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Stability of system of additive functional equations in various Banach spaces: Classical Hyers methods

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Abstract

In this paper, authors proved the generalized Ulam - Hyers stability of system of additive functional equations

$$f\left(\sum_{a=1}^{n} a\ x_a\right) = \sum_{a=1}^{n} (a\ f(x_a)); \qquad n \geq 1$$

$$g\left(\sum_{a=1}^{n} 2a\ y_{2a}\right) = \sum_{a=1}^{n} (2a\ g(y_{2a})); \qquad n \geq 1$$

$$h\left(\sum_{a=1}^{n} (2a-1)\ z_{2a-1}\right) = \sum_{a=1}^{n} ((2a-1)\ h(z_{2a-1})); \qquad n \geq 1$$

where n is a positive integer, which is originating from sum of first n, natural numbers, even natural numbers and odd natural numbers, respectively in various Banach spaces.

Keywords

Additive functional equations, Generalized Ulam - Hyers stability, Banach spaces, 2-Banach space, Quasi 2 - Banach space, Quasi - Beta - 2- Banach space, Fuzzy Quasi - Beta - 2 - Banach space, Random Quasi - Beta - 2 - Banach space.

AMS Subject Classification

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1. Introduction

The stability of functional equations is a burning theme that has been dealt in the last seven decades. In 1940, S.M. Ulam [40], gave a spacious collection of talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed the number of significant unsolved problems. One of them is the initial spot of a new line of investigation, the Stability Problem.

Ulam Problem : *Let G be a group and H be a metric group with metric $d(.\,,\,.)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality*

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with

$$d(f(x), a(x)) < \varepsilon$$

for all $x \in G$?

For the case where the answer is affirmative, the functional equation for homomorphisms will be called stable.

The first result pertaining to the stability of functional equations was presented by D.H.Hyers [19] in 1941. He has comprehensively answered the question of Ulam by assuming G and H are Banach Spaces. He proved the following celebrated theorem.

Theorem 1.1. *[19] Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \varepsilon \tag{1.3}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function a is linear.

This pioneer result can be expressed as "Cauchy functional equation and it is stable for any pair of Banach spaces". The method which was provided by Hyers and which produces the additive function $a(x)$ will be called a **direct method**. This method is the most important and most powerful tool for studying the stability of various functional equations.

In 1951, T. Aoki [3] generalized the Hyers theorem for approximately linear transformation in Banach spaces, by weakening the condition for the Cauchy difference for sum of powers of norms. Then Th.M. Rassias [32] in 1978, investigated a similar case (see L. Maligranda [29]). Both proved the following Hyers-Ulam-Aoki-Rassias theorem for the "sum".

Theorem 1.2. *[3, 32] Let X and Y be two Banach spaces. Let $\theta \in [0, \infty)$ and $p \in [0, 1)$. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(||x||^p + ||y||^p\right) \tag{1.4}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p \tag{1.5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function T is linear.

This result is known as the Modified Hyers - Ulam Stability or Generalized Hyers- Ulam Stability for the additive functional equation.

In 1982-84, J.M. Rassias [31] replaced the sum by the product of powers of norms. Infact, he proved the following theorem.

Theorem 1.3. *[31] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \|x\|^p \|y\|^p \tag{1.6}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.7}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{1.8}$$

for all $x \in E$. If $p < 0$, then the inequality (1.6) holds for $x, y \neq 0$ and (1.8) for $x \neq 0$.
If $p > \frac{1}{2}$ the inequality (1.6) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{1.9}$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{2p} - 2} \|x\|^{2p} \tag{1.10}$$

for all $x \in E$. If in addition $f : E \to E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}-$ linear mapping.

In 1994, P. Gavruta [18] generalized all the above mentioned results by considering the control function as a function of variables and proved the following theorem.

Theorem 1.4. *[18] Let E be a abelian group, F be a Banach space and let $\phi : E \times E \to [0, \infty)$ be a function satisfying*

$$\Phi(x, y) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \phi\left(2^k x, 2^k y\right) < +\infty \tag{1.11}$$

for all $x, y \in E$. If a function $f : E \to F$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \phi(x, y) \tag{1.12}$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to F$ which satisfies

$$\|f(x) - T(x)\| \leq \Phi(x, y) \tag{1.13}$$

for all $x \in E$. If moreover $f(tx)$ is continuous in t for fixed $x \in E$, then T is linear.

This stability property is called Generalized Hyers - Ulam - Rassias Stability of functional equation.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by K. Ravi, M. Arunkumar and J.M. Rassias [34] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach is called J.M. Rassias Stability of functional equation.

Theorem 1.5. *[34] Let $(E,\perp)$ denote an orthogonality normed space with norm $\|.\|_E$ and $(F,\|.\|_F)$ is a Banach space and $f: E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned} &\|f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)\\ &\quad -2(m^2-2)f(x)+2f(y)\big\|_F\\ &\qquad \leq \varepsilon\left\{\|x\|_E^p\,\|y\|_E^p+\left(\|x\|_E^{2p}+\|y\|_E^{2p}\right)\right\} \end{aligned} \tag{1.14}$$

for all $x,y \in E$ with $x\perp y$, where ε and p are constants with $\varepsilon, p>0$ and either $m>1; p<1$ or $m<1; p>1$ with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$.
Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \tag{1.15}$$

exists for all $x \in E$ and $Q: E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x)-Q(x)|_F \leq \frac{\varepsilon}{2|m^2-m^{2p}|}\,\|x\|_E^{2p} \tag{1.16}$$

for all $x \in E$.

A number is a mathematical object used to count, measure, and label. The most familiar numbers are the natural numbers (sometimes called whole numbers or counting numbers): 1, 2, 3, and so on. An even number is an integer that is *evenly divisible* by two, that is divisible by two without remainder; an odd number is an integer that is not even. (The old-fashioned term *evenly divisible* is now almost always shortened to *divisible*.) Equivalently, another way of defining an odd number is that it is an integer of the form $n = 2k-1$, where k is an integer, and an even number has the form $n = 2k$, where k is an integer.

The sum of first n natural numbers, sum of first n even natural numbers and sum of first n odd natural numbers, are

$$1+2+3+4+........+N = \frac{N(N+1)}{2}$$

$$2+4+6+8+........+2N = N(N+1)$$

$$1+3+5+7+........+(2N-1) = N^2$$

The above sum of observation can be taken as a functional equation of the following forms

$$\begin{aligned} &f(x_1+2x_2+3x_3+4x_4+\cdots+nx_n)\\ &= 1f(x_1)+2f(x_2)+3f(x_3)+4f(x_4)+\cdots+nf(x_n)\\ &g(2y_2+4y_4+6y_6+8y_8+\cdots+2ny_{2n})\\ &= 2g(y_2)+4g(y_4)+6g(y_6)+8g(y_8)+\cdots+2ng(y_{2n})\\ &h(1z_1+3z_3+5z_5+7z_7+\cdots+(2n-1)z_{2n-1})\\ &= 1h(z_1)+3h(z_3)+5h(z_5)+7h(z_7)+\cdots+(2n-1)h(z_{2n-1}) \end{aligned}$$

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x)+f(y) \tag{1.17}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in tribute of A.L. Cauchy. The theory of additive functional equations is commonly useful to the growth of theories of further functional equations. Moreover, the properties of additive functional equations are powerful tackle in almost every field of natural and social sciences. Every solution of the additive functional equation (1.17) is called an additive function.

The solution and stability of various additive functional equations

$$f(2x-y)+f(x-2y) = 3f(x)-3f(y), \tag{1.18}$$

$$f(x+y-2z)+f(2x+2y-z) = 3f(x)+3f(y)-3f(z), \tag{1.19}$$

$$\begin{aligned} &f(m(x+y)-2mz)+f(2m(x+y)-mz)\\ &\quad = 3m[f(x)+f(y)-f(z)]\ m\geq 1, \end{aligned} \tag{1.20}$$

$$\begin{aligned} &f\left(a\sum_{i=1}^{n-1}x_i-2ax_n\right)+f\left(2a\sum_{i=1}^{n-1}x_i-ax_n\right)\\ &\quad = 3a\left(\sum_{i=1}^{n-1}f(x_i)-f(x_n)\right)\ n\geq 3, \end{aligned} \tag{1.21}$$

$$f(2x\pm y\pm z) = f(x\pm y)+f(x\pm z) \tag{1.22}$$

$$f(x)+f(x) = f(2x) \tag{1.23}$$

$$f(y) = \frac{f(y+z)+f(y-z)}{2} \tag{1.24}$$

$$f(x) = \sum_{\ell=1}^{n}\left(\frac{f(x+\ell y_\ell)+f(x-\ell y_\ell)}{2\,\ell}\right) \tag{1.25}$$

$$f\left(nx_0\pm\sum_{i=1}^{n}x_i\right) = \sum_{i=1}^{n}f(x_0\pm x_i) \tag{1.26}$$

were discussed by D.O. Lee [27], K. Ravi, M. Arunkumar [33], M. Arunkumar [4, 5, 7, 8, 11].

In this paper, authors proved the generalized Ulam - Hyers stability of system of additive functional equations

$$f\left(\sum_{a=1}^{n} a\, x_a\right) = \sum_{a=1}^{n} \left(a\, f(x_a)\right), \;\; n \geq 1 \tag{1.27}$$

$$g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) = \sum_{a=1}^{n} \left(2a\, g(y_{2a})\right), \;\; n \geq 1 \tag{1.28}$$

$$h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) = \sum_{a=1}^{n} \left((2a-1)\, h(z_{2a-1})\right), \quad n \geq 1 \tag{1.29}$$

where n is a positive integer, which is originating from, sum of first n natural numbers, sum of first n even natural numbers and sum of first n odd natural numbers in various Banach spaces and having solutions

$$f(x) = b_1\, x; \tag{1.30}$$

$$g(y) = b_2\, y; \tag{1.31}$$

$$h(z) = b_3\, z \tag{1.32}$$

where b_1, b_2, b_3 are constants.

In this paper, authors proved the generalized Ulam - Hyers stability of system of additive functional equations where n is a positive integer, which is originating from sum of first n, natural numbers, even natural numbers and odd natural numbers, respectively in various Banach spaces.

2. General Solution

In this section, we give the general solution of the functional equations (1.27), (1.28) and (1.29). To prove the general solution, we let us take A and B be real vector spaces.

The proof of the following theorems are proved by the additive property of functions. Hence the details of the proof are omitted.

Theorem 2.1. *An additive function $f : A \to B$ satisfies the functional equation (1.17) for all $x, y \in A$ if and only if $f : A \to B$ satisfies the functional equation (1.27) for all $x_1, x_2, \cdots, x_n \in A$.*

Theorem 2.2. *An additive function $g : A \to B$ satisfies the functional equation*

$$g(x+y) = g(x) + g(y) \tag{2.1}$$

for all $x, y \in A$ if and only if $g : A \to B$ satisfies the functional equation (1.28) for all $y_2, y_4, \cdots, y_{2n} \in A$.

Theorem 2.3. *An additive function $h : A \to B$ satisfies the functional equation*

$$h(x+y) = h(x) + h(y) \tag{2.2}$$

for all $x, y \in A$ if and only if $h : A \to B$ satisfies the functional equation (1.29) for all $z_1, z_3, \cdots, z_{2n-1} \in A$.

3. Stability Results In Banach Space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in Banach spaces using Hyers Method.

Now, we give basic definitions and notations in Banach Space.

Definition 3.1. *Let X be a set. A function $||.|| : X \times X \to [0, \infty)$ is called a **Normed Linear space** on X if $||.||$ satisfies the following conditions*

(i) $||x|| = 0$ if and only if $x = 0$ for all $x \in X$;

(ii) $||x, y|| = 0$ if and only if $x = y$ for all $x, y \in X$;

(iii) $||\lambda x|| = |\lambda| ||x||$ for all $x \in X$ and $\lambda \in R$;

(iv) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$;

(v) $||x, z|| \leq ||x, y|| + ||y, z||$ for all $x, y, z \in X$;

Definition 3.2. *A sequence $\{x_n\}$ in a normed linear space X is called a **convergent** sequence if there is an $x \in X$ such that* $\lim_{n\to\infty} ||x_n - x|| = 0$

Definition 3.3. *A sequence $\{x_n\}$ in a normed linear space X is called a **Cauchy sequence** if there exists a point, $x_m \in X$ such that* $\lim_{n,m\to\infty} ||x_n - x_m|| = 0$

Definition 3.4. *A normed linear space is said to be **Complete** if every Cauchy sequence converges.*

Definition 3.5. *A complete normed linear space is said to be a **Banach Space***

To prove stability results, let us consider $\mathscr{A}$ and $\mathscr{B}$ be Banach spaces.

Theorem 3.6. *Let $f, g, h : \mathscr{A} \to \mathscr{B}$ be a mapping satisfying the following inequalities*

$$\left\| f\left(\sum_{a=1}^{n} a\, x_a\right) - \sum_{a=1}^{n} \left(a\, f(x_a)\right) \right\| \leq N(x_1, x_2, \cdots, x_n); \tag{3.1}$$

$$\left\| g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) - \sum_{a=1}^{n} \left(2a\, g(y_{2a})\right) \right\| \leq E(y_2, y_4, \cdots, y_{2n}); \tag{3.2}$$

$$\left\| h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) - \sum_{a=1}^{n} \left((2a-1)\, h(z_{2a-1})\right) \right\| \leq O(z_1, z_3, \cdots, z_{2n-1}); \tag{3.3}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$, where $N : \mathscr{A}^n \to [0, \infty)$, $E : \mathscr{A}^{2n} \to [0, \infty)$ and $O : \mathscr{A}^{2n-1} \to [0, \infty)$ satisfying the conditions

$$\lim_{\alpha\to\infty} \frac{N(\kappa^{\alpha\gamma} x_1, \kappa^{\alpha\gamma} x_2, \cdots, \kappa^{\alpha\gamma} x_n)}{\kappa^{\alpha\gamma}} = 0; \tag{3.4}$$

$$\lim_{\alpha\to\infty} \frac{E(\rho^{\alpha\gamma} y_2, \rho^{\alpha\gamma} y_4, \cdots, \rho^{\alpha\gamma} y_{2n})}{\rho^{\alpha\gamma}} = 0; \tag{3.5}$$

$$\lim_{\alpha\to\infty} \frac{O(\tau^{\alpha\gamma} z_1, \tau^{\alpha\gamma} z_3, \cdots, \tau^{\alpha\gamma} z_{2n-1})}{\tau^{\alpha\gamma}} = 0; \tag{3.6}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$ with $\gamma = \pm 1$. Then there exists one and only additive mapping $A_f, A_g, A_h : \mathscr{A} \to \mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\left\|A_f(x) - f(x)\right\| \leq \frac{1}{\kappa} \sum_{\beta=\frac{1-\gamma}{2}}^{\infty} \frac{N(\kappa^{\beta\gamma}x, \kappa^{\beta\gamma}x, \cdots, \kappa^{\beta\gamma}x)}{\kappa^{\beta\gamma}}; \tag{3.7}$$

$$\left\|A_g(y) - g(y)\right\| \leq \frac{1}{\rho} \sum_{\beta=\frac{1-\gamma}{2}}^{\infty} \frac{E(\rho^{\beta\gamma}y, \rho^{\beta\gamma}y, \cdots, \rho^{\beta\gamma}y)}{\rho^{\beta\gamma}}; \tag{3.8}$$

$$\left\|A_h(z) - h(z)\right\| \leq \frac{1}{\tau} \sum_{\beta=\frac{1-\gamma}{2}}^{\infty} \frac{O(\tau^{\beta\gamma}z, \tau^{\beta\gamma}z, \cdots, \tau^{\beta\gamma}z)}{\tau^{\beta\gamma}}; \tag{3.9}$$

for all $x, y, z \in \mathscr{A}$, respectively. The mappings A_f, A_g, A_h are respectively defined as

$$A_f(x) = \lim_{\alpha\to\infty} \frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}}; \tag{3.10}$$

$$A_g(y) = \lim_{\alpha\to\infty} \frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}}; \tag{3.11}$$

$$A_h(z) = \lim_{\alpha\to\infty} \frac{h(\tau^{\alpha\gamma}z)}{\tau^{\alpha\gamma}}; \tag{3.12}$$

for all $x, y, z \in \mathscr{A}$.

