Malaya Journal of Matematik, Vol. 6, No. 1, 113-128, 2018

https://doi.org/10.26637/MJM0601/0015

Stability of general quadratic − cubic − quartic functional equation in quasi beta Banach space via two dissimilar methods

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Abstract
In this paper, authors proved the generalized Ulam - Hyers stability of mixed type general quartic - cubic -quartic functional equation

$$f(x+my)+f(x-my)=m^2f(x+y)+m^2f(x-y)+2(1-m^2)f(x)+\frac{m^2(m^2-1)}{6}(f(2y)+2f(-y)-6f(y))$$

where $m \neq 0, \pm 1$ in Quasi beta Banach space via two dissimilar methods.

Keywords
Quadratic, cubic, quartic functional equations, Generalized Ulam - Hyers stability, Quasi beta Banach space, fixed point.

AMS Subject Classification
39B52, 32B72, 32B82.

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Article History: Received **11** October **2017**; Accepted **27** December **2017**

Contents

1. Introduction

The learning of stability problems for functional equations is interconnected to a question of Ulam [58] pertaining to the stability of group homomorphisms and confidently answered for Banach spaces by Hyers [35]. It was advanced and admirable results obtained by number of authors [2, 34, 45, 50, 53]

Over the last seven decades, the stability problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involve only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [5–8, 10–15, 17, 19–21, 23, 24, 26–31, 33, 38, 43, 46–49, 61].

M. Eshaghi Gordji et.al., [32], introduce and establish the solution and stability of the generalized Ulam - Hyers stability of mixed type general quartic - cubic -quartic functional equation

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)$$
$$+2(1-k^2)f(x)+\frac{k^2(k^2-1)}{6}(f(2y)+2f(-y)-6f(y)) \quad (1.1)$$

where $k \neq 0, \pm 1$ in Non-Archimedean normed space.

In this paper, authors proved the generalized Ulam - Hyers stability of mixed type general quartic - cubic - quartic functional equation

$$f(x+my)+f(x-my)=m^2f(x+y)+m^2f(x-y)$$
$$+2(1-m^2)f(x)+\frac{m^2(m^2-1)}{6}(f(2y)+2f(-y)-6f(y)) \quad (1.2)$$

where $m \neq 0, \pm 1$ in Quasi beta Banach space via two dissimilar techniques.

2. Definitions and Notations On Quasi Beta Banach space

In this section, we present some basic facts concerning quasi-β-Normed spaces and some preliminary results.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:*

(Q1) $\| x \| \geq 0$ *for all* $x \in X$ *and* $\| x \| = 0$ *if and only if* $x = 0$.

(Q2) $\| \lambda x \| = | \lambda |^{\beta} . \| x \|$ *for all* $\lambda \in \mathbb{K}$ *and all* $x \in X$.

(Q3) *There is a constant $K \geq 1$ such that* $\| x+y \| \leq K(\| x \| + \| y \|)$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 2.3. *A qusi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if*

$$\| x+y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

More details, one can refer [18, 54, 57] for the concepts of quasi-normed spaces and p-Banach space. Given a $p-$norm, the formula $d(x,y) := ||x-y||^p$ gives us a translation invariant metric on X.

By the Aoki-Rolewicz theorem [54], each quasi-norm is equivalent to some $p-$norm. Since it is much easier to work with $p-$norms than quasi-norms, henceforth we restrict our attention mainly to $p-$norms.

In [57], J. Tabor has investigated a version of the Hyers-Rassias-Gajda theorem in quasi-Banach spaces.

In order to prove the stability results let us consider $\mathscr{U}$ be a Linear Space over $\mathbb{R}$ and $\mathscr{V}$ be a quasi - beta Banach space with $||\cdot||_{\mathscr{V}}$. Also, throughout this paper we define a function $f : \mathscr{U} \longrightarrow \mathscr{V}$

$$\mathscr{F}_{QCQ}(x,y) = f(x+my)+f(x-my)-m^2f(x+y)$$
$$-m^2f(x-y)-2(1-m^2)f(x)$$
$$-\frac{m^2(m^2-1)}{6}(f(2y)+2f(-y)-6f(y))$$

where $m \neq 0, \pm 1$ for all $x, y \in \mathscr{U}$.

3. Stability Results: Direct Method

3.1 Case 1: f is Even

Theorem 3.1. *Assume $\rho : \mathscr{U}^2 \longrightarrow (0, \infty]$ be a function satisfying the condition*

$$\lim_{q\to\infty} \frac{\rho(2^{qs}x, 2^{qs}y)}{4^{qs}} = 0 \quad (3.1)$$

for all $x, y \in \mathscr{U}$. Also, let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \leq \rho(x,y) \quad (3.2)$$

for all $x, y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f_2(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}} = \|f(2x)-16f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}$$
$$\leq \frac{K^{q-1}}{[4\, m^2(m^2-1)]^{\beta}} \sum_{p=\frac{1-s}{2}}^{\infty} \frac{\Delta(2^{sp}x, 2^{sp}x)}{4^{sp}} \quad (3.3)$$

and

$$\Delta(2^{sp}x, 2^{sp}x) = K^3\Big[12m^2\rho(0, 2^{sp}x)+12(m^2-1)\rho(2^{sp}x, 2^{sp}x)$$
$$+6\rho(0, 2\cdot 2^{sp}x)+12\rho(m\cdot 2^{sp}x, 2^{sp}x)\Big] \quad (3.4)$$

for all $x \in \mathscr{U}$. The mapping $\mathscr{Q}_2(x)$ is defined by

$$\mathscr{Q}_2(x) = \lim_{q\to\infty} \frac{f_2(2^{qs}x)}{4^{qs}} = \lim_{q\to\infty} \frac{f(2^{s(q+1)}x)-16f(2^{qs}x)}{4^{qs}} \quad (3.5)$$

for all $x \in \mathscr{U}$, where $s = \pm 1$.

