_\Omega u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx$$
$$\leq \delta_2 \int_\Omega |u|^2 dx + \frac{1}{4\delta_2} \int_\Omega \left(\int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau \right)^2 dx$$

$$e^{-\sigma\varepsilon t} \cdot \int_\Omega u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx$$
$$\leq \delta_2 e^{-\sigma\varepsilon t} \cdot \int_\Omega |u|^2 dx$$
$$+ \frac{1}{4\delta_2} e^{-\sigma\varepsilon t} \int_\Omega \left(\int_0^t (t-\tau)^{-(\alpha+1)} e^{\frac{-\sigma\varepsilon t}{2}} u_\tau(\tau) d\tau \right)^2 dx,$$

$$e^{-\sigma\varepsilon t} \int_\Omega u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx$$
$$\leq \delta_2 e^{-\sigma\varepsilon t} C_* \int_\Omega |\nabla u|^2 dx$$
$$+ \frac{(\sigma\varepsilon)^\alpha \Gamma(-\alpha)}{4\delta_2} \int_\Omega \int_0^t \frac{e^{-\sigma\varepsilon\tau} u_\tau^2}{(t-\tau)^{(\alpha+1)}} d\tau dx \quad (2.11)$$

Used Sobolev-Poincare inequality.

For the third term on the right side of (2.9), using the Young and Sobolev-Poincare inequalities, we get

$$\begin{aligned} \int_\Omega u u_t dx &\leq \delta_3 \int_\Omega |u|^2 dx + \frac{1}{4\delta_3} \int_\Omega |u_t|^2 dx \\ &\leq \delta_3 C_* \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta_3} \int_\Omega |u_t|^2 dx \quad (2.12) \end{aligned}$$

By (2.9), (2.10), (2.11) and (2.12), it yields

$$H'(t)$$
$$\geq \left(\frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2} - \mu^\alpha \Gamma(-\alpha) - \frac{\sigma\varepsilon^2}{4\delta_3} - \frac{\delta_1}{\Gamma(-\alpha)} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega u_t^2 dx$$
$$+ \left(\frac{\sigma}{2} + 1 \right) \varepsilon e^{-\sigma\varepsilon t} \cdot \int_\Omega |\Delta u|^2 dx$$
$$- \left(\sigma\varepsilon^2 \delta_3 C_{p_1} + \frac{\varepsilon \delta_2 C_{p_1}}{\Gamma(-\alpha)} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |\nabla u|^2 dx$$
$$+ \left(\varepsilon - \frac{\sigma\varepsilon}{p+1} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |u|^{p+1} dx$$
$$+ \left(\mu - \frac{(\sigma\varepsilon)^\alpha}{4\delta_1} - \frac{(\sigma\varepsilon)^\alpha \varepsilon}{4\delta_2} \right) \int_\Omega \int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx$$
$$- \mu\beta F(t).$$