Proof. Changing

$$(x_1, x_2, \cdots, x_n) = (x, x, \cdots, x) \quad in \quad (3.1);$$
$$(y_2, y_4, \cdots, y_{2n}) = (y, y, \cdots, y) \quad in \quad (3.2);$$
$$(z_1, z_3, \cdots, z_{2n-1}) = (z, z, \cdots, z) \quad in \quad (3.3);$$

we arrive the following inequalities

$$\left\| f\left(\sum_{a=1}^{n} a\, x\right) - \left(\sum_{a=1}^{n} a\right) f(x) \right\| \leq N(x, x, \cdots, x); \tag{3.13}$$

$$\left\| g\left(\sum_{a=1}^{n} 2a\, y\right) - \left(\sum_{a=1}^{n} 2a\right) g(y) \right\| \leq E(y, y, \cdots, y); \tag{3.14}$$

$$\left\| h\left(\sum_{a=1}^{n} (2a-1)\, z\right) - \left(\sum_{a=1}^{n} (2a-1)\right) h(z) \right\| \leq O(z, z, \cdots, z); \tag{3.15}$$

for all $x, y, z \in \mathscr{A}$. Define

$$\sum_{a=1}^{n} a = \frac{n(n+1)}{2} = \kappa \quad in \quad (3.13);$$
$$\sum_{a=1}^{n} 2a = n(n+1) = \rho \quad in \quad (3.14);$$
$$\sum_{a=1}^{n} (2a-1) = n^2 = \tau \quad in \quad (3.15);$$

we obtain the succeeding inequalities

$$\|f(\kappa\, x) - \kappa\, f(x)\| \leq N(x, x, \cdots, x); \tag{3.16}$$
$$\|g(\rho\, y) - \rho\, g(y)\| \leq E(y, y, \cdots, y); \tag{3.17}$$
$$\|h(\tau\, z) - \tau\, h(z)\| \leq O(z, z, \cdots, z); \tag{3.18}$$

for all $x, y, z \in \mathscr{A}$. It follows from above inequalities

$$\left\| \frac{f(\kappa\, x)}{\kappa} - f(x) \right\| \leq \frac{N(x, x, \cdots, x)}{\kappa}; \tag{3.19}$$
$$\left\| \frac{g(\rho\, y)}{\rho} - g(y) \right\| \leq \frac{E(y, y, \cdots, y)}{\rho}; \tag{3.20}$$
$$\left\| \frac{h(\tau\, z)}{\tau} - h(z) \right\| \leq \frac{O(z, z, \cdots, z)}{\tau}; \tag{3.21}$$

for all $x, y, z \in \mathscr{A}$. Replacing

$$x = \kappa\, x \quad and \quad \div \quad \kappa \quad in \quad (3.19);$$
$$y = \rho\, y \quad and \quad \div \quad \rho \quad in \quad (3.20);$$
$$z = \tau\, z \quad and \quad \div \quad \tau \quad in \quad (3.21);$$

we arrive the following inequalities

$$\left\| \frac{f(\kappa^2\, x)}{\kappa^2} - \frac{f(\kappa\, x)}{\kappa} \right\| \leq \frac{N(\kappa\, x, \kappa\, x, \cdots, \kappa\, x)}{\kappa^2}; \tag{3.22}$$
$$\left\| \frac{g(\rho^2\, y)}{\rho^2} - \frac{g(\rho\, y)}{\rho} \right\| \leq \frac{E(\rho\, y, \rho\, y, \cdots, \rho\, y)}{\rho^2}; \tag{3.23}$$
$$\left\| \frac{h(\tau^2\, z)}{\tau^2} - \frac{h(\tau\, z)}{\tau} \right\| \leq \frac{O(\tau\, z, \tau\, z, \cdots, \tau\, z)}{\tau^2}; \tag{3.24}$$

for all $x, y, z \in \mathscr{A}$. With the help of triangle inequality from

(3.19) *and* (3.22); (3.20) *and* (3.23); (3.21) *and* (3.24);

we achieve the subsequent inequalities

$$\begin{aligned}
&\left\| \frac{f(\kappa^2\, x)}{\kappa^2} - f(x) \right\| \\
&\leq \left\| \frac{f(\kappa^2\, x)}{\kappa^2} - \frac{f(\kappa\, x)}{\kappa} \right\| + \left\| \frac{f(\kappa\, x)}{\kappa} - f(x) \right\| \\
&\leq \frac{1}{\kappa}\left(N(x, x, \cdots, x) + \frac{N(\kappa\, x, \kappa\, x, \cdots, \kappa\, x)}{\kappa} \right);
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\left\| \frac{g(\rho^2\, y)}{\rho^2} - g(y) \right\| \\
&\leq \left\| \frac{g(\rho^2\, y)}{\rho^2} - \frac{g(\rho\, y)}{\rho} \right\| + \left\| \frac{g(\rho\, y)}{\rho} - g(y) \right\| \\
&\leq \frac{1}{\rho}\left(E(y, y, \cdots, y) + \frac{E(\rho\, y, \rho\, y, \cdots, \rho\, y)}{\rho} \right);
\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\left\|\frac{h\left(\tau^2 z\right)}{\tau^2}-h(z)\right\|\\&\leq\left\|\frac{h\left(\tau^2 z\right)}{\tau^2}-\frac{h(\tau z)}{\tau}\right\|+\left\|\frac{h(\tau z)}{\tau}-h(z)\right\|\\&\leq\frac{1}{\tau}\left(E(z,z,\cdots,z)+\frac{E(\tau z,\tau z,\cdots,\tau)}{\tau}\right);\end{aligned}\tag{3.27}$$

for all $x,y,z\in\mathscr{A}$. Generalizing for a positive integer α, we arrive

$$\left\|\frac{f\left(\kappa^\alpha x\right)}{\kappa^\alpha}-f(x)\right\|\leq\frac{1}{\kappa}\sum_{\beta=0}^{\alpha}\frac{N(\kappa^\beta x,\kappa^\beta x,\cdots,\kappa^\beta x)}{\kappa^\beta};\tag{3.28}$$

$$\left\|\frac{g\left(\rho^\alpha y\right)}{\rho^\alpha}-g(y)\right\|\leq\frac{1}{\rho}\sum_{\beta=0}^{\alpha}\frac{E(\rho^\beta y,\rho^\beta y,\cdots,\rho^\beta y)}{\rho^\beta};\tag{3.29}$$

$$\left\|\frac{h\left(\tau^\alpha z\right)}{\tau^\alpha}-h(z)\right\|\leq\frac{1}{\tau}\sum_{\beta=0}^{\alpha}\frac{O(\tau^\beta z,\tau^\beta z,\cdots,\tau^\beta z)}{\tau^\beta};\tag{3.30}$$

for all $x,y,z\in\mathscr{A}$. It is easy to verify that the sequences

$$\left\{\frac{f\left(\kappa^\alpha x\right)}{\kappa^\alpha}\right\};\left\{\frac{g\left(\rho^\alpha y\right)}{\rho^\alpha}\right\};\left\{\frac{h\left(\tau^\alpha z\right)}{\tau^\alpha}\right\}$$

are Cauchy sequences. Indeed replacing

$$\begin{aligned}&x=\kappa^\delta x \quad and \quad \div \quad \kappa^\delta \quad in \quad (3.28);\\&y=\rho^\delta y \quad and \quad \div \quad \rho^\delta \quad in \quad (3.29);\\&z=\tau^\delta z \quad and \quad \div \quad \tau^\delta \quad in \quad (3.30);\end{aligned}$$

we obtain

$$\begin{aligned}&\left\|\frac{f\left(\kappa^{\alpha+\delta} x\right)}{\kappa^{\alpha+\delta}}-\frac{f\left(\kappa^\delta x\right)}{\kappa^\delta}\right\|\\&\leq\frac{1}{\kappa}\sum_{\beta=0}^{\alpha}\frac{N(\kappa^{\beta+\delta} x,\kappa^{\beta+\delta} x,\cdots,\kappa^{\beta+\delta} x)}{\kappa^{\beta+\delta}}\\&\to 0 \quad as \quad \delta \quad \to \quad \infty;\\&\left\|\frac{g\left(\rho^{\alpha+\delta} y\right)}{\rho^{\alpha+\delta}}-\frac{g\left(\rho^\delta y\right)}{\rho^\delta}\right\|\\&\leq\frac{1}{\rho}\sum_{\beta=0}^{\alpha}\frac{E(\rho^{\beta+\delta} y,\rho^{\beta+\delta} y,\cdots,\rho^{\beta+\delta} y)}{\rho^{\beta+\delta}}\\&\to 0 \quad as \quad \delta \quad \to \quad \infty;\\&\left\|\frac{h\left(\tau^{\alpha+\delta} z\right)}{\tau^{\alpha+\delta}}-\frac{h\left(\tau^\delta z\right)}{\tau^\delta}\right\|\\&\leq\frac{1}{\tau}\sum_{\beta=0}^{\alpha}\frac{O(\tau^{\beta+\delta} z,\tau^{\beta+\delta} z,\cdots,\tau^{\beta+\delta} z)}{\tau^{\beta+\delta}}\\&\to 0 \quad as \quad \delta \quad \to \quad \infty;\end{aligned}$$

for all $x,y,z\in\mathscr{A}$. This proves the existence of Cauchy sequences. Since $\mathscr{B}$ is a Banach space, this sequences converges to a point $A_f;A_g;A_h$ respectively and it defined by

$$\begin{aligned}A_f(x)&=\lim_{\alpha\to\infty}\frac{f(\kappa^\alpha x)}{\kappa^\alpha};\\A_g(y)&=\lim_{\alpha\to\infty}\frac{g(\rho^\alpha y)}{\rho^\alpha};\\A_h(z)&=\lim_{\alpha\to\infty}\frac{h(\tau^\alpha z)}{\tau^\alpha};\end{aligned}$$

for all $x,y,z\in\mathscr{A}$. Taking limit as α tends to infinity in (3.28), (3.29) and (3.30), we see that (3.7), (3.8) and (3.9) holds respectively for all $x,y,z\in\mathscr{A}$ with $\gamma=1$.

To show the mappings $A_f(x);A_g(y);A_h(z)$ satisfies the functional equations (1.27), (1.28) and (1.29) by replacing

$$\begin{aligned}&(x_1,x_2,\cdots,x_n)=(\kappa^\alpha x_1,\kappa^\alpha x_2,\cdots,\kappa^\alpha x_n) \quad in \quad (3.1)\\&(y_2,y_4,\cdots,y_{2n})=(\rho^\alpha y_2,\rho^\alpha y_4,\cdots,\rho^\alpha y_{2n}) \quad in \quad (3.2)\\&(z_1,z_3,\cdots,z_{2n-1})=(\tau^\alpha z_1,\tau^\alpha z_3,\cdots,\tau^\alpha z_{2n-1}) \quad in \quad (3.3)\end{aligned}$$

respectively, we arrive

$$\begin{aligned}&\left\|A_f\left(\sum_{a=1}^{n} a\,x_a\right)-\sum_{a=1}^{n}\left(a\,A_f(x_a)\right)\right\|\\&\qquad=\frac{1}{\kappa^\alpha}\left\|f\left(\sum_{a=1}^{n} a\,\kappa^\alpha x_a\right)-\sum_{a=1}^{n}\left(a\,f(\kappa^\alpha x_a)\right)\right\|\\&\leq\frac{1}{\kappa^\alpha}N(\kappa^\alpha x_1,\kappa^\alpha x_2,\cdots,\kappa^\alpha x_n)\\&\left\|A_g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,A_g(y_{2a})\right)\right\|\\&\qquad=\frac{1}{\rho^\alpha}\left\|g\left(\sum_{a=1}^{n} 2a\,\rho^\alpha y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,g(\rho^\alpha y_{2a})\right)\right\|\\&\leq\frac{1}{\rho^\alpha}E(\rho^\alpha y_2,\rho^\alpha y_4,\cdots,\rho^\alpha y_{2n})\\&\left\|A_h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}\left((2a-1)\,A-h(z_{2a-1})\right)\right\|\\&=\frac{1}{\tau^\alpha}\left\|h\left(\sum_{a=1}^{n}(2a-1)\,\tau^\alpha z_{2a-1}\right)-\sum_{a=1}^{n}\left((2a-1)\,h(\tau^\alpha z_{2a-1})\right)\right\|\\&\leq\frac{1}{\tau^\alpha}O(\tau^\alpha z_1,\tau^\alpha z_3,\cdots,\tau^\alpha z_{2n-1})\end{aligned}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$. Taking limit as α tends to infinity in the above inequalities, it proves that $A_f(x);A_g(y);A_h(z)$ satisfies the functional equations (1.27), (1.28) and (1.29) for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n}$, $z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$.

In order to prove the being $A_f(x);A_g(y);A_h(z)$ are unique. Let $A'_f(x);A'_g(y);A'_h(z)$ be another additive mappings satisfying (3.10), (3.11), (3.12) and (1.27), (1.28), (1.29) respectively.

Now

$$
\begin{aligned}
&\left\|A_f(x)-A'_f(x)\right\| \\
&\leq \frac{1}{\kappa^\alpha}\left\{\left\|A_f(\kappa^\alpha x)-f(\kappa^\alpha x)\right\|+\left\|A'_f(\kappa^\alpha x)-f(\kappa^\alpha x)\right\|\right\} \\
&\leq \frac{2}{\kappa}\sum_{\beta=0}^{\infty}\frac{N(\kappa^{\beta+\alpha}x,\kappa^{\beta+\alpha}x,\cdots,\kappa^{\beta+\alpha}x)}{\kappa^{\beta+\alpha}} \\
&\to 0 \quad as \quad \alpha \quad \to \quad \infty; \\
&\left\|A_g(y)-A'_g(y)\right\| \\
&\leq \frac{1}{\rho^\alpha}\left\{\left\|A_g(\rho^\alpha y)-g(\rho^\alpha y)\right\|+\left\|A'_g(\rho^\alpha y)-g(\rho^\alpha y)\right\|\right\} \\
&\leq \frac{2}{\rho}\sum_{\beta=0}^{\infty}\frac{E(\rho^{\beta+\alpha}y,\rho^{\beta+\alpha}y,\cdots,\rho^{\beta+\alpha}y)}{\rho^{\beta+\alpha}} \\
&\to 0 \quad as \quad \alpha \quad \to \quad \infty; \\
&\left\|A_h(z)-A'_h(z)\right\| \\
&\leq \frac{1}{\tau^\alpha}\left\{\left\|A_h(\tau^\alpha z)-h(\tau^\alpha z)\right\|\right\}+\left\|A'_h(\tau^\alpha z)-h(\tau^\alpha z)\right\| \\
&\leq \frac{2}{\tau}\sum_{\beta=0}^{\infty}\frac{O(\tau^{\beta+\alpha}z,\tau^{\beta+\alpha}z,\cdots,\tau^{\beta+\alpha}z)}{\tau^{\beta+\alpha}} \\
&\to 0 \quad as \quad \alpha \quad \to \quad \infty;
\end{aligned}
$$

for all $x,y,z \in \mathscr{A}$. Hence the mappings $A_f(x);A_g(y);A_h(z)$ are unique. Thus the theorem holds for $\gamma = 1$.