Proof. Interchanging x and y in (3.1) and using evenness of f, one can see that

$$\Big\| f(mx+y)+f(mx-y)-m^2f(x+y)-m^2f(x-y)$$
$$-2(1-m^2)f(y)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\Big\|_{\mathscr{V}} \leq \rho(y,x) \quad (3.6)$$

for all $x,y \in \mathscr{U}$. Substituting y by 0 in (3.6), we find that

$$\left\|2f(mx)-2m^2f(x)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq \rho(0,x) \tag{3.7}$$

for all $x \in \mathscr{U}$. Again substituting y by x in (3.6), we observe that

$$\left\| f((m+1)x)+f((m-1)x)-m^2f(2x)-2(1-m^2)f(x) -\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq \rho(x,x) \tag{3.8}$$

for all $x \in \mathscr{U}$. Replacing (x,y) by $(2x,0)$ in (3.6), we notice that

$$\left\|2f(2mx)-2m^2f(2x)-\frac{m^2(m^2-1)}{6}(f(4x)-4f(2x))\right\|_{\mathscr{V}} \leq \rho(0,2x) \tag{3.9}$$

for all $x \in \mathscr{U}$. Again replacing (x,y) by (x,mx) in (3.6) and using evenness of f, we witness that

$$\left\| f(2mx)-m^2f((m+1)x)-m^2f((m-1)x) -2(1-m^2)f(mx)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq \rho(mx,x) \tag{3.10}$$

for all $x \in \mathscr{U}$. Multiplying the inequalities (3.7), (3.8), (3.9) and (3.10) by $12m^2$, $12(m^2-1)$, 6 and 12 respectively, we arrive the following inequalities

$$12m^2\left\|2f(mx)-2m^2f(x)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq 12m^2\rho(0,x) \tag{3.11}$$

$$12(m^2-1)\left\| f((m+1)x)+f((m-1)x)-m^2f(2x) -2(1-m^2)f(x)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq 12(m^2-1)\rho(x,x) \tag{3.12}$$

$$6\left\|2f(2mx)-2m^2f(2x) -\frac{m^2(m^2-1)}{6}(f(4x)-4f(2x))\right\|_{\mathscr{V}} \leq 6\rho(0,2x) \tag{3.13}$$

$$12\left\| f(2mx)-m^2f((m+1)x)-m^2f((m-1)x) --2(1-m^2)f(mx)\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\right\|_{\mathscr{V}} \leq 12\rho(mx,x) \tag{3.14}$$

for all $x \in \mathscr{U}$. Combining (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
&\left[m^2(m^2-1)\right]^{\beta}\|f(4x)-20f(2x)+64f(x)\|_{\mathscr{V}} \\
&\leq K^3\Big\{12m^2\Big\|2f(mx)-2m^2f(x) -\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\Big\|_{\mathscr{V}} \\
&+12(m^2-1)\Big\|f((m+1)x)+f((m-1)x)-m^2f(2x) -2(1-m^2)f(x)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\Big\|_{\mathscr{V}} \\
&+6\Big\|2f(2mx)-2m^2f(2x)-\frac{m^2(m^2-1)}{6}(f(4x)-4f(2x))\Big\|_{\mathscr{V}} \\
&+12\Big\|f(2mx)-m^2f((m+1)x)-m^2f((m-1)x) -2(1-m^2)f(mx)-\frac{m^2(m^2-1)}{6}(f(2x)-4f(x))\Big\|_{\mathscr{V}}\Big\} \\
&\leq K^3\Big[12m^2\rho(0,x)+12(m^2-1)\rho(x,x) +6\rho(0,2x)+12\rho(mx,x)\Big]=\Delta(x,x)
\end{aligned} \tag{3.15}$$

for all $x \in \mathscr{U}$. It follows from (3.15), we reach

$$\left[m^2(m^2-1)\right]^{\beta}\|[f(4x)-16f(2x)]-4[f(2x)-16f(x)]\|_{\mathscr{V}} \leq \Delta(x,x) \tag{3.16}$$

for all $x \in \mathscr{U}$. Define a function

$$f_2:\mathscr{U}\longrightarrow\mathscr{V} \qquad by \qquad f_2(x)=f(2x)-16f(x) \tag{3.17}$$

for all $x \in \mathscr{U}$. Using (3.17) in (3.16), we achieve

$$\left[m^2(m^2-1)\right]^{\beta}\|f_2(2x)-4f_2(x)\|_{\mathscr{V}} \leq \Delta(x,x) \quad or \quad \|f_2(2x)-4f_2(x)\|_{\mathscr{V}} \leq \frac{\Delta(x,x)}{[m^2(m^2-1)]^{\beta}} \tag{3.18}$$

for all $x \in \mathscr{U}$. Now, from the above inequality, we have

$$\left\|\frac{f_2(2x)}{4}-f_2(x)\right\|_{\mathscr{V}} \leq \frac{\Delta(x,x)}{[4\,m^2(m^2-1)]^{\beta}} \tag{3.19}$$

for all $x \in \mathscr{U}$. Letting x by $2x$ and dividing by 4 in (3.19), we observe

$$\left\|\frac{f_2(2^2x)}{4^2}-\frac{f_2(2x)}{4}\right\|_{\mathscr{V}} \leq \frac{\Delta(2x,2x)}{4\cdot[4\,m^2(m^2-1)]^{\beta}} \tag{3.20}$$