Adding and subtracting $C_1 H(t)$, we get

$$H'(t)$$
$$\geq C_1 H(t) + \left[\frac{C_1}{2} + \frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2} \right.$$
$$\left. - \mu\beta^\alpha \Gamma(-\alpha) - \frac{\sigma\varepsilon^2}{4\delta_3} - \frac{\delta_1}{\Gamma(-\alpha)} \right] e^{-\sigma\varepsilon t} \int_\Omega u_t^2 dx$$
$$+ \left[\left(\frac{\sigma}{2} + 1 \right) \varepsilon + \frac{C_1}{2} \right] e^{-\sigma\varepsilon t} \cdot \int_\Omega |\Delta u|^2 dx$$
$$- \left(\sigma\varepsilon^2 \delta_3 C_{p_1} + \frac{\varepsilon \delta_2 C_{p_1}}{\Gamma(-\alpha)} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |\nabla u|^2 dx$$
$$+ \left(\varepsilon - \frac{C_1}{p+1} - \frac{\sigma\varepsilon}{p+1} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |u|^{p+1} dx$$
$$- C_1 \varepsilon e^{-\sigma\varepsilon t} \cdot \int_\Omega u u_t dx$$
$$+ \left(\mu - \frac{(\sigma\varepsilon)^\alpha}{4\delta_1} - \frac{(\sigma\varepsilon)^\alpha \varepsilon}{4\delta_2} \right) \int_\Omega \int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx$$
$$+ (C_1 - \beta) \mu F(t) + C_1 d$$
$$\geq C_1 H(t) + \left[\frac{C_1}{2} + \frac{\sigma\varepsilon}{2} + \frac{\varepsilon}{2} \right.$$
$$\left. - \mu\beta^\alpha \Gamma(-\alpha) - \frac{\sigma\varepsilon^2}{4\delta_3} - \frac{\delta_1}{\Gamma(-\alpha)} - \frac{C_1 \varepsilon}{4\delta_3} \right] e^{-\sigma\varepsilon t} \int_\Omega u_t^2 dx$$
$$+ \left[\left(\frac{\sigma}{2} + 1 \right) \varepsilon + \frac{C_1}{2} \right] e^{-\sigma\varepsilon t} \cdot \int_\Omega |\Delta u|^2 dx$$
$$+ \left(\varepsilon - \frac{C_1}{p+1} - \frac{\sigma\varepsilon}{p+1} \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |u|^{p+1} dx$$
$$- \left(\sigma\varepsilon^2 \delta_3 C_{p_1} + \frac{\varepsilon \delta_2 C_{p_1}}{\Gamma(-\alpha)} + C_1 \varepsilon C_{p_1} \delta_3 \right) e^{-\sigma\varepsilon t} \cdot \int_\Omega |\nabla u|^2 dx$$
$$+ \left(\mu - \frac{(\sigma\varepsilon)^\alpha}{4\delta_1} - \frac{(\sigma\varepsilon)^\alpha \varepsilon}{4\delta_2} \right) \int_\Omega \int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx$$
$$+ (C_1 - \beta) \mu F(t) + C_1 d$$

We choose $C_1 = \frac{p+1}{2}\varepsilon,\ \delta_1 = \delta_2 = \frac{\Gamma(-\alpha)\varepsilon}{2}$ and $\delta_3 = \frac{1}{2}$, we obtain

$$H'(t)$$
$$\geq \frac{p+1}{2} \varepsilon H(t) - \frac{p+2}{2} \varepsilon^2 C_{p_1} e^{-\sigma\varepsilon t} \cdot \int_\Omega |\nabla u|^2 dx$$
$$+ \left[\frac{p+1}{2} \varepsilon (1-\varepsilon) - \mu\beta^\alpha \Gamma(-\alpha) \right] e^{-\sigma\varepsilon t} \cdot \int_\Omega u_t^2 dx$$
$$+ \frac{p+3}{2} \varepsilon e^{-\sigma\varepsilon t} \cdot \int_\Omega |\Delta u|^2 dx$$
$$+ \left[\mu - \frac{(p+1)^\alpha \varepsilon^{\alpha-1}}{2^{\alpha+1} \Gamma(-\alpha)} (1+\varepsilon) \right] \int_\Omega \int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx$$
$$+ \left(\frac{p+1}{2} \varepsilon - \beta \right) \mu F(t) + \frac{p+1}{2} \varepsilon d.$$

For the third term on the right side of (2), using the Sobolev-

Poincare inequality, we get

$$\begin{aligned}
&H'(t)\\
&\geq \frac{p+1}{2}\varepsilon H(t) - \frac{p+2}{2}\varepsilon^2 C_{p_1} e^{-\sigma\varepsilon t} C_{p_2}\int_\Omega |\Delta u|^2 dx\\
&+\left[\frac{p+1}{2}\varepsilon(1-\varepsilon) - \mu\beta^\alpha\Gamma(-\alpha)\right]e^{-\sigma\varepsilon t}.\int_\Omega u_t^2 dx\\
&+\frac{p+3}{2}\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx\\
&+\left[\mu - \frac{(p+1)^\alpha\varepsilon^{\alpha-1}}{2^{\alpha+1}\Gamma(-\alpha)}(1+\varepsilon)\right]\int_\Omega\int_0^t \frac{e^{-\sigma\varepsilon\tau}u_\tau^2}{(t-\tau)^{(\alpha+1)}}d\tau dx\\
&+\left(\frac{p+1}{2}\varepsilon - \beta\right)\mu F(t) + \frac{p+1}{2}\varepsilon d\\
&\geq \frac{p+1}{2}\varepsilon H(t)\\
&+\left[\frac{p+1}{2}\varepsilon(1-\varepsilon) - \mu\beta^\alpha\Gamma(-\alpha)\right]e^{-\sigma\varepsilon t}.\int_\Omega u_t^2 dx\\
&+\left[\frac{p+3-(p+2)\varepsilon C_p}{2}\right]\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx\\
&+\left[\mu - \frac{(p+1)^\alpha\varepsilon^{\alpha-1}}{2^{\alpha+1}\Gamma(-\alpha)}(1+\varepsilon)\right]\int_\Omega\int_0^t \frac{e^{-\sigma\varepsilon\tau}u_\tau^2}{(t-\tau)^{(\alpha+1)}}d\tau dx\\
&+\left(\frac{p+1}{2}\varepsilon - \beta\right)\mu F(t) + \frac{p+1}{2}\varepsilon d
\end{aligned}$$