Also, if we replace

$$
\begin{aligned}
x &= \frac{x}{\kappa} \quad in \quad (3.16); \\
y &= \frac{y}{\rho} \quad in \quad (3.17); \\
z &= \frac{z}{\tau} \quad in \quad (3.18);
\end{aligned}
$$

we arrive

$$\left\|f(x)-\kappa\, f\left(\frac{x}{\kappa}\right)\right\| \leq N\left(\frac{x}{\kappa},\frac{x}{\kappa},\cdots,\frac{x}{\kappa}\right); \tag{3.31}$$

$$\left\|g(y)-\rho\, g\left(\frac{y}{\rho}\right)\right\| \leq E\left(\frac{y}{\rho},\frac{y}{\rho},\cdots,\frac{y}{\rho}\right); \tag{3.32}$$

$$\left\|h(z)-\tau\, h\left(\frac{z}{\tau}\right)\right\| \leq O\left(\frac{z}{\tau},\frac{z}{\tau},\cdots,\frac{z}{\tau}\right); \tag{3.33}$$

for all $x,y,z \in \mathscr{A}$. The rest of proof is similar to that of the case $\gamma = 1$. Thus the theorem holds for $\gamma = -1$ also. This completes the proof of the Theorem. □

The following corollary is an immediate consequence of Theorem 3.6 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and J.M.Rassias stabilities of the functional equations (1.27), (1.28) and (1.29).

Corollary 3.7. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be a mapping satisfying the following inequalities*

$$\left\|f\left(\sum_{a=1}^{n} a\, x_a\right)-\sum_{a=1}^{n}(a\, f(x_a))\right\| \leq \begin{cases} \lambda, & \\ \sum_{a=1}^{n}\lambda\left\|x_a\right\|^\mu, & \mu\neq 1; \\ \prod_{a=1}^{n}\lambda\left\|x_a\right\|^\mu+\sum_{a=1}^{n}\lambda\left\|x_a\right\|^{n\mu}, & n\mu\neq 1; \end{cases} \tag{3.34}$$

$$\left\|g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right)-\sum_{a=1}^{n}(2a\, g(y_{2a}))\right\| \leq \begin{cases} \lambda, & \\ \sum_{a=1}^{n}\lambda\left\|y_{2a}\right\|^\mu, & \mu\neq 1; \\ \prod_{a=1}^{n}\lambda\left\|y_{2a}\right\|^\mu+\sum_{a=1}^{n}\lambda\left\|y_{2a}\right\|^{n\mu}, & n\mu\neq 1; \end{cases} \tag{3.35}$$

$$\left\|h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right)-\sum_{a=1}^{n}((2a-1)\, h(z_{2a-1}))\right\| \leq \begin{cases} \lambda, & \\ \sum_{a=1}^{n}\lambda\left\|z_{2a-1}\right\|^\mu, & \mu\neq 1; \\ \prod_{a=1}^{n}\lambda\left\|z_{2a-1}\right\|^\mu+\sum_{a=1}^{n}\lambda\left\|z_{2a-1}\right\|^{n\mu}, & n\mu\neq 1; \end{cases} \tag{3.36}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$, where λ and μ are positive constants. Then there exists one and only additive mapping $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\left\|A_f(x)-f(x)\right\| \leq \begin{cases} \dfrac{\lambda}{|\kappa-1|}; \\ \dfrac{n\lambda\left\|x\right\|^\mu}{|\kappa-\kappa^\mu|}; \\ \dfrac{(n+1)\lambda\left\|x\right\|^{n\mu}}{|\kappa-\kappa^{n\mu}|}; \end{cases} \tag{3.37}$$

$$\left\|A_g(y)-g(y)\right\| \leq \begin{cases} \dfrac{\lambda}{|\rho-1|}; \\ \dfrac{n\lambda\left\|y\right\|^\mu}{|\rho-\rho^\mu|}; \\ \dfrac{(n+1)\lambda\left\|y\right\|^{n\mu}}{|\rho-\rho^{n\mu}|}; \end{cases} \tag{3.38}$$

$$\left\|A_h(z)-h(z)\right\| \leq \begin{cases} \dfrac{\lambda}{|\tau-1|}; \\ \dfrac{n\lambda\left\|z\right\|^\mu}{|\tau-\tau^\mu|}; \\ \dfrac{(n+1)\lambda\left\|z\right\|^{n\mu}}{|\tau-\tau^{n\mu}|}; \end{cases} \tag{3.39}$$

for all $x,y,z\in\mathscr{A}$, respectively.

Proof. If we replace

$$N(x_1,x_2,\cdots,x_n)=\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|x_a\|^{\mu},\\ \prod_{a=1}^{n}\lambda\,\|x_a\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|x_a\|^{n\mu},\end{cases}$$

$$E(y_2,y_4,\cdots,y_{2n})=\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|y_{2a}\|^{\mu},\\ \prod_{a=1}^{n}\lambda\,\|y_{2a}\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|y_{2a}\|^{n\mu},\end{cases}$$

$$O(z_1,z_3,\cdots,z_{2n-1})=\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|z_{2a-1}\|^{\mu},\\ \prod_{a=1}^{n}\lambda\,\|z_{2a-1}\|^{\mu}\\ \qquad+\sum_{a=1}^{n}\lambda\,\|z_{2a-1}\|^{n\mu},\end{cases}$$

in Theorem 3.6, we arrive our desired result. □

4. Stability Results In 2- Banach Space

In this section, we establish the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in 2- Banach spaces using Hyers Method.

Now, we give basic definitions and notations in 2- Banach spaces.

Definition 4.1. *Let X be a linear space of dimension greater than 1. Suppose $\|(\bullet,\bullet)\|$ is a real-valued function on $X\times X$ satisfying the following conditions:*

(2N1) $\|(x,y)\|=0$ *if and only if x,y are linearly dependent vectors,*

(2N2) $\|(x,y)\|=\|(y,x)\|$ *for all $x,y\in X$,*

(2N3) $\|(\lambda x,y)\|=|\lambda|\|(x,y)\|$ *for all $\lambda\in R$ and for all $x,y\in X$,*

(2N4) $\|(x+y,z)\|\le\|(x,z)\|+\|(y,z)\|$ *for all $x,y,z\in X$.*

Then $\|(\bullet,\bullet)\|$ is called a 2-norm on X and the pair $(X,\|(\bullet,\bullet)\|)$ is called 2-normed linear space.

Definition 4.2. *A sequence $\{x_n\}$ in a linear 2-normed space X is called a Cauchy sequence if there are two points $y,z\in X$ such that y and z are linearly independent,*

$$\lim_{l,m\to\infty}\|(x_l-x_m,y)\|=0 \qquad \text{and} \qquad \lim_{l,m\to\infty}\|(x_l-x_m,z)\|=0.$$

A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there is an $x,w\in X$ such that

$$\lim_{n\to\infty}\|(x_n-x,w)\|=0$$

for all $y\in X$. If $\{x_n\}$ converges to x, write $x_n\to x$ as $n\to\infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty}x_n=x$. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

To prove stability results, let us consider $\mathscr{A}$ be a 2-normed space and $\mathscr{B}$ be a 2-Banach space. The proof of the following theorem and corollary is similar lines to the of Theorem 3.6 and Corollary 3.7.

Theorem 4.3. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be a mapping satisfying the following inequalities*

$$\left\|\left(f\left(\sum_{a=1}^{n}a\,x_a\right)-\sum_{a=1}^{n}(a\,f(x_a)),u\right)\right\| \le N(x_1,x_2,\cdots,x_n); \tag{4.1}$$

$$\left\|\left(g\left(\sum_{a=1}^{n}2a\,y_{2a}\right)-\sum_{a=1}^{n}(2a\,g(y_{2a})),u\right)\right\| \le E(y_2,y_4,\cdots,y_{2n}); \tag{4.2}$$

$$\left\|\left(h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}((2a-1)\,h(z_{2a-1})),u\right)\right\| \le O(z_1,z_3,\cdots,z_{2n-1}); \tag{4.3}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ and all $u\in\mathscr{B}$, where $N:\mathscr{A}^n\to[0,\infty)$, $E:\mathscr{A}^{2n}\to[0,\infty)$ and $O:\mathscr{A}^{2n-1}\to[0,\infty)$ satisfying the conditions

$$\lim_{\alpha\to\infty}\frac{N(\kappa^{\alpha\gamma}x_1,\kappa^{\alpha\gamma}x_2,\cdots,\kappa^{\alpha\gamma}x_n)}{\kappa^{\alpha\gamma}}=0; \tag{4.4}$$

$$\lim_{\alpha\to\infty}\frac{E(\rho^{\alpha\gamma}y_2,\rho^{\alpha\gamma}y_4,\cdots,\rho^{\alpha\gamma}y_{2n})}{\rho^{\alpha\gamma}}=0; \tag{4.5}$$

$$\lim_{\alpha\to\infty}\frac{O(\tau^{\alpha\gamma}z_1,\tau^{\alpha\gamma}z_3,\cdots,\tau^{\alpha\gamma}z_{2n-1})}{\tau^{\alpha\gamma}}=0; \tag{4.6}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ with $\gamma=\pm1$. Then there exists one and only additive mapping $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\|(A_f(x)-f(x),u)\|\le\frac{1}{\kappa}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{N(\kappa^{\beta\gamma}x,\kappa^{\beta\gamma}x,\cdots,\kappa^{\beta\gamma}x)}{\kappa^{\beta\gamma}}; \tag{4.7}$$

$$\|(A_g(y)-g(y),u)\|\le\frac{1}{\rho}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{E(\rho^{\beta\gamma}y,\rho^{\beta\gamma}y,\cdots,\rho^{\beta\gamma}y)}{\rho^{\beta\gamma}}; \tag{4.8}$$

$$\|(A_h(z)-h(z),u)\|\le\frac{1}{\tau}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{O(\tau^{\beta\gamma}z,\tau^{\beta\gamma}z,\cdots,\tau^{\beta\gamma}z)}{\tau^{\beta\gamma}}; \tag{4.9}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$, respectively. The mappings A_f,A_g,A_h are respectively defined as

$$A_f(x)=\lim_{\alpha\to\infty}\frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}}; \tag{4.10}$$

$$A_g(y)=\lim_{\alpha\to\infty}\frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}}; \tag{4.11}$$

$$A_h(z)=\lim_{\alpha\to\infty}\frac{h(\tau^{\alpha\gamma}z)}{\tau^{\alpha\gamma}}; \tag{4.12}$$

for all $x,y,z\in\mathscr{A}$.

Corollary 4.4. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be a mapping satisfying the following inequalities*

$$\left\|\left(f\left(\sum_{a=1}^{n} a\,x_a\right)-\sum_{a=1}^{n}\left(a\,f(x_a)\right),u\right)\right\| \leq \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}, & \mu\neq 1;\\ \prod_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|x_a,u\|^{n\mu}, & n\mu\neq 1;\end{cases} \tag{4.13}$$

$$\left\|\left(g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,g(y_{2a})\right),u\right)\right\| \leq \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}, & \mu\neq 1;\\ \prod_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{n\mu}, & n\mu\neq 1;\end{cases} \tag{4.14}$$

$$\left\|\left(h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}\left((2a-1)\,h(z_{2a-1})\right),u\right)\right\| \leq \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}, & \mu\neq 1;\\ \prod_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{n\mu}, & n\mu\neq 1;\end{cases} \tag{4.15}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ and all $u\in\mathscr{B}$, where λ and μ are positive constants. Then there exists one and only additive mapping $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\|(A_f(x)-f(x),u)\| \leq \begin{cases}\dfrac{\lambda}{|\kappa-1|};\\ \dfrac{n\lambda\,\|x,u\|^{\mu}}{|\kappa-\kappa^{\mu}|};\\ \dfrac{(n+1)\lambda\,\|x,u\|^{n\mu}}{|\kappa-\kappa^{n\mu}|};\end{cases} \tag{4.16}$$

$$\|(A_g(y)-g(y),u)\| \leq \begin{cases}\dfrac{\lambda}{|\rho-1|};\\ \dfrac{n\lambda\,\|y,u\|^{\mu}}{|\rho-\rho^{\mu}|};\\ \dfrac{(n+1)\lambda\,\|y,u\|^{n\mu}}{|\rho-\rho^{n\mu}|};\end{cases} \tag{4.17}$$

$$\|(A_h(z)-h(z),u)\| \leq \begin{cases}\dfrac{\lambda}{|\tau-1|};\\ \dfrac{n\lambda\,\|z,u\|^{\mu}}{|\tau-\tau^{\mu}|};\\ \dfrac{(n+1)\lambda\,\|z,u\|^{n\mu}}{|\tau-\tau^{n\mu}|};\end{cases} \tag{4.18}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$, respectively.

5. Stability Results In Quasi 2-Banach Space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in Quasi 2-Banach Space using Hyers Method.

Now, we give basic definitions and notations in Quasi 2-Banach Space.

Definition 5.1. *Let X be a linear space of dimension greater than or equal to 2 . Suppose $||(\bullet,\bullet)||$ is a real-valued function on $X\times X$ satisfying the following conditions:*
(Q2N1) $||(x,y)||=0$ if and only if x,y are linearly dependent vectors,
(Q2N2) $||(x,y)||=||(y,x)||$ for all $x,y\in X$,
(Q2N3) $||(\lambda x,y)||=|\lambda|||(x,y)||$ for all $\lambda\in R$ and for all $x,y\in X$,
(Q2N4) It exists a constant $K\geq 1$ such that

$$||(x+y,z)||\leq K(||(x,z)||+||(y,z)||)$$

for all $x,y,z\in X$.
Then $||(\bullet,\bullet)||$ is called a quasi 2-norm on X and the pair $(X,||(\bullet,\bullet)||)$ is called quasi 2-normed linear space. The smallest possible number K such that it satisfies the condition (Q2N4) is called a modulus of concavity of the quasi 2-norm $||(\bullet,\bullet)||$.

Sometimes the condition (Q2N4) called the triangle inequality. Further, M. Kir and M. Acikgoz [26] gave few examples of trivial quasi 2-normed spaces and consider the question about completing the quasi 2-normed space. A quasi 2-normed space in which every Cauchy sequence is a convergent sequence is called a quasi 2-Banach space.

To prove stability results, let us consider $\mathscr{A}$ be a Quasi 2-normed space and $\mathscr{B}$ be a Quasi 2-Banach space. The proof of the following theorem and corollary is similar lines to the of Theorem 3.6 and Corollary 3.7.