for all $x \in \mathscr{U}$. Combining (3.19) and (3.20), one can notice that

$$\begin{aligned}
&\left\|\frac{f_2(2^2x)}{4^2}-f_2(x)\right\|_{\mathscr{V}} \\
&\leq K\left\{\left\|\frac{f_2(2x)}{4}-f_2(x)\right\|_{\mathscr{V}}+\left\|\frac{f_2(2^2x)}{4^2}-\frac{f_2(2x)}{4}\right\|_{\mathscr{V}}\right\} \\
&\leq K\left\{\frac{\Delta(x,x)}{[4\,m^2(m^2-1)]^{\beta}}+\frac{\Delta(2x,2x)}{4\cdot[4\,m^2(m^2-1)]^{\beta}}\right\} \\
&=\frac{K}{[4\,m^2(m^2-1)]^{\beta}}\left\{\Delta(x,x)+\frac{\Delta(2x,2x)}{4}\right\}
\end{aligned} \tag{3.21}$$

for all $x \in \mathscr{U}$. Generalizing for a positive integer q, one can verify that

$$\left\| \frac{f_2(2^q x)}{4^q} - f_2(x) \right\|_{\mathscr{V}} \le \frac{K^{q-1}}{[4\,m^2(m^2-1)]^\beta} \sum_{p=0}^{q-1} \frac{\Delta(2^p x, 2^p x)}{4^p} \tag{3.22}$$

for all $x \in \mathscr{U}$. It is easy to prove that the sequence $\left\{ \frac{f_2(2^q x)}{4^q} \right\}$ is a Cauchy sequence and it converges to a point $\mathscr{Q}_2(x)$ in $\mathscr{V}$. So, we define a mapping $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$ by

$$\mathscr{Q}_2(x) = \lim_{q\to\infty} \frac{f_2(2^q x)}{4^q} = \lim_{q\to\infty} \frac{f(2^{q+1}x) - 16 f(2^q x)}{4^q} \tag{3.23}$$

for all $x \in \mathscr{U}$. Approaching q tends to infinity in (3.22), we arrive (3.3) holds for $s = 1$. It is easy to show that $\mathscr{Q}_2(x)$ satisfies the functional equation (1.2). Indeed, replacing (x, y) by $(2^q x, 2^q y)$ in (3.2) and using (3.1), (3.23), one can see that $\mathscr{Q}_2(x)$ satisfies the functional equation (1.2) for all $x, y \in \mathscr{U}$. To prove the existence of $\mathscr{Q}_2(x)$ is unique, let $\mathscr{R}_2(x)$ be another quadratic mapping satisfying (3.3), (3.23). Now, for any positive integer q_1

$$\begin{aligned}
&\|\mathscr{Q}_2(x) - \mathscr{R}_2(x)\|_{\mathscr{V}} \\
&= \frac{1}{4^{q_1}} \|\mathscr{Q}_2(2^{q_1}x) - \mathscr{R}_2(2^{q_1}x)\|_{\mathscr{V}} \\
&\le \frac{K}{4^{q_1}} \{\|\mathscr{Q}_2(2^{q_1}x) - f_2(2^{q_1}x)\|_{\mathscr{V}} \\
&\qquad + \|\mathscr{R}_2(2^{q_1}x) - f_2(2^{q_1}x)\|_{\mathscr{V}}\} \\
&\le \frac{2K^q}{[4\,m^2(m^2-1)]^\beta} \sum_{p=0}^{q-1} \frac{\Delta(2^{p+q_1}x, 2^{p+q_1}x)}{4^{p+q_1}} \\
&\longrightarrow 0 \quad as \quad q_1 \quad \longrightarrow \quad \infty.
\end{aligned}$$

for all $x \in \mathscr{U}$. Thus, $\mathscr{Q}_2(x)$ is unique. So, the theorem holds for $s = 1$.

Replacing x by $\frac{x}{2}$ in (3.19), we arrive

$$\left\| f_2(x) - 4 f_2\left(\frac{x}{2}\right) \right\|_{\mathscr{V}} \le \frac{4\Delta\left(\frac{x}{2}, \frac{x}{2}\right)}{[4\,m^2(m^2-1)]^\beta} \tag{3.24}$$

for all $x \in \mathscr{U}$. Again replacing x by $\frac{x}{2}$ in (3.24) and multiplying by 4, we find that

$$\left\| 4 f_2\left(\frac{x}{2}\right) - 4^2 f_2\left(\frac{x}{2^2}\right) \right\|_{\mathscr{V}} \le \frac{4^2\Delta\left(\frac{x}{2^2}, \frac{x}{2^2}\right)}{[4\,m^2(m^2-1)]^\beta} \tag{3.25}$$

for all $x \in \mathscr{U}$. Combining (3.24) and (3.25), we reach

$$\begin{aligned}
&\left\| f_2(x) - 4^2 f_2\left(\frac{x}{2^2}\right) \right\|_{\mathscr{V}} \\
&\le K\left\{ \left\| f_2(x) - 4 f_2\left(\frac{x}{2}\right) \right\|_{\mathscr{V}} + \left\| 4 f_2\left(\frac{x}{2}\right) - 4^2 f_2\left(\frac{x}{2^2}\right) \right\|_{\mathscr{V}} \right\} \\
&\le K\left\{ \frac{4\Delta\left(\frac{x}{2}, \frac{x}{2}\right)}{[4\,m^2(m^2-1)]^\beta} + \frac{4^2\Delta\left(\frac{x}{2^2}, \frac{x}{2^2}\right)}{[4\,m^2(m^2-1)]^\beta} \right\} \\
&\le \frac{K}{[4\,m^2(m^2-1)]^\beta} \left\{ 4\Delta\left(\frac{x}{2}, \frac{x}{2}\right) + 4^2\Delta\left(\frac{x}{2^2}, \frac{x}{2^2}\right) \right\}
\end{aligned} \tag{3.26}$$

for all $x \in \mathscr{U}$. Generalizing, for a positive integer q, we have

$$\left\| f_2(x) - 4^q f_2\left(\frac{x}{2^q}\right) \right\|_{\mathscr{V}} \le \frac{K^{q-1}}{[4\,m^2(m^2-1)]^\beta} \sum_{p=1}^{q} 4^p \Delta\left(\frac{x}{2^p}, \frac{x}{2^p}\right) \tag{3.27}$$

for all $x \in \mathscr{U}$. The rest of the proof is parallel clues that of $s = 1$. Thus the theorem holds for $s = -1$. This completes the proof of the theorem. □