Next, we choose $\beta = 1$ and assuming that

$$\varepsilon < \varepsilon_1 = \min\left(1, \frac{p+3}{2(p+2)C_p}\right)$$

it appears that the third coefficient is nonnegative. We can choose μ so that the second coefficient is nonnegative and the forth coefficient greater than $\frac{(p+1)^\alpha}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}$. Also, if p sufficiently large $\frac{p+1}{2}\varepsilon - \beta$ is positive. Consequently, we find

$$\begin{aligned}
H'(t) \quad &\geq \quad \frac{p+1}{2}\varepsilon H(t) + \frac{p+3}{4}\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx\\
&+ \quad \frac{(p+1)^\alpha}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}\int_\Omega\int_0^t \frac{e^{-\sigma\varepsilon\tau}u_\tau^2}{(t-\tau)^{(\alpha+1)}} \qquad (2.13)
\end{aligned}$$

If we select $E_\varepsilon(0) < -d$, then $H(0) > 0$. Consequently of (2.13), that $H(t) > 0$ and $H'(t) > 0$. □

Now, we state the local existence theorem.

Theorem 2.2. *(Local Existence) [5]. Suppose that*

$$\begin{cases} 1 < p < \infty, \ n = 1, 2, \\ 1 < p \leq \frac{2n}{n-2}, \ n \geq 3. \end{cases}$$

For every initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, *there is* $T > 0$ *and a unique weak solution* $u(t)$ *of (1.1) such that* $u \in C([0,T); H_0^2(\Omega)) \cap C^1([0,T); L^2(\Omega))$ *and* $u_t \in L^2((0,T)\times\Omega)$.

3. Blow up

In this section, we state and prove the blow up result for negative initial energy.

Theorem 3.1. *Assume that* $-1 < \alpha < 0$,

$$E(0) < 0 \text{ and } \int_\Omega u_1 u_0 dx \geq 0.$$

Then the solution u of the problem (1.1) blow up in finite time for sufficiently large values of p.

Proof. Set

$$\Psi(t) = H^{1-\gamma}(t) + \varphi e^{-\sigma\varepsilon t}.\int_\Omega u u_t dx$$

where

$$\gamma = \frac{p-1}{2(p+1)}$$

and φ is a positive constant to be determined later. Our goal is to show that $\Psi(t)$ satisfies a differential inequality of the form

$$\Psi^{\frac{1}{1-\gamma}}(t) \leq K\Psi'(t).$$

This, of course, will lead to a blow up result in finite time.

Now, by taking a derivative of $\Psi(t)$ and using (1.1), we have

$$\begin{aligned}
\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t) - \varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega u u_t dx\\
&+\varphi e^{-\sigma\varepsilon t}.\left(\int_\Omega u_t^2 dx + \int_\Omega u u_{tt} dx\right)\\
&= (1-\gamma)H^{-\gamma}(t)H'(t) - \varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega u u_t dx\\
&+\varphi e^{-\sigma\varepsilon t}\left[\int_\Omega |u|^{p+1}dx + \int_\Omega |\Delta u|^2 dx\right.\\
&\left.-\frac{1}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\right]\\
&+\varphi e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx.
\end{aligned}$$