Theorem 5.2. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be a mapping satisfying the following inequalities*

$$\left\|\left(f\left(\sum_{a=1}^{n} a\,x_a\right)-\sum_{a=1}^{n}\left(a\,f(x_a)\right),u\right)\right\| \leq N(x_1,x_2,\cdots,x_n); \tag{5.1}$$

$$\left\|\left(g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,g(y_{2a})\right),u\right)\right\| \leq E(y_2,y_4,\cdots,y_{2n}); \tag{5.2}$$

$$\left\|\left(h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}\left((2a-1)\,h(z_{2a-1})\right),u\right)\right\| \leq O(z_1,z_3,\cdots,z_{2n-1}); \tag{5.3}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ and all $u\in\mathscr{B}$, where $N:\mathscr{A}^n\to[0,\infty)$, $E:\mathscr{A}^{2n}\to[0,\infty)$ and

$O:\mathscr{A}^{2n-1}\to[0,\infty)$ *satisfying the conditions*

$$\lim_{\alpha\to\infty}\frac{N(\kappa^{\alpha\gamma}x_1,\kappa^{\alpha\gamma}x_2,\cdots,\kappa^{\alpha\gamma}x_n)}{\kappa^{\alpha\gamma}}=0; \tag{5.4}$$
$$\lim_{\alpha\to\infty}\frac{E(\rho^{\alpha\gamma}y_2,\rho^{\alpha\gamma}y_4,\cdots,\rho^{\alpha\gamma}y_{2n})}{\rho^{\alpha\gamma}}=0; \tag{5.5}$$
$$\lim_{\alpha\to\infty}\frac{O(\tau^{\alpha\gamma}z_1,\tau^{\alpha\gamma}z_3,\cdots,\tau^{\alpha\gamma}z_{2n-1})}{\tau^{\alpha\gamma}}=0; \tag{5.6}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ *with* $\gamma=\pm1$. *Then there exists one and only additive mapping* $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ *satisfying (1.27), (1.28), (1.29), and*

$$\left\|\left(A_f(x)-f(x),u\right)\right\| \le \frac{K^{\alpha-1}}{\kappa}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{N(\kappa^{\beta\gamma}x,\kappa^{\beta\gamma}x,\cdots,\kappa^{\beta\gamma}x)}{\kappa^{\beta\gamma}}; \tag{5.7}$$
$$\left\|\left(A_g(y)-g(y),u\right)\right\| \le \frac{K^{\alpha-1}}{\rho}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{E(\rho^{\beta\gamma}y,\rho^{\beta\gamma}y,\cdots,\rho^{\beta\gamma}y)}{\rho^{\beta\gamma}}; \tag{5.8}$$
$$\left\|\left(A_h(z)-h(z),u\right)\right\| \le \frac{K^{\alpha-1}}{\tau}\sum_{\beta=\frac{1-\gamma}{2}}^{\infty}\frac{O(\tau^{\beta\gamma}z,\tau^{\beta\gamma}z,\cdots,\tau^{\beta\gamma}z)}{\tau^{\beta\gamma}}; \tag{5.9}$$

for all $x,y,z\in\mathscr{A}$ *and all* $u\in\mathscr{B}$, *respectively. The mappings* A_f,A_g,A_h *are respectively defined as*

$$A_f(x)=\lim_{\alpha\to\infty}\frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}}; \tag{5.10}$$
$$A_g(y)=\lim_{\alpha\to\infty}\frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}}; \tag{5.11}$$
$$A_h(z)=\lim_{\alpha\to\infty}\frac{h(\tau^{\alpha\gamma}z)}{\tau^{\alpha\gamma}}; \tag{5.12}$$

for all $x,y,z\in\mathscr{A}$.

Corollary 5.3. *Let* $f,g,h:\mathscr{A}\to\mathscr{B}$ *be a mapping satisfying the following inequalities*

$$\left\|\left(f\left(\sum_{a=1}^{n}a\,x_a\right)-\sum_{a=1}^{n}\left(a\,f(x_a)\right),u\right)\right\| \le \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|x_a,u\|^{n\mu}, & n\mu\neq1;\end{cases} \tag{5.13}$$

$$\left\|\left(g\left(\sum_{a=1}^{n}2a\,y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,g(y_{2a})\right),u\right)\right\| \le \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{n\mu}, & n\mu\neq1;\end{cases} \tag{5.14}$$

$$\left\|\left(h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}\left((2a-1)\,h(z_{2a-1})\right),u\right)\right\| \le \begin{cases}\lambda, & \\ \sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{n\mu}, & n\mu\neq1;\end{cases} \tag{5.15}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ *and all* $u\in\mathscr{B}$, *where* λ *and* μ *are positive constants. Then there exists one and only additive mapping* $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ *satisfying (1.27), (1.28), (1.29), and*

$$\left\|\left(A_f(x)-f(x),u\right)\right\| \le \begin{cases}\dfrac{K^{\alpha-1}\lambda}{|\kappa-1|};\\ \dfrac{K^{\alpha-1}n\lambda\,\|x,u\|^{\mu}}{|\kappa-\kappa^{\mu}|};\\ \dfrac{K^{\alpha-1}(n+1)\lambda\,\|x,u\|^{n\mu}}{|\kappa-\kappa^{n\mu}|};\end{cases} \tag{5.16}$$

$$\left\|\left(A_g(y)-g(y),u\right)\right\| \le \begin{cases}\dfrac{K^{\alpha-1}\lambda}{|\rho-1|};\\ \dfrac{K^{\alpha-1}n\lambda\,\|y,u\|^{\mu}}{|\rho-\rho^{\mu}|};\\ \dfrac{K^{\alpha-1}(n+1)\lambda\,\|y,u\|^{n\mu}}{|\rho-\rho^{n\mu}|};\end{cases} \tag{5.17}$$

$$\left\|\left(A_h(z)-h(z),u\right)\right\| \le \begin{cases}\dfrac{K^{\alpha-1}\lambda}{|\tau-1|};\\ \dfrac{K^{\alpha-1}n\lambda\,\|z,u\|^{\mu}}{|\tau-\tau^{\mu}|};\\ \dfrac{K^{\alpha-1}(n+1)\lambda\,\|z,u\|^{n\mu}}{|\tau-\tau^{n\mu}|};\end{cases} \tag{5.18}$$

for all $x,y,z\in\mathscr{A}$ *and all* $u\in\mathscr{B}$, *respectively.*

6. Stability Results In Quasi-Beta-2-Banach Space

In this section, we discussed the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in quasi-β-2-Banach space using Hyers Method.

Now, we give basic definitions and notations in quasi-β-2-Banach space.

Definition 6.1. *Let* X *be a linear space of dimension greater than or equal to 2. Suppose* $\|(\bullet,\bullet)\|$ *is a real-valued function on* $X\times X$ *satisfying the following conditions:*

(QB2N1) $||(x,y)|| = 0$ *if and only if* x,y *are linearly dependent vectors,*

(QB2N2) $||(x,y)|| = ||(y,x)||$ *for all* $x,y \in X$,

(QB2N3) $||(\lambda x,y)|| = |\lambda|^{\beta}||(x,y)||$ *for all* $\lambda \in R$ *and for all* $x,y \in X$ *where* β *is a a real number with* $0 < \beta \leq 1$

(QB2N4) *If exists a constant* $K \geq 1$ *such that* $||(x+y,z)|| \leq K(||(x,z)||+||(y,z)||)$ *for all* $x,y,z \in X$.

The pair $(X,||(\bullet,\bullet)||)$ *is called quasi-*β*-normed space if* $||(\bullet,\bullet)||$ *is a quasi-*β*-2-norm on* X. *The smallest possible* K *is called the modulus of concavity of* $\|\cdot\|$.

Definition 6.2. *A quasi-*β*-2-Banach space is a complete quasi-*β*-normed space.*

To prove stability results, let us consider $\mathscr{A}$ as a quasi-β-2-Banach space and $\mathscr{B}$ as a quasi-β-2-Banach space.

Theorem 6.3. *Let* $f,g,h : \mathscr{A} \to \mathscr{B}$ *be a mapping satisfying the following inequalities*

$$\left\|\left(f\left(\sum_{a=1}^{n} a\, x_a\right) - \sum_{a=1}^{n}\left(a\, f(x_a)\right),u\right)\right\| \leq N(x_1,x_2,\cdots,x_n); \tag{6.1}$$

$$\left\|\left(g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) - \sum_{a=1}^{n}\left(2a\, g(y_{2a})\right),u\right)\right\| \leq E(y_2,y_4,\cdots,y_{2n}); \tag{6.2}$$

$$\left\|\left(h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) - \sum_{a=1}^{n}\left((2a-1)\, h(z_{2a-1})\right),u\right)\right\| \leq O(z_1,z_3,\cdots,z_{2n-1}); \tag{6.3}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ *and all* $u \in \mathscr{B}$, *where* $N : \mathscr{A}^n \to [0,\infty)$, $E : \mathscr{A}^{2n} \to [0,\infty)$ *and* $O : \mathscr{A}^{2n-1} \to [0,\infty)$ *satisfying the conditions*

$$\lim_{\alpha\to\infty} \frac{N(\kappa^{\alpha\gamma}x_1,\kappa^{\alpha\gamma}x_2,\cdots,\kappa^{\alpha\gamma}x_n)}{\kappa^{\alpha\gamma}} = 0; \tag{6.4}$$

$$\lim_{\alpha\to\infty} \frac{E(\rho^{\alpha\gamma}y_2,\rho^{\alpha\gamma}y_4,\cdots,\rho^{\alpha\gamma}y_{2n})}{\rho^{\alpha\gamma}} = 0; \tag{6.5}$$

$$\lim_{\alpha\to\infty} \frac{O(\tau^{\alpha\gamma}z_1,\tau^{\alpha\gamma}z_3,\cdots,\tau^{\alpha\gamma}z_{2n-1})}{\tau^{\alpha\gamma}} = 0; \tag{6.6}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ *with* $\gamma = \pm 1$. *Then there exists one and only additive mapping* $A_f,A_g,A_h : \mathscr{A} \to \mathscr{B}$ *satisfying (1.27), (1.28), (1.29), and*

$$\left\|\left(A_f(x) - f(x),u\right)\right\| \leq \frac{K^{\alpha-1}}{\kappa^{\beta}} \sum_{\eta=\frac{1-\gamma}{2}}^{\infty} \frac{N(\kappa^{\eta\gamma}x,\kappa^{\eta\gamma}x,\cdots,\kappa^{\eta\gamma}x)}{\kappa^{\eta\gamma}}; \tag{6.7}$$

$$\left\|\left(A_g(y) - g(y),u\right)\right\| \leq \frac{K^{\alpha-1}}{\rho^{\beta}} \sum_{\eta=\frac{1-\gamma}{2}}^{\infty} \frac{E(\rho^{\eta\gamma}y,\rho^{\eta\gamma}y,\cdots,\rho^{\eta\gamma}y)}{\rho^{\eta\gamma}}; \tag{6.8}$$

$$\left\|\left(A_h(z) - h(z),u\right)\right\| \leq \frac{K^{\alpha-1}}{\tau^{\beta}} \sum_{\eta=\frac{1-\gamma}{2}}^{\infty} \frac{O(\tau^{\eta\gamma}z,\tau^{\eta\gamma}z,\cdots,\tau^{\eta\gamma}z)}{\tau^{\eta\gamma}}; \tag{6.9}$$

for all $x,y,z \in \mathscr{A}$ *and all* $u \in \mathscr{B}$, *respectively. The mappings* A_f,A_g,A_h *are respectively defined as*

$$A_f(x) = \lim_{\alpha\to\infty} \frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}}; \tag{6.10}$$

$$A_g(y) = \lim_{\alpha\to\infty} \frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}}; \tag{6.11}$$

$$A_h(z) = \lim_{\alpha\to\infty} \frac{h(\tau^{\alpha\gamma}z)}{\tau^{\alpha\gamma}}; \tag{6.12}$$

for all $x,y,z \in \mathscr{A}$.

Proof. Changing

$$\begin{aligned}
(x_1,x_2,\cdots,x_n) &= (x,x,\cdots,x) \quad in \quad (6.1);\\
(y_2,y_4,\cdots,y_{2n}) &= (y,y,\cdots,y) \quad in \quad (6.2);\\
(z_1,z_3,\cdots,z_{2n-1}) &= (z,z,\cdots,z) \quad in \quad (6.3);
\end{aligned}$$

we arrive the following inequalities

$$\left\|f\left(\sum_{a=1}^{n} a\, x\right) - \left(\sum_{a=1}^{n} a\right)\, f(x)\right\| \leq N(x,x,\cdots,x); \tag{6.13}$$

$$\left\|g\left(\sum_{a=1}^{n} 2a\, y\right) - \left(\sum_{a=1}^{n} 2a\right)\, g(y)\right\| \leq E(y,y,\cdots,y); \tag{6.14}$$

$$\left\|h\left(\sum_{a=1}^{n} (2a-1)\, z\right) - \left(\sum_{a=1}^{n} (2a-1)\right)\, h(z)\right\| \leq O(z,z,\cdots,z); \tag{6.15}$$

for all $x,y,z \in \mathscr{A}$. Define

$$\begin{aligned}
\sum_{a=1}^{n} a &= \frac{n(n+1)}{2} = \kappa \quad in \quad (6.13);\\
\sum_{a=1}^{n} 2a &= n(n+1) = \rho \quad in \quad (6.14);\\
\sum_{a=1}^{n} (2a-1) &= n^2 = \tau \quad in \quad (6.15);
\end{aligned}$$

we obtain the succeeding inequalities

$$\|f(\kappa x)-\kappa f(x)\|\leq N(x,x,\cdots,x); \tag{6.16}$$
$$\|g(\rho y)-\rho g(y)\|\leq E(y,y,\cdots,y); \tag{6.17}$$
$$\|h(\tau z)-\tau h(z)\|\leq O(z,z,\cdots,z); \tag{6.18}$$

for all $x,y,z\in\mathscr{A}$. It follows from above inequalities

$$\left\|\frac{f(\kappa x)}{\kappa}-f(x)\right\|\leq\frac{N(x,x,\cdots,x)}{\kappa^{\beta}}; \tag{6.19}$$
$$\left\|\frac{g(\rho y)}{\rho}-g(y)\right\|\leq\frac{E(y,y,\cdots,y)}{\rho^{\beta}}; \tag{6.20}$$
$$\left\|\frac{h(\tau z)}{\tau}-h(z)\right\|\leq\frac{O(z,z,\cdots,z)}{\tau^{\beta}}; \tag{6.21}$$

for all $x,y,z\in\mathscr{A}$. Replacing

$$x=\kappa x \quad and \quad \div \quad \kappa \quad in \quad (6.19);$$
$$y=\rho y \quad and \quad \div \quad \rho \quad in \quad (6.20);$$
$$z=\tau z \quad and \quad \div \quad \tau \quad in \quad (6.21);$$

we arrive the following inequalities

$$\left\|\frac{f(\kappa^2 x)}{\kappa^2}-\frac{f(\kappa x)}{\kappa}\right\|\leq\frac{N(\kappa x,\kappa x,\cdots,\kappa x)}{\kappa^{\beta}\kappa}; \tag{6.22}$$
$$\left\|\frac{g(\rho^2 y)}{\rho^2}-\frac{g(\rho y)}{\rho}\right\|\leq\frac{E(\rho y,\rho y,\cdots,\rho y)}{\rho^{\beta}\rho}; \tag{6.23}$$
$$\left\|\frac{h(\tau^2 z)}{\tau^2}-\frac{h(\tau z)}{\tau}\right\|\leq\frac{O(\tau z,\tau z,\cdots,\tau z)}{\tau^{\beta}\tau}; \tag{6.24}$$