Corollary 3.2. *Let r and t be positive integers and $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \begin{cases} r; & \\ r(\|x\|^t + \|y\|^t) & t \ne 2; \\ r\|x\|^t\|y\|^t & 2t \ne 2; \\ r(\|x\|^t\|y\|^t + \|x\|^{2t} + \|y\|^{2t}) & 2t \ne 2; \end{cases} \tag{3.28}$$

for all $x, y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} = \|f(2x) - 16f(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}}$$

$$\le \begin{cases} \frac{K^{3+q-1} r_{2c}}{3}; \\ \frac{K^{3+q-1} r_{2s} \|x\|^t}{|4 - 2^{\beta t}|}; \\ \frac{K^{3+q-1} r_{2p} \|x\|^{2t}}{|4 - 2^{2\beta t}|}; \\ \frac{K^{3+q-1} r_{2sp} \|x\|^{2t}}{|4 - 2^{2\beta t}|}; \end{cases} \tag{3.29}$$

where

$$\begin{aligned}
r_{2c} &= \frac{(24m^2+6)|r|}{[4\,m^2(m^2-1)]^\beta}; \\
r_{2s} &= \frac{4(36m^2 + 6\cdot 2^t + 12m^t - 12)r}{[4\,m^2(m^2-1)]^\beta}; \\
r_{2p} &= \frac{4(12m^2 + 12m^t)r}{[4\,m^2(m^2-1)]^\beta}; \\
r_{2sp} &= \frac{4(48m^2 + 6\cdot 2^{2t} + 12(m^{2t} + m^t) - 12)r}{[4\,m^2(m^2-1)]^\beta};
\end{aligned} \tag{3.30}$$

for all $x \in \mathscr{U}$.

Theorem 3.3. *Assume $\rho : \mathscr{U}^2 \longrightarrow (0,\infty]$ be a function satisfying the condition*

$$\lim_{q\to\infty} \frac{\rho(2^{qs}x, 2^{qs}y)}{16^{qs}} = 0 \tag{3.31}$$

for all $x,y \in \mathscr{U}$. Also, let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \rho(x,y) \tag{3.32}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned} &\|f_4(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \\ &= \|f(2x) - 4f(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \\ &\le \frac{K^{q-1}}{[16\, m^2(m^2-1)]^\beta} \sum_{p=\frac{1-s}{2}}^{\infty} \frac{\Delta(2^{sp}x, 2^{sp}x)}{16^{sp}} \end{aligned} \tag{3.33}$$

where $\Delta(2^{sp}x, 2^{sp}x)$ is defined in (3.4) for all $x \in \mathscr{U}$. The mapping $\mathscr{Q}_4(x)$ is defined by

$$\mathscr{Q}_4(x) = \lim_{q\to\infty} \frac{f_4(2^{qs}x)}{16^{qs}} = \lim_{q\to\infty} \frac{f(2^{s(q+1)}x) - 4f(2^{qs}x)}{16^{qs}} \tag{3.34}$$

for all $x \in \mathscr{U}$, where $s = \pm 1$.

Proof. It follows from (3.15), we reach

$$\begin{aligned} &\left[m^2(m^2-1)\right]^\beta \|[f(4x) - 4f(2x)] - 16[f(2x) - 4f(x)]\|_{\mathscr{V}} \\ &\le \Delta(x,x) \end{aligned} \tag{3.35}$$

for all $x \in \mathscr{U}$. Define a function

$$f_4 : \mathscr{U} \longrightarrow \mathscr{V} \qquad by \qquad f_4(x) = f(2x) - 4f(x) \tag{3.36}$$

for all $x \in \mathscr{U}$. Using (3.36) in (3.35), we achieve

$$\begin{aligned} &\left[m^2(m^2-1)\right]^\beta \|f_4(2x) - 16 f_4(x)\|_{\mathscr{V}} \le \Delta(x,x) \quad or \\ &\|f_4(2x) - 16 f_4(x)\|_{\mathscr{V}} \le \frac{\Delta(x,x)}{[m^2(m^2-1)]^\beta} \end{aligned} \tag{3.37}$$

for all $x \in \mathscr{U}$. Now, from the above inequality, we have

$$\left\| \frac{f_4(2x)}{16} - f_4(x) \right\|_{\mathscr{V}} \le \frac{\Delta(x,x)}{[16\, m^2(m^2-1)]^\beta} \tag{3.38}$$

for all $x \in \mathscr{U}$. Letting x by $2x$ and dividing by 16 in (3.38), we observe

$$\left\| \frac{f_4(2^2x)}{16^2} - \frac{f_4(2x)}{16} \right\|_{\mathscr{V}} \le \frac{\Delta(2x,2x)}{16 \cdot [16\, m^2(m^2-1)]^\beta} \tag{3.39}$$