By using the inequalities (2.12) and (2.11) with the constant $\delta_4, \delta_5 > 0$ we obtain

$$\begin{aligned}
\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t) - \varphi\sigma\varepsilon\delta_4 e^{-\sigma\varepsilon t}.\int_\Omega |u|^2 dx\\
&+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx\\
&-\frac{\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}}{4\delta_4}\int_\Omega u_t^2 dx + \varphi e^{-\sigma\varepsilon t}.\int_\Omega u_t^2 dx\\
&+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |\Delta u|^2 dx - \frac{\varphi\delta_5 e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega |u|^2 dx\\
&-\frac{\varphi(\sigma\varepsilon)^\alpha}{4\delta_5}\int_\Omega\int_0^t (t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2 d\tau dx.
\end{aligned}$$

The last term of this inequality is derived from the (2.13), we see

$$\begin{aligned}
&\Psi'(t)\\
&\geq \left[(1-\gamma)H^{-\gamma}(t)-\frac{\varphi\Gamma(-\alpha)\varepsilon}{2\delta_5}\right]H'(t)\\
&+\frac{\varphi(p+1)\Gamma(-\alpha)\varepsilon^2}{4\delta_5}H(t)\\
&+\varphi\left(1-\frac{(p+1)\varepsilon}{8\delta_4}\right)e^{-\sigma\varepsilon t}.\int_\Omega u_t^2dx\\
&+\varphi e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx\\
&+\varphi\left[1-\frac{(p+3)\Gamma(-\alpha)\varepsilon^2}{8\delta_5}-\left(\frac{p+1}{2}\varepsilon\delta_4+\frac{\delta_5}{\Gamma(-\alpha)}\right)C_p\right]\\
&.e^{-\sigma\varepsilon t}\int_\Omega|\Delta u|^2dx.
\end{aligned}$$

We pick $\delta_5=L\Gamma(-\alpha)H^{\gamma}(t)$, we have

$$\begin{aligned}
\Psi'(t) \quad \geq \quad & \left[(1-\gamma)-\frac{\varphi\varepsilon}{2L}\right]H^{-\gamma}(t)H'(t) && (3.1)\\
&+\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)H(t)\\
&+\varphi\left(1-\frac{(p+1)\varepsilon}{8\delta_4}\right)e^{-\sigma\varepsilon t}\int_\Omega u_t^2dx && (3.2)\\
&+\varphi e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1}dx\\
&+\varphi\left[1-\frac{(p+3)\varepsilon^2}{8LH^{\gamma}(t)}-\left(\frac{p+1}{2}\varepsilon\delta_4+LH^{\gamma}(t)\right)C_p\right]\\
&.e^{-\sigma\varepsilon t}\int_\Omega|\Delta u|^2dx && (3.3)
\end{aligned}$$

Adding and substracting $H(t)$ to the right hand side of (3.3), we have

$$\begin{aligned}
\Psi'(t) \quad \geq \quad & \left[(1-\gamma)-\frac{\varphi\varepsilon}{2L}\right]H^{-\gamma}(t)H'(t)\\
&+\left[\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)+1\right]H(t)\\
&+\left[\varphi\left(1-\frac{(p+1)\varepsilon}{8\delta_4}\right)+\frac{1}{2}-\frac{\varepsilon}{4\delta_6}\right]\int_\Omega u_t^2dx\\
&+\left(\varphi-\frac{1}{p+1}\right)e^{-\sigma\varepsilon t}.\int_\Omega |u|^{p+1}dx\\
&+\varphi\left[1-\frac{(p+3)\varepsilon^2}{8LH^{\gamma}(t)}-\left(\frac{p+1}{2}\varepsilon\delta_4+LH^{\gamma}(t)\right)C_p\right]\\
&.e^{-\sigma\varepsilon t}\int_\Omega|\Delta u|^2dx+\mu F(t)+d.
\end{aligned}$$