for all $x,y,z\in\mathscr{A}$. With the help of triangle inequality from

$$(6.19)\ and\ (6.22); \quad (6.20)\ and\ (6.23); \quad (6.21)\ and\ (6.24);$$

we achieve the subsequent inequalities

$$\begin{aligned}&\left\|\frac{f(\kappa^2 x)}{\kappa^2}-f(x)\right\|\\ &\leq\left\|\frac{f(\kappa^2 x)}{\kappa^2}-\frac{f(\kappa x)}{\kappa}\right\|+\left\|\frac{f(\kappa x)}{\kappa}-f(x)\right\|\\ &\leq\frac{K}{\kappa^{\beta}}\left(N(x,x,\cdots,x)+\frac{N(\kappa x,\kappa x,\cdots,\kappa x)}{\kappa}\right);\end{aligned} \tag{6.25}$$

$$\begin{aligned}&\left\|\frac{g(\rho^2 y)}{\rho^2}-g(y)\right\|\\ &\leq\left\|\frac{g(\rho^2 y)}{\rho^2}-\frac{g(\rho y)}{\rho}\right\|+\left\|\frac{g(\rho y)}{\rho}-g(y)\right\|\\ &\leq\frac{K}{\rho^{\beta}}\left(E(y,y,\cdots,y)+\frac{E(\rho y,\rho y,\cdots,\rho y)}{\rho}\right);\end{aligned} \tag{6.26}$$

$$\begin{aligned}&\left\|\frac{h(\tau^2 z)}{\tau^2}-h(z)\right\|\\ &\leq\left\|\frac{h(\tau^2 z)}{\tau^2}-\frac{h(\tau z)}{\tau}\right\|+\left\|\frac{h(\tau z)}{\tau}-h(z)\right\|\\ &\leq\frac{K}{\tau^{\beta}}\left(E(z,z,\cdots,z)+\frac{E(\tau z,\tau z,\cdots,\tau\,)}{\tau}\right);\end{aligned} \tag{6.27}$$

for all $x,y,z\in\mathscr{A}$. Generalizing for a positive integer α, we arrive

$$\left\|\frac{f(\kappa^{\alpha} x)}{\kappa^{\alpha}}-f(x)\right\|\leq\frac{K^{\alpha-1}}{\kappa^{\beta}}\sum_{\eta=0}^{\alpha}\frac{N(\kappa^{\eta} x,\kappa^{\eta} x,\cdots,\kappa^{\eta} x)}{\kappa^{\eta}}; \tag{6.28}$$

$$\left\|\frac{g(\rho^{\alpha} y)}{\rho^{\alpha}}-g(y)\right\|\leq\frac{K^{\alpha-1}}{\rho^{\beta}}\sum_{\eta=0}^{\alpha}\frac{E(\rho^{\eta} y,\rho^{\eta} y,\cdots,\rho^{\eta} y)}{\rho^{\eta}}; \tag{6.29}$$

$$\left\|\frac{h(\tau^{\alpha} z)}{\tau^{\alpha}}-h(z)\right\|\leq\frac{K^{\alpha-1}}{\tau^{\beta}}\sum_{\eta=0}^{\alpha}\frac{O(\tau^{\eta} z,\tau^{\eta} z,\cdots,\tau^{\eta} z)}{\tau^{\eta}}; \tag{6.30}$$

for all $x,y,z\in\mathscr{A}$. The rest of proof is similar lines to the of Theorem 3.6. □

Corollary 6.4. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be a mapping satisfying the following inequalities*

$$\begin{aligned}&\left\|\left(f\left(\sum_{a=1}^{n} a\,x_a\right)-\sum_{a=1}^{n}(a\,f(x_a)),u\right)\right\|\\ &\leq\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|x_a,u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|x_a,u\|^{n\mu}, & n\mu\neq1;\end{cases}\end{aligned} \tag{6.31}$$

$$\begin{aligned}&\left\|\left(g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)-\sum_{a=1}^{n}(2a\,g(y_{2a})),u\right)\right\|\\ &\leq\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|y_{2a},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|y_{2a},u\|^{n\mu}, & n\mu\neq1;\end{cases}\end{aligned} \tag{6.32}$$

$$\begin{aligned}&\left\|\left(h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)-\sum_{a=1}^{n}((2a-1)\,h(z_{2a-1})),u\right)\right\|\\ &\leq\begin{cases}\lambda,\\ \sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}, & \mu\neq1;\\ \prod_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{\mu}+\sum_{a=1}^{n}\lambda\,\|z_{2a-1},u\|^{n\mu}, & n\mu\neq1;\end{cases}\end{aligned} \tag{6.33}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ and all $u\in\mathscr{B}$, where λ and μ are positive constants. Then there exists one and only additive mapping $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$

satisfying (1.27), (1.28), (1.29), and

$$\left\|\left(A_f(x)-f(x),u\right)\right\| \leq \begin{cases} \dfrac{K^{\alpha-1}\lambda\kappa}{\kappa^{\beta}|\kappa-1|}; \\ \dfrac{K^{\alpha-1}n\lambda\kappa\|x,u\|^{\mu}}{\kappa^{\beta}|\kappa-\kappa^{\mu}|}; \\ \dfrac{K^{\alpha-1}(n+1)\lambda\kappa\|x,u\|^{n\mu}}{\kappa^{\beta}|\kappa-\kappa^{n\mu}|}; \end{cases} \tag{6.34}$$

$$\left\|\left(A_g(y)-g(y),u\right)\right\| \leq \begin{cases} \dfrac{K^{\alpha-1}\lambda\rho}{\rho^{\beta}|\rho-1|}; \\ \dfrac{K^{\alpha-1}n\lambda\rho\|y,u\|^{\mu}}{\rho^{\beta}|\rho-\rho^{\mu}|}; \\ \dfrac{K^{\alpha-1}(n+1)\lambda\rho\|y,u\|^{n\mu}}{\rho^{\beta}|\rho-\rho^{n\mu}|}; \end{cases} \tag{6.35}$$

$$\left\|\left(A_h(z)-h(z),u\right)\right\| \leq \begin{cases} \dfrac{K^{\alpha-1}\lambda\tau}{\tau^{\beta}|\tau-1|}; \\ \dfrac{K^{\alpha-1}n\lambda\tau\|z,u\|^{\mu}}{\tau^{\beta}|\tau-\tau^{\mu}|}; \\ \dfrac{K^{\alpha-1}(n+1)\lambda\tau\|z,u\|^{n\mu}}{\tau^{\beta}|\tau-\tau^{n\mu}|}; \end{cases} \tag{6.36}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$, respectively.

7. Stability Results In Fuzzy Quasi-Beta-2-Banach Space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in fuzzy quasi-β-2-Banach spaces using Hyers Method.

Now, we give basic definitions and notations in Fuzzy quasi-β-2-Banach space.

Definition 7.1. *Let X be a linear space of dimension greater than or equal to 2 . A function $N : X \times X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy quasi-β-2-norm on X if for all $x,y,z \in X$ and all $s,t \in \mathbb{R}$,*
(2QBFN1) $N(x,z,c) = 0$ for $c \leq 0$;
(2QBFN2) $x = 0$ if and only if $N(x,z,c) = 1$ for all $c > 0$;
(2QBFN3) $N(cx,z,t) = N\left(x,z,\frac{t}{|c^{\beta}|}\right)$ if $c \neq 0$ where β is a a real number with $0 < \beta \leq 1$
(2QBFN4) $N(x+y,z,s+t) \geq min\{N(x,z,Ks),N(y,z,Kt)\}$; a constant $K \geq 1$
(2QBFN5) $N(x,z,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty}N(x,,z,t) = 1$;
(2QBFN6) *for $x \neq 0, N(x,z,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.*
The pair (X,X,N) is called a fuzzy quasi-β-2-Banach space.

Example 7.2. *Let X be a linear space. Then*

$$N(x,z,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0, \ x,z \in X, \\ 0, & t \leq 0, \ x,z \in X \end{cases}$$

is a fuzzy quasi-β-2-normed space on X.

Example 7.3. *Let X be a linear space. Then*

$$N(x,z,t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{t}{\|x\|}, & 0 < t \leq \|x\|, z \in X \\ 1, & t > \|x\|, z \in X \end{cases}$$

is a fuzzy quasi-β-2-normed space on X.

Definition 7.4. *Let X be a fuzzy quasi-β-2-normed space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x,z \in X$ such that $\lim_{n\to\infty} N(x_n - x,z,t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 7.5. *A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n,z,t) > 1-\varepsilon$.*

Definition 7.6. *Every convergent sequence in a fuzzy quasi-β-2-normed space is Cauchy. If each Cauchy sequence is convergent, then the 2- norm is said to be complete and the fuzzy quasi-β-2-normed space is called a fuzzy quasi-β-2-Banach space.*

To prove stability results, let us consider $\mathscr{A}$ is a fuzzy quasi-β-2-normed space and $\mathscr{B}$ is a fuzzy quasi-β-2-Banach space.

Theorem 7.7. *Let $f,g,h : \mathscr{A} \to \mathscr{B}$ be a mapping satisfying the following inequalities*

$$\mathscr{N}\left(\left(f\left(\sum_{a=1}^{n} a\,x_a\right) - \sum_{a=1}^{n}\left(a\,f(x_a)\right),u\right),t\right) \geq \mathscr{N}'\left(N(x_1,x_2,\cdots,x_n),u,t\right); \tag{7.1}$$

$$\mathscr{N}\left(\left(g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right) - \sum_{a=1}^{n}\left(2a\,g(y_{2a})\right),u\right),t\right) \geq \mathscr{N}'\left(E(y_2,y_4,\cdots,y_{2n}),u,t\right); \tag{7.2}$$

$$\mathscr{N}\left(\left(h\left(\sum_{a=1}^{n} (2a-1)\,z_{2a-1}\right) - \sum_{a=1}^{n}\left((2a-1)\,h(z_{2a-1})\right),u\right),t\right) \geq \mathscr{N}'\left(O(z_1,z_3,\cdots,z_{2n-1}),u,t\right); \tag{7.3}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, where $N : \mathscr{A}^n \to [0,\infty)$, $E : \mathscr{A}^{2n} \to [0,\infty)$ and $O : \mathscr{A}^{2n-1} \to [0,\infty)$ satisfying the conditions

$$\lim_{\alpha\to\infty} \mathscr{N}'\left(N(\kappa^{\alpha\gamma}x_1,\kappa^{\alpha\gamma}x_2,\cdots,\kappa^{\alpha\gamma}x_n),u,\kappa^{\alpha\gamma}t\right) = 1; \tag{7.4}$$

$$\lim_{\alpha\to\infty} \mathscr{N}'\left(E(\rho^{\alpha\gamma}y_2,\rho^{\alpha\gamma}y_4,\cdots,\rho^{\alpha\gamma}y_{2n}),u,\rho^{\alpha\gamma}t\right) = 1; \tag{7.5}$$

$$\lim_{\alpha\to\infty} \mathscr{N}'\left(O(\tau^{\alpha\gamma}z_1,\tau^{\alpha\gamma}z_3,\cdots,\tau^{\alpha\gamma}z_{2n-1}),u,\tau^{\alpha\gamma}t\right) = 1; \tag{7.6}$$

with the conditions

$$\mathcal{N}'\left(N(\kappa^\alpha x_1,\kappa^\alpha x_2,\cdots,\kappa^\alpha x_n),u,t\right) = \mathcal{N}'\left(\nu^\alpha N(x_1,x_2,\cdots,x_n),u,t\right); \tag{7.7}$$

$$\mathcal{N}'\left(E(\rho^\alpha y_2,\rho^\alpha y_4,\cdots,\rho^\alpha y_{2n}),u,t\right) = \mathcal{N}'\left(\nu^\alpha E(y_2,y_4,\cdots,y_{2n}),u,t\right); \tag{7.8}$$

$$\mathcal{N}'\left(O(\tau^\alpha z_1,\tau^\alpha z_3,\cdots,\tau^\alpha z_{2n-1}),u,t\right) = \mathcal{N}'\left(\nu^\alpha O(z_1,z_3,\cdots,z_{2n-1}),u,t\right); \tag{7.9}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ all $u\in\mathscr{B}$ and $t>0$ with $\gamma=\pm1$. Then there exists one and only additive mapping $A_f,A_g,A_h:\mathscr{A}\to\mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\mathcal{N}\left(\left(A_f(x)-f(x),u\right),t\right) \geq \mathcal{N}'\left(N(x,x,\cdots,x),u,K\kappa^{\beta-1}|\kappa-\nu|t\right); \tag{7.10}$$

$$\mathcal{N}\left(\left(A_g(y)-g(y),u\right),t\right) \geq \mathcal{N}'\left(E(y,y,\cdots,y),u,K\rho^{\beta-1}|\rho-\nu|t\right); \tag{7.11}$$

$$\mathcal{N}\left(\left(A_h(z)-h(z),u\right),t\right) \geq \mathcal{N}'\left(O(z,z,\cdots,z),u,K\tau^{\beta-1}|\tau-\nu|t\right); \tag{7.12}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$, respectively. The mappings A_f,A_g,A_h are respectively defined as

$$\lim_{\alpha\to\infty}\mathcal{N}\left(\left(A_f(x)-\frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}},u\right),t\right)=1; \tag{7.13}$$

$$\lim_{\alpha\to\infty}\mathcal{N}\left(\left(A_g(y)-\frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}},u\right),t\right)=1; \tag{7.14}$$

$$\lim_{\alpha\to\infty}\mathcal{N}\left(\left(A_h(y)-\frac{h(\tau^{\alpha\gamma}y)}{\tau^{\gamma}},u\right),t\right)=1; \tag{7.15}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$.