for all $x \in \mathscr{U}$. Combining (3.38) and (3.39) one can notice that

$$\begin{aligned} &\left\| \frac{f_4(2^2x)}{16^2} - f_4(x) \right\|_{\mathscr{V}} \\ &\le K \left\{ \left\| \frac{f_4(2x)}{16} - f_4(x) \right\|_{\mathscr{V}} + \left\| \frac{f_4(2^2x)}{16^2} - \frac{f_4(2x)}{16} \right\|_{\mathscr{V}} \right\} \\ &\le K \left\{ \frac{\Delta(x,x)}{[16\, m^2(m^2-1)]^\beta} + \frac{\Delta(2x,2x)}{16 \cdot [16\, m^2(m^2-1)]^\beta} \right\} \\ &= \frac{K}{[16\, m^2(m^2-1)]^\beta} \left\{ \Delta(x,x) + \frac{\Delta(2x,2x)}{16} \right\} \end{aligned} \tag{3.40}$$

for all $x \in \mathscr{U}$. Generalizing for a positive integer q, one can verify that

$$\left\| \frac{f_4(2^qx)}{16^q} - f_4(x) \right\|_{\mathscr{V}} \le \frac{K^{q-1}}{[16\, m^2(m^2-1)]^\beta} \sum_{p=0}^{q-1} \frac{\Delta(2^px, 2^px)}{16^p} \tag{3.41}$$

for all $x \in \mathscr{U}$. The rest of the proof is similar ideas to that of Theorem 3.1. □

Corollary 3.4. *Let r and t be positive integers and $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \begin{cases} r; & \\ r(\|x\|^t + \|y\|^t) & t \ne 4; \\ r\|x\|^t\|y\|^t & 2t \ne 4; \\ r(\|x\|^t\|y\|^t + \|x\|^{2t} + \|y\|^{2t}) & 2t \ne 4; \end{cases} \tag{3.42}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned} \|f_4(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} &= \|f(2x) - 4f(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \\ &\le \begin{cases} \dfrac{K^{3+q-1} r_{4c}}{15}; \\ \dfrac{K^{3+q-1} r_{4s}\|x\|^t}{|16 - 2^{\beta t}|}; \\ \dfrac{K^{3+q-1} r_{4p}\|x\|^{2t}}{|16 - 2^{2\beta t}|}; \\ \dfrac{K^{3+q-1} r_{4sp}\|x\|^{2t}}{|16 - 2^{2\beta t}|}; \end{cases} \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} r_{4c} &= \frac{16(24m^2+6)|r|}{[16\, m^2(m^2-1)]^\beta}; \\ r_{4s} &= \frac{16(36m^2 + 6\cdot 2^t + 12m^t - 12)r}{[16\, m^2(m^2-1)]^\beta}; \\ r_{4p} &= \frac{16(12m^2 + 12m^t)r}{[16\, m^2(m^2-1)]^\beta}; \\ r_{4sp} &= \frac{16(48m^2 + 6\cdot 2^{2t} + 12(m^{2t}+m^t) - 12)r}{[16\, m^2(m^2-1)]^\beta}; \end{aligned} \tag{3.44}$$

for all $x \in \mathscr{U}$.

Theorem 3.5. *Assume $\rho : \mathscr{U}^2 \longrightarrow (0,\infty]$ be a function satisfying the conditions (3.1) and (3.31) for all $x,y \in \mathscr{U}$. Also, let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QQ}(x,y)\|_{\mathscr{V}} \le \rho(x,y) \tag{3.45}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$ and a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned}&\|f(x)-\mathscr{Q}_2(x)-\mathscr{Q}_4(x)\|_{\mathscr{V}}\\ &\le \frac{K^{3+q}}{[12\,m^2(m^2-1)]^\beta}\left\{\frac{1}{4^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{4^{sp}}\right.\\ &\qquad\left.+\frac{1}{16^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{16^{sp}}\right\}\end{aligned} \tag{3.46}$$

where $\Delta(2^{sp}x,2^{sp}x)$ is defined in (3.4) for all $x \in \mathscr{U}$. The mappings $\mathscr{Q}_2(x)$ and $\mathscr{Q}_4(x)$ are defined in (3.5) and (3.34) for all $x \in \mathscr{U}$, where $s = \pm 1$.

Proof. Given, $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality (3.45) for all $x,y \in \mathscr{U}$. Hence By Theorem 3.1 there exists a unique quadratic mapping such that

$$\begin{aligned}&\left\|f(2x)-16f(x)-\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\\ &\le \frac{K^{q-1}}{[4\,m^2(m^2-1)]^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{4^{sp}}\end{aligned} \tag{3.47}$$

for all $x \in \mathscr{U}$. Also, by Theorem 3.3 there exists a unique quartic mapping such that

$$\begin{aligned}&\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\\ &\le \frac{K^{q-1}}{[16\,m^2(m^2-1)]^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{16^{sp}}\end{aligned} \tag{3.48}$$

for all $x \in \mathscr{U}$. Now,

$$\begin{aligned}&\left\|12f(x)+\mathscr{Q}_2'(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\\ &=\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)-f(2x)+16f(x)+\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\\ &\le K\left\{\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\right.\\ &\qquad\left.+\left\|f(2x)-16f(x)-\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\right\}\\ &\le \frac{K^{q}}{[m^2(m^2-1)]^\beta}\left\{\frac{1}{4^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{4^{sp}}\right.\\ &\qquad\left.+\frac{1}{16^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{16^{sp}}\right\}\end{aligned} \tag{3.49}$$

for all $x \in \mathscr{U}$. Thus, it follows from the above inequality that

$$\begin{aligned}&\left\|f(x)+\frac{1}{12}\mathscr{Q}_2'(x)-\frac{1}{12}\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\\ &\le \frac{K^{q}}{[12\,m^2(m^2-1)]^\beta}\left\{\frac{1}{4^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{4^{sp}}\right.\\ &\qquad\left.+\frac{1}{16^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{16^{sp}}\right\}\end{aligned} \tag{3.50}$$