By using the inequality

$$\varepsilon<\varepsilon_2=\frac{2L(1-\gamma)}{\varphi}$$

and

$$\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)\geq 0.$$

Also, we take

$$\varphi=\frac{p+3}{4(p+1)},\ \delta_4=\delta_6=\frac{1}{2}\text{ and }\varepsilon<\varepsilon_3=\frac{4(p+3)}{(p+1)(p+11)},$$

to get

$$\varphi\left(1-\frac{(p+1)\varepsilon}{8\delta_4}\right)-\frac{\varepsilon}{4\delta_6}\geq 0$$

and

$$\varphi-\frac{1}{p+1}=\frac{p-1}{4(p+1)}.$$

The fifth coefficient is nonnegative as soon as ε and C_p is chosen small enough, namely

$$1-\frac{(p+3)\varepsilon^2}{8LH^{\gamma}(t)}-\left(\frac{p+1}{2}\varepsilon\delta_4+LH^{\gamma}(t)\right)C_p\geq 0.$$

Consequently, we have

$$\Psi'(t)\geq H(t)+\frac{1}{2}\int_\Omega u_t^2dx+\frac{p-1}{4(p+1)}\int_\Omega|u|^{p+1}dx. \quad (3.4)$$

On the other hand, from the $\Psi(t)$, we have

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}\left(\int_\Omega uu_tdx\right)^{\frac{1}{1-\gamma}}\right]. \quad (3.5)$$

By the Cauchy-Schwarz and Hölder inequalities, we have

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}B\left(\int_\Omega|u_t|^2dx+\int_\Omega|u|^{p+1}dx\right)\right]. \quad (3.6)$$

If K is chosen large enough so that

$$\begin{aligned}
&2^{\frac{\gamma}{1-\gamma}}\leq K,\\
&2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}B\leq\frac{K}{2},\\
&2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}B\leq\frac{p-1}{4(p+1)}K,
\end{aligned}$$

That is K has to be selected so that

$$K\geq 2^{\frac{\gamma}{1-\gamma}}\max\left\{1,2\varphi^{\frac{1}{1-\gamma}}B,\frac{4(p+1)}{p-1}b^{\frac{1}{1-\gamma}}B\right\}.$$

Combining (3.4) and (3.6), we have

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq K\Psi'(t), \quad (3.7)$$

From (3.4) it is clear that $\Psi'(t)\geq 0$. Hence, by the definition of $\Psi(t)$ and the hypotheses on the initial data, we have

$$\Psi(t)\geq\Psi(0)>\varphi\int_\Omega u_1u_0dx\geq 0.$$

Thus $\Psi(t) > 0$. Integrating (3.7) over $(0,t)$, we find

$$\Psi^{\frac{\gamma}{1-\gamma}}(t) \geq \frac{1}{\Psi^{\frac{-\gamma}{1-\gamma}}(0) - \frac{\gamma}{K(1-\gamma)}t}$$

Consequently, $\Psi(t)$ blows up at some time

$$T^* \leq \frac{K(1-\gamma)\Psi^{\frac{-\gamma}{1-\gamma}}(0)}{\gamma}.$$

□

References

[1] M.R. Alaimia, N.E.Tatar, Blow up for the wave equation with a fractional damping, J. Appl. Anal. 11(1) (2005) 133-144.

[2] W. Chen, Y. Zhou, Global nonexistence for a semilinear Petrovsky equation, Nonlinear Anal. 70 (2009) 3203-3208.

[3] S. Chen, R. Triggiani, Proof of extension of two conjectures on structural damping for elastic systems: the case $1/2 \leq \alpha \leq 1$, Pacific J. Math. 136 (1989) 15-55.

[4] G. Li, Y. Sun, W. Liu, Global existence and blow up of solutions for a strongly damped Petrovsky system with nonlinear damping, Appl. Anal. 91(3) (2012) 575-586.

[5] S. A. Messaoudi, Global existence and nonexistence in a system of Petrovsky, J. Math. Anal. Appl. 265 (2) (2002) 296-308.

[6] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, 1974.

[7] E. Pişkin, N. Polat, On the decay of solutions for a nonlinear Petrovsky equation, Math. Sci. Letters, 3(1) (2014) 43-47.

[8] I.Podlubny, Fractional Differential Equations, Math. Sci. Engrg. 198, Academic Press, San Diago, CA,1999.

[9] N.E. Tatar, A wave equation with fractional damping, Z. Anal. Anw. 22(3) (2003) 609-617.

[10] N.E. Tatar, A blow up result for a fractionally damped wave equation, NoDEA, 12 (2005) 215-226.

[11] S.T. Wu, L.Y. Tsai, On global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system, Taiwanese J. Math. 13 (2A) (2009) 545-558.

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