Proof. Changing

$$(x_1,x_2,\cdots,x_n)=(x,x,\cdots,x) \quad in \quad (7.1);$$
$$(y_2,y_4,\cdots,y_{2n})=(y,y,\cdots,y) \quad in \quad (7.2);$$
$$(z_1,z_3,\cdots,z_{2n-1})=(z,z,\cdots,z) \quad in \quad (7.3);$$

we arrive the following inequalities

$$\mathcal{N}\left(\left(f\left(\sum_{a=1}^{n} a\,x\right)-\left(\sum_{a=1}^{n} a\right)\,f(x),u\right),t\right) \geq \mathcal{N}'\left(N(x,x,\cdots,x),u,t\right); \tag{7.16}$$

$$\mathcal{N}\left(\left(g\left(\sum_{a=1}^{n} 2a\,y\right)-\left(\sum_{a=1}^{n} 2a\right)\,g(y),u\right),t\right) \geq \mathcal{N}'\left(E(y,y,\cdots,y),u,t\right); \tag{7.17}$$

$$\mathcal{N}\left(\left(h\left(\sum_{a=1}^{n} (2a-1)\,z\right)-\left(\sum_{a=1}^{n}(2a-1)\right)\,h(z),u\right),t\right) \geq \mathcal{N}'\left(O(z,z,\cdots,z),u,t\right); \tag{7.18}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and all $t>0$. Define

$$\sum_{a=1}^{n} a=\frac{n(n+1)}{2}=\kappa \quad in \quad (7.16);$$
$$\sum_{a=1}^{n} 2a=n(n+1)=\rho \quad in \quad (7.17);$$
$$\sum_{a=1}^{n} (2a-1)=n^2=\tau \quad in \quad (7.18);$$

we obtain the succeeding inequalities

$$\mathcal{N}\left(\left(f(\kappa\,x)-\kappa\,f(x),u\right),t\right)\geq\mathcal{N}'\left(N(x,x,\cdots,x),u,t\right); \tag{7.19}$$

$$\mathcal{N}\left(\left(g(\rho\,y)-\rho\,g(y),u\right),t\right)\geq\mathcal{N}'\left(E(y,y,\cdots,y),u,t\right); \tag{7.20}$$

$$\mathcal{N}\left(\left(h(\tau\,z)-\tau\,h(z),u\right),t\right)\geq\mathcal{N}'\left(O(z,z,\cdots,z),u,t\right); \tag{7.21}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$. Using (2QBFN3) it follows from above inequalities

$$\mathcal{N}\left(\left(\frac{f(\kappa\,x)}{\kappa}-f(x),u\right),t\right)\geq\mathcal{N}'\left(N(x,x,\cdots,x),u,\kappa^\beta t\right); \tag{7.22}$$

$$\mathcal{N}\left(\left(\frac{g(\rho\,y)}{\rho}-g(y),u\right),t\right)\geq\mathcal{N}'\left(E(y,y,\cdots,y),u,\rho^\beta t\right); \tag{7.23}$$

$$\mathcal{N}\left(\left(\frac{h(\tau\,z)}{\tau}-h(z),u\right),t\right)\geq\mathcal{N}'\left(O(z,z,\cdots,z),u,\tau^\beta t\right); \tag{7.24}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$. Replacing, using and substituting

$$x=\kappa^\alpha\,x,\ (2QBFN3),(7.7)\ in\ (7.22)\ and\ t=\nu^{\alpha\beta}t\ in\ (7.22);$$
$$y=\rho^\alpha\,y,\ (2QBFN3),(7.8)\ in\ (7.23)\ and\ t=\nu^{\alpha\beta}t\ in\ (7.23);$$
$$z=\tau^\alpha\,x,\ (2QBFN3),(7.9)\ in\ (7.24)\ and\ t=\nu^{\alpha\beta}t\ in\ (7.24);$$

we arrive the following inequalities

$$\mathcal{N}\left(\left(\frac{f\left(\kappa^{\alpha+1}\,x\right)}{\kappa^{\alpha+1}}-\frac{f(\kappa^\alpha x)}{\kappa^\alpha},u\right),\frac{\nu^{\alpha\beta}t}{\kappa^{\alpha\beta}}\right) \geq \mathcal{N}'\left(N(x,x,\cdots,x),u,\kappa^\beta t\right); \tag{7.25}$$

$$\mathcal{N}\left(\left(\frac{g\left(\rho^{\alpha+1}\,y\right)}{\rho^{\alpha+1}}-\frac{g(\rho^\alpha y)}{\rho^\alpha},u\right),\frac{\nu^{\alpha\beta}t}{\rho^{\alpha\beta}}\right) \geq \mathcal{N}'\left(E(y,y,\cdots,y),u,\rho^\beta t\right); \tag{7.26}$$

$$\mathcal{N}\left(\left(\frac{h\left(\tau^{\alpha+1}\,z\right)}{\tau^{\alpha+1}}-\frac{h(\tau^\alpha z)}{\tau^\alpha},u\right),\frac{\nu^{\alpha\beta}t}{\tau^{\alpha\beta}}\right) \geq \mathcal{N}'\left(O(z,z,\cdots,z),u,\tau^\beta t\right); \tag{7.27}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. It is easy to see that

$$\frac{f(\kappa^{\alpha}x)}{\kappa^{\alpha}} - f(x) = \sum_{\sigma=0}^{\alpha-1} \frac{f(\kappa^{\sigma+1}x)}{\kappa^{\sigma+1}} - \frac{f(\kappa^{\sigma}x)}{\kappa^{\sigma}}; \quad (7.28)$$

$$\frac{g(\rho^{\alpha}y)}{\rho^{\alpha}} - g(y) = \sum_{\sigma=0}^{\alpha-1} \frac{g(\rho^{\sigma+1}y)}{\rho^{\sigma+1}} - \frac{g(\kappa^{\sigma}y)}{\kappa^{\sigma}}; \quad (7.29)$$

$$\frac{h(\kappa^{\alpha}z)}{\kappa^{\alpha}} - h(z) = \sum_{\sigma=0}^{\alpha-1} \frac{h(\tau^{\sigma+1}z)}{\tau^{\sigma+1}} - \frac{h(\tau^{\sigma}z)}{\tau^{\sigma}}; \quad (7.30)$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. From

$$(7.28) \text{ and } (7.25); \quad (7.29) \text{ and } (7.26); \quad (7.30) \text{ and } (7.27);$$

we achieve the subsequent inequalities

$$\mathscr{N}\left(\left(\frac{f(\kappa^{\alpha}x)}{\kappa^{\alpha}} - f(x), u\right), \sum_{\sigma=0}^{\alpha-1} \frac{\nu^{\sigma\beta}t}{\kappa^{\sigma\beta}}\right) \geq \min \bigcup_{\sigma=0}^{\alpha-1} \left\{ \mathscr{N}\left(\left(\frac{f(\kappa^{\sigma+1}x)}{\kappa^{\sigma+1}} - \frac{f(\kappa^{\sigma}x)}{\kappa^{\sigma}}, u\right), \frac{K\nu^{\sigma\beta}t}{\kappa^{\sigma\beta}}\right)\right\} \geq \mathscr{N}'\left(N(x,x,\cdots,x),u,K\,\kappa^{\beta}t\right); \quad (7.31)$$

$$\mathscr{N}\left(\left(\frac{g(\rho^{\alpha}y)}{\rho^{\alpha}} - g(y), u\right), \sum_{\sigma=0}^{\alpha-1} \frac{\nu^{\sigma\beta}t}{\rho^{\sigma\beta}}\right) \geq \min \bigcup_{\sigma=0}^{\alpha-1} \left\{ \mathscr{N}\left(\left(\frac{g(\rho^{\sigma+1}y)}{\rho^{\sigma+1}} - \frac{g(\rho^{\sigma}y)}{\rho^{\sigma}}, u\right), \frac{K\nu^{\sigma\beta}t}{\rho^{\sigma\beta}}\right)\right\} \geq \mathscr{N}'\left(N(y,y,\cdots,y),u,K\,\rho^{\beta}t\right); \quad (7.32)$$

$$\mathscr{N}\left(\left(\frac{h(\tau^{\alpha}z)}{\tau^{\alpha}} - h(z), u\right), \sum_{\sigma=0}^{\alpha-1} \frac{\nu^{\sigma\beta}t}{\tau^{\sigma\beta}}\right) \geq \min \bigcup_{\sigma=0}^{\alpha-1} \left\{ \mathscr{N}\left(\left(\frac{h(\tau^{\sigma+1}z)}{\tau^{\sigma+1}} - \frac{h(\tau^{\sigma}z)}{\tau^{\sigma}}, u\right), \frac{K\nu^{\sigma\beta}t}{\tau^{\sigma\beta}}\right)\right\} \geq \mathscr{N}'\left(N(z,z,\cdots,z),u,K\,\tau^{\beta}t\right); \quad (7.33)$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. It is easy to verify that the sequences

$$\left\{\frac{f(\kappa^{\alpha}x)}{\kappa^{\alpha}}\right\}; \left\{\frac{g(\rho^{\alpha}y)}{\rho^{\alpha}}\right\}; \left\{\frac{h(\tau^{\alpha}z)}{\tau^{\alpha}}\right\}$$

are Cauchy sequences. Indeed, changing

$$x = \kappa^{\delta}x \text{ and using } (2QBFN3), (7.7) \text{ in } (7.31);$$
$$y = \rho^{\delta}y \text{ and using } (2QBFN3), (7.8) \text{ in } (7.32);$$
$$z = \tau^{\delta}z \text{ and using } (2QBFN3), (7.9) \text{ in } (7.33);$$

we have

$$\mathscr{N}\left(\left(\frac{f(\kappa^{\alpha+\delta}x)}{\kappa^{\alpha+\delta}} - \frac{f(\kappa^{\delta}x)}{\kappa^{\delta}}, u\right), t\right) \geq \mathscr{N}'\left(N(x,x,\cdots,x),u,\frac{K\,\kappa^{\beta}t}{\sum_{\sigma=0}^{\alpha-1}\frac{\nu^{(\sigma+\delta)\beta}}{\kappa^{(\sigma+\delta)\beta}}}\right); \quad (7.34)$$

$$\mathscr{N}\left(\left(\frac{g(\rho^{\alpha+\delta}y)}{\rho^{\alpha+\delta}} - \frac{g(\rho^{\delta}y)}{\rho^{\delta}}, u\right), t\right) \geq \mathscr{N}'\left(N(y,y,\cdots,y),u,\frac{K\,\rho^{\beta}t}{\sum_{\sigma=0}^{\alpha-1}\frac{\nu^{(\sigma+\delta)\beta}}{\rho^{(\sigma+\delta)\beta}}}\right); \quad (7.35)$$

$$\mathscr{N}\left(\left(\frac{h(\tau^{\alpha+\delta}z)}{\tau^{\alpha+\delta}} - \frac{h(\tau^{\delta}z)}{\tau^{\delta}}, u\right), t\right) \geq \mathscr{N}'\left(N(z,z,\cdots,z),u,\frac{K\,\tau^{\beta}t}{\sum_{\sigma=0}^{\alpha-1}\frac{\nu^{(\sigma+\delta)\beta}}{\tau^{(\sigma+\delta)\beta}}}\right); \quad (7.36)$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$ and all $\alpha, \delta > 0$. Since

$$0 < \nu < \kappa \quad \text{and} \quad \sum_{\sigma=0}^{\alpha}\left(\frac{\nu}{\kappa}\right)^{\sigma} < \infty;$$
$$0 < \nu < \rho \quad \text{and} \quad \sum_{\sigma=0}^{\alpha}\left(\frac{\nu}{\rho}\right)^{\sigma} < \infty;$$
$$0 < \nu < \tau \quad \text{and} \quad \sum_{\sigma=0}^{\alpha}\left(\frac{\nu}{\tau}\right)^{\sigma} < \infty;$$

the Cauchy criterion for convergence and (2QBFN5) implies that our sequences are Cauchy sequences. Due to the completeness of $\mathscr{B}$, this sequences converges to some points $A_f; A_g; A_h$ respectively and its is defined by

$$\lim_{\alpha\to\infty} \mathscr{N}\left(\left(A_f(x) - \frac{f(\kappa^{\alpha}x)}{\kappa^{\alpha}}, u\right), t\right) = 1; \quad (7.37)$$

$$\lim_{\alpha\to\infty} \mathscr{N}\left(\left(A_g(y) - \frac{g(\rho^{\alpha}y)}{\rho^{\alpha}}, u\right), t\right) = 1; \quad (7.38)$$

$$\lim_{\alpha\to\infty} \mathscr{N}\left(\left(A_h(y) - \frac{h(\tau^{\alpha}y)}{\tau^{\alpha}}, u\right), t\right) = 1; \quad (7.39)$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. Letting $\delta = 0$ and letting $\alpha \to \infty$ in (7.34), (7.35) and (7.36) respectively, we arrive

$$\begin{aligned}
&\mathcal{N}\left(\left(A_f(x)-f(x),u\right),t\right)\\
&\qquad\geq \mathcal{N}'\left(N(x,x,\cdots,x),u,K\kappa^{\beta-1}(\kappa-\nu)t\right);\\
&\mathcal{N}\left(\left(A_g(y)-g(y),u\right),t\right)\\
&\qquad\geq \mathcal{N}'\left(E(y,y,\cdots,y),u,K\rho^{\beta-1}(\rho-\nu)t\right);\\
&\mathcal{N}\left(\left(A_h(z)-h(z),u\right),t\right)\\
&\qquad\geq \mathcal{N}'\left(O(z,z,\cdots,z),u,K\tau^{\beta-1}(\tau-\nu)t\right);
\end{aligned}$$

for all $x,y,z\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$, respectively with $\gamma=1$.

To show the mappings $A_f(x);A_g(y);A_h(z)$ satisfies the functional equations (1.27), (1.28) and (1.29) by replacing

$$\begin{aligned}
&(x_1,x_2,\cdots,x_n)=(\kappa^\alpha x_1,\kappa^\alpha x_2,\cdots,\kappa^\alpha x_n) \quad in \quad (7.1)\\
&(y_2,y_4,\cdots,y_{2n})=(\rho^\alpha y_2,\rho^\alpha y_4,\cdots,\rho^\alpha y_{2n}) \quad in \quad (7.2)\\
&(z_1,z_3,\cdots,z_{2n-1})=(\tau^\alpha z_1,\tau^\alpha z_3,\cdots,\tau^\alpha z_{2n-1}) \quad in \quad (7.3)
\end{aligned}$$

respectively, we arrive

$$\begin{aligned}
&\mathcal{N}\left(\frac{1}{\kappa^\alpha}\left(f\left(\sum_{a=1}^{n} a\,\kappa^\alpha x_a\right)-\sum_{a=1}^{n}\left(a\,f(\kappa^\alpha x_a)\right),u\right),t\right)\\
&\geq \mathcal{N}'\left(N(\kappa^\alpha x_1,\kappa^\alpha x_2,\cdots,\kappa^\alpha x_n),u,\kappa^{\alpha\beta}t\right); \qquad (7.40)\\
&\mathcal{N}\left(\frac{1}{\rho^\alpha}\left(g\left(\sum_{a=1}^{n} 2a\,\rho^\alpha y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,g(\rho^\alpha y_{2a})\right),u\right),t\right)\\
&\geq \mathcal{N}'\left(E(\rho^\alpha y_2,\rho^\alpha y_4,\cdots,\rho^\alpha y_{2n}),u,\rho^{\alpha\beta}t\right); \qquad (7.41)\\
&\mathcal{N}\left(\frac{1}{\tau^\alpha}\left(h\left(\sum_{a=1}^{n}(2a-1)\,\tau^\alpha z_{2a-1}\right)\right.\right.\\
&\qquad\left.\left.-\sum_{a=1}^{n}\left((2a-1)\,h(\tau^\alpha z_{2a-1})\right),u\right),t\right)\\
&\geq \mathcal{N}'\left(O(\tau^\alpha z_1,\tau^\alpha z_3,\cdots,\tau^\alpha z_{2n-1}),u,\tau^{\alpha\beta}t\right); \qquad (7.42)
\end{aligned}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1}\in\mathscr{A}$ and all $u\in\mathscr{B}$ and $t>0$. Now