for all $x \in \mathscr{U}$. Hence, we obtain (3.46) by defining

$$\mathscr{Q}_2(x) = -\frac{1}{12}\mathscr{Q}_2'(x); \qquad \mathscr{Q}_4(x) = \frac{1}{12}\mathscr{Q}_4'(x);$$

for all $x \in \mathscr{U}$. □

Corollary 3.6. *Let r and t be positive integers and $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QQ}(x,y)\|_{\mathscr{V}} \le \begin{cases} r; & \\ r(||x||^t+||y||^t) & t\ne 2,4;\\ r||x||^t||y||^t & 2t\ne 2,4;\\ r(||x||^t||y||^t+||x||^{2t}+||y||^{2t}) & 2t\ne 2,4;\end{cases} \tag{3.51}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$ and a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned}&\|f(x)-\mathscr{Q}_2(x)-\mathscr{Q}_4(x)\|_{\mathscr{V}}\\ &\le \begin{cases}\dfrac{K^{3+q}}{12^\beta}\left[\dfrac{r_{2c}}{3}+\dfrac{r_{4c}}{15}\right];\\ \dfrac{K^{3+q}||x||^t}{12^\beta}\left[\dfrac{r_{2s}}{|4-2^{\beta t}|}+\dfrac{r_{4s}}{|16-2^{\beta t}|}\right];\\ \dfrac{K^{3+q}||x||^{2t}}{12^\beta}\left[\dfrac{r_{2p}}{|4-2^{2\beta t}|}+\dfrac{r_{4p}}{|16-2^{2\beta t}|}\right];\\ \dfrac{K^{3+q}||x||^{2t}}{12^\beta}\left[\dfrac{r_{2sp}}{|4-2^{2\beta t}|}+\dfrac{r_{4sp}}{|16-2^{2\beta t}|}\right];\end{cases}\end{aligned} \tag{3.52}$$

where $r_{2c}, r_{4c}, r_{2s}, r_{4s}, r_{2p}, r_{4p}, r_{2sp}, r_{4sp}$ are respectively defined in (3.30) and (3.44) for all $x \in \mathscr{U}$.

3.2 Case 2: f is Odd

Theorem 3.7. *Assume $\rho : \mathscr{U}^2 \longrightarrow (0,\infty]$ be a function satisfying the condition*

$$\lim_{q\to\infty}\frac{\rho(2^{qs}x,2^{qs}y)}{8^{qs}} = 0 \tag{3.53}$$

for all $x,y \in \mathscr{U}$. Also, let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an odd mapping satisfying the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \rho(x,y) \tag{3.54}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique cubic function $\mathscr{C}(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x)-\mathscr{C}(x)\|_{\mathscr{V}} \le K^{q-1}\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta} \sum_{p=\frac{1-s}{2}}^{\infty} \frac{\rho(0,2^{sp}x)}{8^{sp}} \tag{3.55}$$

for all $x \in \mathscr{U}$. The mapping $\mathscr{C}(x)$ is defined by

$$\mathscr{C}(x) = \lim_{q\to\infty} \frac{f(2^{qs}x)}{8^{qs}} \tag{3.56}$$

for all $x \in \mathscr{U}$, where $s = \pm 1$.

Proof. Changing (x,y) by $(0,x)$ in (3.54) and using oddness of f, we arrive

$$\left\|-\frac{m^2(m^2-1)}{6}(f(2x)-2f(x)-6f(x))\right\|_{\mathscr{V}} \le \rho(0,x)$$

$$\left[\frac{m^2(m^2-1)}{6}\right]^{\beta}\|f(2x)-8f(x)\|_{\mathscr{V}} \le \rho(0,x) \tag{3.57}$$

for all $x \in \mathscr{U}$. It follows from above inequality that

$$\left\|\frac{f(2x)}{8}-f(x)\right\|_{\mathscr{V}} \le \left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\rho(0,x) \tag{3.58}$$

for all $x \in \mathscr{U}$. Letting x by $2x$ and dividing by 8 in (3.58), we observe

$$\left\|\frac{f(2^2x)}{8^2}-\frac{f(2x)}{8}\right\|_{\mathscr{V}} \le \left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\frac{\rho(0,2x)}{8} \tag{3.59}$$

for all $x \in \mathscr{U}$. Combining (3.58) and (3.59) one can notice that

$$\begin{aligned}&\left\|\frac{f(2^2x)}{8^2}-f(x)\right\|_{\mathscr{V}}\\ &\le K\left\{\left\|\frac{f(2x)}{8}-f(x)\right\|_{\mathscr{V}}+\left\|\frac{f(2^2x)}{8^2}-\frac{f(2x)}{8}\right\|_{\mathscr{V}}\right\}\\ &= K\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\left\{\rho(0,x)+\frac{\rho(0,2x)}{8}\right\}\end{aligned} \tag{3.60}$$

for all $x \in \mathscr{U}$. Generalizing for a positive integer q, one can verify that

$$\begin{aligned}&\left\|\frac{f(2^qx)}{8^q}-f(x)\right\|_{\mathscr{V}}\\ &\le K^{q-1}\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\sum_{p=0}^{q-1}\frac{\rho(0,2^px)}{8^p}\end{aligned} \tag{3.61}$$

for all $x \in \mathscr{U}$. The rest of the proof is similar to that of Theorem 3.1. □

Corollary 3.8. *Let r and t be positive integers and $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \begin{cases} r; & \\ r(||x||^t+||y||^t) & t \ne 3;\\ r(||x||^t||y||^t+||x||^{2t}+||y||^{2t}) & 2t \ne 3;\end{cases} \tag{3.62}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique cubic function $\mathscr{C}(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x)-\mathscr{C}(x)\|_{\mathscr{V}} \le \begin{cases} \dfrac{K^{q-1}r_{3c}}{7}; \\ \dfrac{K^{q-1}r_{3s}||x||^t}{|8-2^{\beta t}|}; \\ \dfrac{K^{q-1}r_{3sp}||x||^{2t}}{|8-2^{2\beta t}|}; \end{cases} \tag{3.63}$$

where

$$\begin{aligned} r_{3c} &= \left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\cdot 8|r|;\\ r_{3s} &= \left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\cdot 8r;\\ r_{3sp} &= \left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\cdot 8r;\end{aligned} \tag{3.64}$$

for all $x \in \mathscr{U}$.