$$\begin{aligned}
&\mathcal{N}\left(\left(A_f\left(\sum_{a=1}^{n} a\,x_a\right)-\sum_{a=1}^{n}\left(a\,A_f(x_a)\right),u\right),t\right)\\
&=\min\left\{\mathcal{N}\left(\left(A_f\left(\sum_{a=1}^{n} a\,x_a\right)\right.\right.\right.\\
&\qquad\left.\left.-\frac{1}{\kappa^\alpha}f\left(\sum_{a=1}^{n} a\,\kappa^\alpha x_a\right),u\right),\frac{Kt}{3}\right)\\
&\mathcal{N}\left(\left(-\sum_{a=1}^{n}\left(a\,A_f(x_a)\right)\right.\right.\\
&\qquad\left.\left.\frac{1}{\kappa^\alpha}\sum_{a=1}^{n}\left(a\,f(\kappa^\alpha x_a)\right),u\right),\frac{Kt}{3}\right),\\
&\mathcal{N}\left(\frac{1}{\kappa^\alpha}\left(f\left(\sum_{a=1}^{n} a\,\kappa^\alpha x_a\right)\right.\right.\\
&\qquad\left.\left.\left.-\sum_{a=1}^{n}\left(a\,f(\kappa^\alpha x_a)\right),u\right),\frac{Kt}{3}\right)\right\}; \qquad (7.43)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{N}\left(\left(A_g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)-\sum_{a=1}^{n}\left(2a\,A_g(y_{2a})\right),u\right),t\right)\\
&=\min\left\{\mathcal{N}\left(\left(A_g\left(\sum_{a=1}^{n} 2a\,y_{2a}\right)\right.\right.\right.\\
&\qquad\left.\left.-\frac{1}{\rho^\alpha}g\left(\sum_{a=1}^{n} 2a\,\rho^\alpha y_{2a}\right),u\right),\frac{Kt}{3}\right),\\
&\mathcal{N}\left(\left(-\sum_{a=1}^{n}\left(2a\,A_g(y_{2a})\right)\right.\right.\\
&\qquad\left.\left.+\frac{1}{\rho^\alpha}\sum_{a=1}^{n}\left(2a\,g(\rho^\alpha y_{2a})\right),u\right),\frac{Kt}{3}\right),\\
&\mathcal{N}\left(\frac{1}{\rho^\alpha}\left(g\left(\sum_{a=1}^{n} 2a\,\rho^\alpha y_{2a}\right)\right.\right.\\
&\qquad\left.\left.\left.-\sum_{a=1}^{n}\left(2a\,g(\rho^\alpha y_{2a})\right),u\right),\frac{Kt}{3}\right)\right\}; \qquad (7.44)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{N}\left(\left(A_h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)\right.\right.\\
&\qquad\left.\left.-\sum_{a=1}^{n}\left((2a-1)\,A-h(z_{2a-1})\right),u\right),t\right)\\
&=\min\left\{\mathcal{N}\left(\left(A_h\left(\sum_{a=1}^{n}(2a-1)\,z_{2a-1}\right)\right.\right.\right.\\
&\qquad\left.\left.-\frac{1}{\tau^\alpha}h\left(\sum_{a=1}^{n}(2a-1)\,\tau^\alpha z_{2a-1}\right),u\right),\frac{Kt}{3}\right),\\
&\mathcal{N}\left(\left(-\sum_{a=1}^{n}\left((2a-1)\,A_h(z_{2a-1})\right)\right.\right.\\
&\qquad\left.\left.+\frac{1}{\tau^\alpha}\sum_{a=1}^{n}\left((2a-1)\,h(\tau^\alpha z_{2a-1})\right),u\right),\frac{Kt}{3}\right),\\
&\mathcal{N}\left(\frac{1}{\tau^\alpha}\left(h\left(\sum_{a=1}^{n}(2a-1)\,\tau^\alpha z_{2a-1}\right)\right.\right.\\
&\qquad\left.\left.\left.-\sum_{a=1}^{n}\left((2a-1)\,h(\tau^\alpha z_{2a-1})\right),u\right),\frac{Kt}{3}\right)\right\}; \qquad (7.45)
\end{aligned}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. Letting $\alpha \to \infty$ in

$$\begin{array}{llll} (7.43) & and & using & (7.37), (7.55), (7.4), (2QBFN2) \\ (7.44) & and & using & (7.38), (7.56), (7.5), (2QBFN2) \\ (7.45) & and & using & (7.39), (7.57), (7.6), (2QBFN2) \end{array}$$

we reach the following equations

$$\mathscr{N}\left(\left(A_f\left(\sum_{a=1}^{n} a\, x_a\right) - \sum_{a=1}^{n}\left(a\, A_f(x_a)\right), u\right), t\right) = \min\left\{1, 1, \mathscr{N}'\left(N(\kappa^\alpha x_1, \kappa^\alpha x_2, \cdots, \kappa^\alpha x_n), u, \kappa^{\alpha\beta} t\right)\right\}; \tag{7.46}$$

$$\mathscr{N}\left(\left(A_g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) - \sum_{a=1}^{n}\left(2a\, A_g(y_{2a})\right), u\right), t\right) = \min\left\{1, 1, \mathscr{N}'\left(E(\rho^\alpha y_2, \rho^\alpha y_4, \cdots, \rho^\alpha y_{2n}), u, \rho^{\alpha\beta} t\right)\right\}; \tag{7.47}$$

$$\mathscr{N}\left(\left(A_h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) - \sum_{a=1}^{n}\left((2a-1)\, A - h(z_{2a-1})\right), u\right), t\right) = \min\left\{1, 1, \mathscr{N}'\left(O(\tau^\alpha z_1, \tau^\alpha z_3, \cdots, \tau^\alpha z_{2n-1}), u, \tau^{\alpha\beta} t\right)\right\}; \tag{7.48}$$

$$A_f\left(\sum_{a=1}^{n} a\, x_a\right) = \sum_{a=1}^{n}\left(a\, A_f(x_a)\right); \tag{7.49}$$

$$A_g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) = \sum_{a=1}^{n}\left(2a\, A_g(y_{2a})\right); \tag{7.50}$$

$$A_h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) = \sum_{a=1}^{n}\left((2a-1)\, A - h(z_{2a-1})\right); \tag{7.51}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$. Thus $A_f(x); A_g(y); A_h(z)$ satisfies the functional equations (1.27), (1.28) and (1.29) for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$.

In order to prove the being $A_f(x); A_g(y); A_h(z)$ are unique. Let $A'_f(x); A'_g(y); A'_h(z)$ be another additive mappings satisfying (7.10), (7.11), (7.12) and (7.13), (7.14), (7.15) respectively. Now

$$\begin{aligned} &\mathscr{N}\left(\left(A_f(x) - A'_f(x), u\right), t\right) \\ &= \mathscr{N}\left(\left(\frac{1}{\kappa^\alpha}\left(A_f(x) - A'_f(x)\right), u\right), t\right) \\ &\geq \min\left\{\mathscr{N}\left(\left(A_f(\kappa^\alpha x) - f(\kappa^\alpha x), u\right), \frac{K\kappa^{\alpha\beta} t}{2}\right), \mathscr{N}\left(\left(A'_f(\kappa^\alpha x) - f(\kappa^\alpha x), u\right), \frac{K\kappa^{\alpha\beta} t}{2}\right)\right\} \\ &\geq \mathscr{N}'\left(N(\kappa^\alpha x, \kappa^\alpha x, \cdots, \kappa^\alpha x), u, \frac{K\kappa^{\beta-1}(\kappa - \nu)}{2} t\right) \\ &= \mathscr{N}'\left(N(x, x, \cdots, x), u, \frac{K\kappa^{\beta-1}(\kappa - \nu) t}{2\nu^\alpha}\right) \\ &\to 1 \quad as \quad \alpha \to \infty; \end{aligned} \tag{7.52}$$

$$\begin{aligned} &\mathscr{N}\left(\left(A_g(y) - A'_g(y), u\right), t\right) \\ &= \mathscr{N}\left(\left(\frac{1}{\rho^\alpha}\left(A_g(y) - A'_g(y)\right), u\right), t\right) \\ &\geq \min\left\{\mathscr{N}\left(\left(A_g(\rho^\alpha y) - g(\rho^\alpha y), u\right), \frac{K\rho^{\alpha\beta} t}{2}\right), \mathscr{N}\left(\left(A'_g(\rho^\alpha y) - g(\rho^\alpha y), u\right), \frac{K\rho^{\alpha\beta} t}{2}\right)\right\} \\ &\geq \mathscr{N}'\left(E(\rho^\alpha y, \rho^\alpha y, \cdots, \rho^\alpha y), u, \frac{K\rho^{\beta-1}(\rho - \nu) t}{2}\right) \\ &= \mathscr{N}'\left(E(y, y, \cdots, y), u, \frac{K\rho^{\beta-1}(\rho - \nu) t}{2\nu^\alpha}\right) \\ &\to 1 \quad as \quad \alpha \to \infty; \end{aligned} \tag{7.53}$$

$$\begin{aligned} &\mathscr{N}\left(\left(A_h(z) - A'_h(z), u\right), t\right) \\ &= \mathscr{N}\left(\left(\frac{1}{\tau^\alpha}\left(A_h(z) - A'_h(z)\right), u\right), t\right) \\ &\geq \min\left\{\mathscr{N}\left(\left(A_h(\tau^\alpha z) - h(\tau^\alpha z), u\right), \frac{K\tau^{\alpha\beta} t}{2}\right), \mathscr{N}\left(\left(A'_h(\tau^\alpha z) - h(\tau^\alpha z), u\right), \frac{K\tau^{\alpha\beta} t}{2}\right)\right\} \\ &\geq \mathscr{N}'\left(E(\tau^\alpha z, \tau^\alpha z, \cdots, \tau^\alpha z), u, \frac{K\tau^{\beta-1}(\tau - \nu)}{2} t\right) \\ &= \mathscr{N}'\left(E(z, z, \cdots, z), u, \frac{K\tau^{\beta-1}(\tau - \nu) t}{2\nu^\alpha}\right) \\ &\to 1 \quad as \quad \alpha \to \infty; \end{aligned} \tag{7.54}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. Hence the mappings $A_f(x); A_g(y); A_h(z)$ are unique. Thus the theorem holds for $\gamma = 1$.

Also, if we replace

$$x = \frac{x}{\kappa} \quad in \quad (7.19); \qquad y = \frac{y}{\rho} \quad in \quad (7.20); \qquad z = \frac{z}{\tau} \quad in \quad (7.21);$$

we arrive

$$\mathcal{N}\left(\left(f(x) - \kappa f\left(\frac{x}{\kappa}\right), u\right), t\right) \geq \mathcal{N}'\left(N\left(\frac{x}{\kappa}, \frac{x}{\kappa}, \cdots, \frac{x}{\kappa}\right), u, t\right); \tag{7.55}$$

$$\mathcal{N}\left(\left(g(y) - \rho\, g\left(\frac{y}{\rho}\right), u\right), t\right) \geq \mathcal{N}'\left(E\left(\frac{y}{\rho}, \frac{y}{\rho}, \cdots, \frac{y}{\rho}\right), u, t\right); \tag{7.56}$$

$$\mathcal{N}\left(\left(h(z) - \tau\, h\left(\frac{z}{\tau}\right), u\right), t\right) \geq \mathcal{N}'\left(O\left(\frac{z}{\tau}, \frac{z}{\tau}, \cdots, \frac{z}{\tau}\right), u, t\right); \tag{7.57}$$

for all $x, y, z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. The rest of proof is similar to that of the case $\gamma = 1$. Thus the theorem holds for $\gamma = -1$ also. This completes the proof of the Theorem. □

The following corollary is an immediate consequence of Theorem 7.7 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and J.M.Rassias stabilities of the functional equations (1.27), (1.28) and (1.29).

Corollary 7.8. *Let $f, g, h : \mathscr{A} \to \mathscr{B}$ be a mapping satisfying the following inequalities*

$$\mathcal{N}\left(\left(f\left(\sum_{a=1}^{n} a\, x_a\right) - \sum_{a=1}^{n} (a\, f(x_a)), u\right), t\right) \geq \begin{cases} \mathcal{N}'(\lambda, u, t), \\ \mathcal{N}'\left(\sum_{a=1}^{n} \lambda\, |x_a, u|^{\mu}, u, t\right), & \mu \neq 1; \\ \mathcal{N}'\left(\prod_{a=1}^{n} \lambda\, |x_a, u|^{\mu} + \sum_{a=1}^{n} \lambda\, |x_a, u|^{n\mu}, u, t\right), & n\mu \neq 1; \end{cases} \tag{7.58}$$

$$\mathcal{N}\left(\left(g\left(\sum_{a=1}^{n} 2a\, y_{2a}\right) - \sum_{a=1}^{n} (2a\, g(y_{2a})), u\right), t\right) \geq \begin{cases} \mathcal{N}'(\lambda, u, t), \\ \mathcal{N}'\left(\sum_{a=1}^{n} \lambda\, |y_{2a}, u|^{\mu}, u, t\right), & \mu \neq 1; \\ \mathcal{N}'\left(\prod_{a=1}^{n} \lambda\, |y_{2a}, u|^{\mu} + \sum_{a=1}^{n} \lambda\, |y_{2a}, u|^{n\mu}, u, t\right), & n\mu \neq 1; \end{cases} \tag{7.59}$$

$$\mathcal{N}\left(\left(h\left(\sum_{a=1}^{n} (2a-1)\, z_{2a-1}\right) - \sum_{a=1}^{n} ((2a-1)\, h(z_{2a-1})), u\right), t\right) \geq \begin{cases} \mathcal{N}'(\lambda, u, t) \\ \mathcal{N}'\left(\sum_{a=1}^{n} \lambda\, |z_{2a-1}, u|^{\mu}\, u, t\right), & \mu \neq 1; \\ \mathcal{N}'\left(\prod_{a=1}^{n} \lambda\, |z_{2a-1}, u|^{\mu} + \sum_{a=1}^{n} \lambda\, |z_{2a-1}, u|^{n\mu}, u, t\right), & n\mu \neq 1; \end{cases} \tag{7.60}$$

for all $x_1, x_2, \cdots, x_n, y_2, y_4, \cdots, y_{2n}, z_1, z_3, \cdots, z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, where λ and μ are positive constants. Then there exists one and only additive mapping $A_f, A_g, A_h : \mathscr{A} \to \mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\mathcal{N}\left(\left(A_f(x) - f(x), u\right), t\right) \geq \begin{cases} \mathcal{N}'\left(\lambda, u, K\kappa^{\beta-1}|\kappa - 1|t\right); \\ \mathcal{N}'\left(n\lambda|x, u|^{\mu}, u, K\kappa^{\beta-1}|\kappa - \nu|t\right); \\ \mathcal{N}'\left((n+1)\lambda|x, u|^{n\mu}, u, K\kappa^{\beta-1}|\kappa - \nu|t\right); \end{cases} \tag{7.61}$$

$$\mathcal{N}\left(\left(A_g(y) - g(y), u\right), t\right) \geq \begin{cases} \mathcal{N}'\left(\lambda, u, K\rho^{\beta-1}|\rho - 1|t\right); \\ \mathcal{N}'\left(n\lambda|x, u|^{\mu}, u, K\rho^{\beta-1}|\rho - \nu|t\right); \\ \mathcal{N}'\left((n+1)\lambda|x, u|^{n\mu}, u, K\rho^{\beta-1}|\rho - \nu|t\right); \end{cases} \tag{7.62}$$

$$\mathcal{N}\left(\left(A_h(z) - h(z), u\right), t\right) \geq \begin{cases} \mathcal{N}'\left(\lambda, u, K\tau^{\beta-1}|\tau - 1|t\right); \\ \mathcal{N}'\left(n\lambda|x, u|^{\mu}, u, K\tau^{\beta-1}|\tau - \nu|t\right); \\ \mathcal{N}'\left((n+1)\lambda|x, u|^{n\mu}, u, K\tau^{\beta-1}|\tau - \nu|t\right); \end{cases} \tag{7.63}$$

for all $x, y, z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, respectively.

8. Stability Results In Random Quasi-Beta-2-Banach Space

In this section, we investigate the generalized Ulam - Hyers stability of the functional equations (1.27), (1.28) and (1.29) in random quasi-β-2-Banach spaces using Hyers Method.

Now, we give basic definitions and notations in random quasi-β-2-Banach space.