3.3 Case 3: f is Odd-Even

Theorem 3.9. *Assume $\rho : \mathscr{U}^2 \longrightarrow (0,\infty]$ be a function satisfying the conditions (3.1), (3.31) and (3.53) for all $x,y \in \mathscr{U}$. Also, let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be a mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \rho(x,y) \tag{3.65}$$

for all $x,y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$, a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ and a unique cubic function $\mathscr{C}(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned}&\|f(x)-\mathscr{Q}_2(x)-\mathscr{Q}_4(x)-\mathscr{C}(x)\|_{\mathscr{V}}\\ &\le \frac{K}{2^{\beta}}\left\{\frac{K^{3+q}}{[12\,m^2(m^2-1)]^{\beta}}\right.\\ &\left\{\frac{1}{4^{\beta}}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)+\Delta(-2^{sp}x,-2^{sp}x)}{4^{sp}}\right.\\ &\left.+\frac{1}{16^{\beta}}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,-2^{sp}x)+\Delta(2^{sp}x,-2^{sp}x)}{16^{sp}}\right\}\\ &\left.+K^{q-1}\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\rho(0,2^{sp}x)+\rho(0,-2^{sp}x)}{8^{sp}}\right\}\end{aligned} \tag{3.66}$$

where $\Delta(2^{sp}x, 2^{sp}x)$ is defined in (3.4) for all $x \in \mathscr{U}$. The mappings $\mathscr{Q}_2(x)$, $\mathscr{Q}_4(x)$ and $\mathscr{C}(x)$ are defined in (3.5), (3.34) and (3.56) for all $x \in \mathscr{U}$, where $s = \pm 1$.

Proof. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in \mathscr{U}$. Then it is easy to verify that $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathscr{U}$. By the definition of $f_e(x)$ and Theorem 3.5, we have

$$\|f_e(x) - \mathscr{Q}_2(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \leq \frac{K}{2^\beta}\left\{\frac{K^{3+q}}{[12\, m^2(m^2-1)]^\beta}\left\{\frac{1}{4^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)+\Delta(-2^{sp}x,-2^{sp}x)}{4^{sp}} + \frac{1}{16^\beta}\sum_{p=\frac{1-s}{2}}^{\infty}\frac{\Delta(2^{sp}x,2^{sp}x)}{16^{sp}}\right\}\right\} \tag{3.67}$$

for all $x \in \mathscr{U}$. Also, let $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in \mathscr{U}$. Then it is easy to verify that $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in \mathscr{U}$. By the definition of $f_o(x)$ and Theorem 3.7, we have

$$\|f_o(x) - \mathscr{C}(x)\|_{\mathscr{V}} \leq \frac{K}{2^\beta}\left\{K^{q-1}\left[\frac{3}{4\, m^2(m^2-1)}\right]^\beta \sum_{p=\frac{1-s}{2}}^{\infty}\frac{\rho(0,2^{sp}x)+\rho(0,-2^{sp}x)}{8^{sp}}\right\} \tag{3.68}$$

for all $x \in \mathscr{U}$. Suppose, if we define a function $f(x)$ by

$$f(x) = f_e(x) + f_o(x) \tag{3.69}$$

for all $x \in \mathscr{U}$. It follows from (3.67), (3.68) and (3.69), we arrive our desired result. □

Corollary 3.10. *Let r and t be positive integers and $f : \mathscr{U} \longrightarrow \mathscr{V}$ be a mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \leq \begin{cases} r; & \\ r(||x||^t + ||y||^t) & t \neq 2,3,4; \\ r(||x||^t||y||^t + ||x||^{2t} + ||y||^{2t}) & 2t \neq 2,3,4; \end{cases} \tag{3.70}$$

for all $x, y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$, a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ and a unique cubic function $\mathscr{C}(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x) - \mathscr{Q}_2(x) - \mathscr{Q}_4(x) - \mathscr{C}(x)\|_{\mathscr{V}} \leq \begin{cases} \frac{K}{2^\beta}\left\{\frac{K^{3+q}}{12^\beta}\left[\frac{r_{2c}}{3} + \frac{r_{4c}}{15}\right] + \frac{K^{q-1}r_{3c}}{7}\right\}; \\ \frac{K}{2^\beta}\left\{\frac{K^{3+q}||x||^t}{12^\beta}\left[\frac{r_{2s}}{|4-2^{\beta t}|} + \frac{r_{4s}}{|16-2^{\beta t}|}\right] + \frac{K^{q-1}r_{3s}||x||^t}{|8-2^{\beta t}|}\right\}; \\ \frac{K}{2^\beta}\left\{\frac{K^{3+q}||x||^{2t}}{12^\beta}\left[\frac{r_{2sp}}{|4-2^{2\beta t}|} + \frac{r_{4sp}}{|16-2^{2\beta t}|}\right] + \frac{K^{q-1}r_{3sp}||x||^{2t}}{|8-2^{2\beta t}|}\right\}; \end{cases} \tag{3.71}$$

for all $x \in \mathscr{U}$.