From now on, Δ^+ is the space of distribution functions, that is, the space of all mappings

$$F : R \cup \{-\infty, \infty\} \longrightarrow [0, 1],$$

such that F is leftcontinuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is,

$$l^- f(x) = \lim_{t \to x^-} f(t).$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$

for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & if\ t \le 0, \\ 1, & if\ t \ge 0. \end{cases} \tag{8.1}$$

Definition 8.1. *A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:*

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous $t-$norms are $T_P(a,b) = ab, T_M(a,b) = min(a,b)$ and $T_L(a,b) = max(a+b-1,0)$ (the Lukasiewicz $t-$norm). Recall (see [21, 22]) that if T is a $t-$norm and x_n is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_{n+i}$ is defined recurrently by

$$T_{i=1}^{1} x_i = x_1 \quad and \quad T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) \quad for \quad n \ge 2.$$

$T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [22] that, for the Lukasiewicz $t-$norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty \tag{8.2}$$

Definition 8.2. *A random quasi-β-2-normed space is a quartile $(X, \mathscr{R}, T, t)$, where X is a vector space, T is a continuous $t-$norm, $\mathscr{R}$ is a mapping from X into D^+ and $u \in D^+$ satisfying the following conditions:*

(2QBRN1) $\mathscr{R}_x(u,t) = \varepsilon_0(u,t)$ for all $t > 0$ if and only if $x = 0$;

(2QBRN2) $\mathscr{R}_{\alpha\, x}(u,t) = \mathscr{R}_x(u, t/|\alpha|^{\beta})$ for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \ne 0$;

(2QBRN3) $\mathscr{R}_{x+y}(u,(t+s)) \ge T\left(\mathscr{R}_x(u,Kt), \mathscr{R}_y(u,Ks)\right)$ for all $x,y \in X$, $t,s \ge 0$ and a constant $K \ge 1$.

Example 8.3. *Every normed spaces $(X, ||\cdot||)$ defines a random quasi-β-2-normed space $(X, \mathscr{R}, T_M, u)$, where*

$$\mathscr{R}_x(u,t) = \frac{t}{t + ||x||}$$

and T_M is the minimum $t-$norm.

Definition 8.4. *Let $(X, \mathscr{R}, T, u)$ be a random quasi-β-2-normed space.*

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mathscr{R}_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \ge N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mathscr{R}_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \ge m \ge N$ and all $t > 0$.

(3) A random quasi-β-2-normed space $(X, \mathscr{R}, T, u)$ is said to be complete if every Cauchy sequence in X is convergent to a point in X.

(4) A complete random quasi-β-2-normed space $(X, \mathscr{R}, T, u)$ is is called random quasi-β-2-Banach space.

To prove stability results, let us consider $\mathscr{A}$ be a random quasi-β-2-normed space and $\mathscr{B}$ be a random quasi-β-2-Banach space.

Theorem 8.5. *Let $f,g,h : \mathscr{A} \to \mathscr{B}$ be a mapping satisfying the following inequalities*

$$\mathscr{R}_{f\left(\sum_{a=1}^{n}\ a\ x_a\right) - \sum_{a=1}^{n}(a\ f(x_a))}(u,t) \ge \mathscr{R}'_{\mathscr{N}\ x_1,x_2,\cdots,x_n}(u,t); \tag{8.3}$$

$$\mathscr{R}_{g\left(\sum_{a=1}^{n}\ 2a\ y_{2a}\right) - \sum_{a=1}^{n}(2a\ g(y_{2a}))}(u,t) \ge \mathscr{R}'_{\mathscr{E}\ y_2,y_4,\cdots,y_{2n}}(u,t); \tag{8.4}$$

$$\begin{aligned}&\mathscr{R}_{h\left(\sum_{a=1}^{n}\ (2a-1)\ z_{2a-1}\right) - \sum_{a=1}^{n}((2a-1)\ h(z_{2a-1}))}(u,t) \\ &\ge \mathscr{R}'_{\mathscr{O}\ z_1,z_3,\cdots,z_{2n-1}}(u,t);\end{aligned} \tag{8.5}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, for which there exist a function $\mathscr{R}'_{\mathscr{N}} : \mathscr{A}^n \to D^+$, $\mathscr{R}'_{\mathscr{E}} : \mathscr{A}^{2n} \to D^+$ and $\mathscr{R}'_{\mathscr{O}} : \mathscr{A}^{2n-1} \to D^+$ with the conditions

$$\begin{aligned}&T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{N}\ \kappa^{\alpha\gamma}x,\kappa^{\alpha\gamma}x,\cdots,\kappa^{\alpha\gamma}x}(u, \kappa^{\alpha\gamma}t) = 1 \\ &= \lim_{\alpha\to\infty} \mathscr{R}'_{\mathscr{N}\ \kappa^{\alpha\gamma}x_1,\kappa^{\alpha\gamma}x_2,\cdots,\kappa^{\alpha\gamma}x_n}(u, \kappa^{\alpha\gamma}t);\end{aligned} \tag{8.6}$$

$$\begin{aligned}&T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{E}\ \rho^{\alpha\gamma}y,\rho^{\alpha\gamma}y,\cdots,\rho^{\alpha\gamma}y}(u, \rho^{\alpha\gamma}t) = 1 \\ &= \lim_{\alpha\to\infty} \mathscr{R}'_{\mathscr{E}\ \rho^{\alpha\gamma}y_2,\rho^{\alpha\gamma}y_4,\cdots,\rho^{\alpha\gamma}y_{2n}}(u, \rho^{\alpha\gamma}t);\end{aligned} \tag{8.7}$$

$$\begin{aligned}&T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{O}\ \tau^{\alpha\gamma}z,\tau^{\alpha\gamma}z,\cdots,\tau^{\alpha\gamma}z}(u, \tau^{\alpha\gamma}t) = 1 \\ &= \lim_{\alpha\to\infty} \mathscr{R}'_{\mathscr{O}\ \tau^{\alpha\gamma}z_1,\tau^{\alpha\gamma}z_3,\cdots,\tau^{\alpha\gamma}z_{2n-1}}(u, \tau^{\alpha\gamma}t);\end{aligned} \tag{8.8}$$

for all $x,y,z,x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ all $u \in \mathscr{B}$ and $t > 0$ with $\gamma = \pm 1$. Then there exists one and only additive mapping $A_f, A_g, A_h : \mathscr{A} \to \mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\mathscr{R}_{A_f(x) - f(x)}(u,t) \ge T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{N}\ \kappa^{\alpha\gamma}x,\kappa^{\alpha\gamma}x,\cdots,\kappa^{\alpha\gamma}x}\left(u, K\kappa^{\beta\alpha\gamma}t\right); \tag{8.9}$$

$$\mathscr{R}_{A_g(y) - g(y)}(u,t) \ge T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{E}\ \rho^{\alpha\gamma}y,\rho^{\alpha\gamma}y,\cdots,\rho^{\alpha\gamma}y}\left(u, \rho^{\beta\alpha\gamma}t\right); \tag{8.10}$$

$$\mathscr{R}_{A_h(z) - h(z)}(u,t) \ge T_{\alpha=0}^{\infty} \mathscr{R}'_{\mathscr{O}\ \tau^{\alpha\gamma}z,\tau^{\alpha\gamma}z,\cdots,\tau^{\alpha\gamma}z}\left(u, \tau^{\beta\alpha\gamma}t\right); \tag{8.11}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, respectively. The mappings A_f, A_g, A_h are respectively defined as

$$\mathscr{R}_{A_f(x)}(u,t) = \lim_{\alpha\to\infty} \mathscr{R}_{\frac{f(\kappa^{\alpha\gamma}x)}{\kappa^{\alpha\gamma}}}(u,t); \tag{8.12}$$

$$\mathscr{R}_{A_g(y)}(u,t) = \lim_{\alpha\to\infty} \mathscr{R}_{\frac{g(\rho^{\alpha\gamma}y)}{\rho^{\alpha\gamma}}}(u,t); \tag{8.13}$$

$$\mathscr{R}_{A_h(y)}(u,t) = \lim_{\alpha\to\infty} \mathscr{R}_{\frac{h(\tau^{\alpha\gamma}y)}{\tau^{\gamma}}}(u,t); \tag{8.14}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$.

Proof. Changing

$$\begin{aligned} (x_1,x_2,\cdots,x_n) &= (x,x,\cdots,x) \quad \text{in} \quad (8.3);\\ (y_2,y_4,\cdots,y_{2n}) &= (y,y,\cdots,y) \quad \text{in} \quad (8.4);\\ (z_1,z_3,\cdots,z_{2n-1}) &= (z,z,\cdots,z) \quad \text{in} \quad (8.5); \end{aligned}$$

we arrive the following inequalities

$$\mathscr{R}_{f\left(\sum_{a=1}^n\ a\ x\right)-\left(\sum_{a=1}^n a\right)\ f(x)}(u,t) \geq \mathscr{R}'_{\mathscr{N}\,x,x,\cdots,x}(u,t); \tag{8.15}$$

$$\mathscr{R}_{g\left(\sum_{a=1}^n\ 2a\ y\right)-\left(\sum_{a=1}^n 2a\right)\ g(y)}(u,t) \geq \mathscr{R}'_{\mathscr{E}\,y,y,\cdots,y}(u,t); \tag{8.16}$$

$$\mathscr{R}_{h\left(\sum_{a=1}^n\ (2a-1)\ z\right)-\left(\sum_{a=1}^n (2a-1)\right)\ h(z)}(u,t) \geq \mathscr{R}'_{\mathscr{O}\,z,z,\cdots,z}(u,t); \tag{8.17}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and all $t > 0$. Define

$$\begin{aligned} \sum_{a=1}^n a = \frac{n(n+1)}{2} = \kappa \quad &\text{in} \quad (8.15);\\ \sum_{a=1}^n 2a = n(n+1) = \rho \quad &\text{in} \quad (8.16);\\ \sum_{a=1}^n (2a-1) = n^2 = \tau \quad &\text{in} \quad (8.17); \end{aligned}$$

we obtain the succeeding inequalities

$$\mathscr{R}_{f(\kappa\ x)-\kappa\ f(x)}(u,t) \geq \mathscr{R}'_{\mathscr{N}\,x,x,\cdots,x}(u,t); \tag{8.18}$$

$$\mathscr{R}_{g(\rho\ y)-\rho\ g(y)}(u,t) \geq \mathscr{R}'_{\mathscr{E}\,y,y,\cdots,y}(u,t); \tag{8.19}$$

$$\mathscr{R}_{h(\tau\ z)-\tau\ h(z)}(u,t) \geq \mathscr{R}'_{\mathscr{O}\,z,z,\cdots,z}(u,t); \tag{8.20}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. Using (2QBRN2) it follows from above inequalities

$$\mathscr{R}_{\frac{f(\kappa\ x)}{\kappa}-f(x)}(u,t) \geq \mathscr{R}'_{\mathscr{N}\,x,x,\cdots,x}\left(u,\kappa^{\beta}t\right); \tag{8.21}$$

$$\mathscr{R}_{\frac{g(\rho\ y)}{\rho}-g(y)}(u,t) \geq \mathscr{R}'_{\mathscr{E}\,y,y,\cdots,y}\left(u,\rho^{\beta}t\right); \tag{8.22}$$

$$\mathscr{R}_{\frac{h(\tau\ z)}{\tau}-h(z)}(u,t) \geq \mathscr{R}'_{\mathscr{O}\,z,z,\cdots,z}\left(u,\tau^{\beta}t\right); \tag{8.23}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$. The rest of proof is similar tracing to that of Theorem 7.7. □

The following corollary is an immediate consequence of Theorem 8.5 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and J.M.Rassias stabilities of the functional equations (1.27), (1.28) and (1.29).

Corollary 8.6. *Let $f,g,h : \mathscr{A} \to \mathscr{B}$ be a mapping satisfying the following inequalities*

$$\mathscr{R}\left(\left(f\left(\sum_{a=1}^n\ a\ x_a\right) - \sum_{a=1}^n\left(a\ f(x_a)\right),u\right),t\right) \geq \begin{cases} \mathscr{R}'(\lambda,u,t) \\ \mathscr{R}'\left(\sum_{a=1}^n \lambda\,|x_a,u|^{\mu},u,t\right), & \mu \neq 1; \\ \mathscr{R}'\left(\prod_{a=1}^n \lambda\,|x_a,u|^{\mu} + \sum_{a=1}^n \lambda\,|x_a,u|^{n\mu},u,t\right), & n\mu \neq 1; \end{cases} \tag{8.24}$$

$$\mathscr{R}\left(\left(g\left(\sum_{a=1}^n\ 2a\ y_{2a}\right) - \sum_{a=1}^n\left(2a\ g(y_{2a})\right),u\right),t\right) \geq \begin{cases} \mathscr{R}'(\lambda,u,t) \\ \mathscr{R}'\left(\sum_{a=1}^n \lambda\,|y_{2a},u|^{\mu},u,t\right) & \mu \neq 1; \\ \mathscr{R}'\left(\prod_{a=1}^n \lambda\,|y_{2a},u|^{\mu} + \sum_{a=1}^n \lambda\,|y_{2a},u|^{n\mu},u,t\right), & n\mu \neq 1; \end{cases} \tag{8.25}$$

$$\mathscr{R}\left(\left(h\left(\sum_{a=1}^n\ (2a-1)\ z_{2a-1}\right) - \sum_{a=1}^n\left((2a-1)\ h(z_{2a-1})\right),u\right),t\right) \geq \begin{cases} \mathscr{R}'(\lambda,u,t) \\ \mathscr{R}'\left(\sum_{a=1}^n \lambda\,|z_{2a-1},u|^{\mu}\,u,t\right), & \mu \neq 1; \\ \mathscr{R}'\left(\prod_{a=1}^n \lambda\,|z_{2a-1},u|^{\mu} \right. \\ \quad \left. + \sum_{a=1}^n \lambda\,|z_{2a-1},u|^{n\mu},u,t\right), & n\mu \neq 1; \end{cases} \tag{8.26}$$

for all $x_1,x_2,\cdots,x_n,y_2,y_4,\cdots,y_{2n},z_1,z_3,\cdots,z_{2n-1} \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, where λ and μ are positive constants. Then there exists one and only additive mapping $A_f,A_g,A_h : \mathscr{A} \to \mathscr{B}$ satisfying (1.27), (1.28), (1.29), and

$$\mathscr{R}\left(\left(A_f(x) - f(x),u\right),t\right) \geq \begin{cases} \mathscr{R}'\left(\lambda,u,K\kappa^{\beta-1}|\kappa-1|t\right); \\ \mathscr{R}'\left(n\lambda|x,u|^{\mu},u,K\kappa^{\beta-1}|\kappa-\nu|t\right); \\ \mathscr{R}'\left((n+1)\lambda|x,u|^{n\mu},u,K\kappa^{\beta-1}|\kappa-\nu|t\right); \end{cases} \tag{8.27}$$

$$\mathscr{R}\left(\left(A_g(y) - g(y),u\right),t\right) \geq \begin{cases} \mathscr{R}'\left(\lambda,u,K\rho^{\beta-1}|\rho-1|t\right); \\ \mathscr{R}'\left(n\lambda|x,u|^{\mu},u,K\rho^{\beta-1}|\rho-\nu|t\right); \\ \mathscr{R}'\left((n+1)\lambda|x,u|^{n\mu},u,K\rho^{\beta-1}|\rho-\nu|t\right); \end{cases} \tag{8.28}$$

$$\mathscr{R}\left(\left(A_h(z) - h(z),u\right),t\right) \geq \begin{cases} \mathscr{R}'\left(\lambda,u,K\tau^{\beta-1}|\tau-1|t\right); \\ \mathscr{R}'\left(n\lambda|x,u|^{\mu},u,K\tau^{\beta-1}|\tau-\nu|t\right); \\ \mathscr{R}'\left((n+1)\lambda|x,u|^{n\mu},u,K\tau^{\beta-1}|\tau-\nu|t\right); \end{cases} \tag{8.29}$$

for all $x,y,z \in \mathscr{A}$ and all $u \in \mathscr{B}$ and $t > 0$, respectively.

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