4. Stability Results: Fixed Point Method

Now, we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T : X \longrightarrow X$ which is strictly contractive mapping, that is*
(A_1) $\quad d(Tx,Ty) \leq Ld(x,y)$
for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is
(A_2) $\quad lim_{n\to\infty}T^n x = x^*$,
for any starting point $x \in X$;
(iii) One has the following estimation inequalities:
(A_3) $\quad d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;
(A_4) $\quad d(x,x^*) \leq \frac{1}{1-L}\ d(x,x^*), \forall\ x \in X$.

Theorem 4.2. [41] *(The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \longrightarrow X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
(F_1) $\quad d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0$,
or
(F_2) there exists a natural number n_0 such that:
$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
$(FPC2)$The sequence $(T^n x)$ is convergent to a fixed point y^* of T
$(FPC3)$ y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

4.1 Case 1: f is Even

Theorem 4.3. *Let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping for which there exists a function $\rho, \Delta : \mathscr{U}^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{q\to\infty}\frac{1}{\alpha_b^{2q}}rho(\alpha_b^q x,\alpha_b^q y)=0 \tag{4.1}$$

where

$$\alpha_b=\begin{cases} 2 & if \quad b=0,\\ \frac{1}{2} & if \quad b=1\end{cases} \tag{4.2}$$

such that the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}}\le\rho(x,y) \tag{4.3}$$

holds for all $x,y\in\mathscr{U}$. Assume that there exists $L=L(i)$ such that the function

$$\Delta(x,x)=\frac{4}{[4\,m^2(m^2-1)]^\beta}\Delta\left(\frac{x}{2},\frac{x}{2}\right) \tag{4.4}$$

where $\Delta(x,x)$ is defined in (3.15) with the property

$$\frac{1}{\alpha_b^2}\Delta(\alpha_b x,\alpha_b x)=L\,\Delta(x,x) \tag{4.5}$$

for all $x\in\mathscr{U}$. Then there exists a unique quadratic mapping $\mathscr{Q}_2:\mathscr{U}\longrightarrow\mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned}\|f_2(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}&=\|f(2x)-16f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}\\ &\le\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\end{aligned} \tag{4.6}$$

for all $x\in\mathscr{U}$.

Proof. In order to prove the stability result, let us consider the set

$$\mathscr{B}=\{g|\,g:\mathscr{U}\longrightarrow\mathscr{V},\ g(0)=0\}$$

and introduce the generalized metric on $\mathscr{B}$,

$$d(f_1,f_2)=\inf\{\eta\in(0,\infty):\|f_1(x)-f_2(x)\|_{\mathscr{V}}\le\eta\,\Delta(x,x),x\in\mathscr{U}\}. \tag{4.7}$$

Hence, (4.7) is complete with respect to the defined metric.

Now, define $J:\mathscr{B}\longrightarrow\mathscr{B}$ by

$$Jf(x)=\frac{1}{\alpha_b^2}f(\alpha_b x),\qquad for\ \ all\ \ x\in\mathscr{U}. \tag{4.8}$$

It is easy to prove that J is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant L. Indeed, from (4.7) and $f_1,f_2\in\mathscr{B}$, we arrive

$$\begin{aligned}&d(f_1,f_2)\le\eta\\ &\Longrightarrow\|f_1(x)-f_2(x)\|_{\mathscr{V}}\le\eta\Delta(x,x),\quad x\in\mathscr{U};\\ &\Longrightarrow\left\|\frac{1}{\alpha_b^2}f_1(\alpha_b x)-\frac{1}{\alpha_b^2}f_2(\alpha_b x)\right\|_{\mathscr{V}}\le\frac{\eta}{\alpha_b^2}\Delta(\alpha_b x,\alpha_b x),\quad x\in\mathscr{U};\\ &\Longrightarrow\left\|\frac{1}{\alpha_b^2}f_1(\alpha_b x)-\frac{1}{\alpha_b^2}f_2(\alpha_b x)\right\|_{\mathscr{V}}\le L\eta\Delta(x,x),\quad x\in\mathscr{U};\\ &\Longrightarrow\|Jf_1(x)-Jf_2(x)\|_{\mathscr{V}}\le L\eta\Delta(x,x),\quad x\in\mathscr{U};\\ &\Longrightarrow d(Jf_1,Jf_2)\le L\eta.\end{aligned}$$

It follows from (3.19) that

$$\left\|\frac{f_2(2x)}{4}-f_2(x)\right\|_{\mathscr{V}}\le\frac{\Delta(x,x)}{[4\,m^2(m^2-1)]^\beta} \tag{4.9}$$

for all $x\in\mathscr{U}$. From (4.7), (4.8) (4.2) for the case $b=0$, we reach

$$d(Jf_2,f_2)\le L\Delta(x,x)=L^{1-0}\Delta(x,x),\qquad x\in\mathscr{U}. \tag{4.10}$$

It follows from (3.24) that

$$\left\|f_2(x)-4f_2\left(\frac{x}{2}\right)\right\|_{\mathscr{V}}\le\frac{4\Delta\left(\frac{x}{2},\frac{x}{2}\right)}{[4\,m^2(m^2-1)]^\beta} \tag{4.11}$$

for all $x\in\mathscr{U}$. From (4.7), (4.8) (4.2) for the case $b=1$, we reach

$$d(f_2,Jf_2)\le\Delta(x,x)=L^{1-1}\Delta(x,x),\qquad x\in\mathscr{U}. \tag{4.12}$$

Thus, from (4.10) and (4.12), we have

$$d(Jf_2,f_2)\le L\Delta(x,x)=L^{1-0}\Delta(x,x),\qquad x\in\mathscr{U}. \tag{4.13}$$

Hence property (FPC1) holds. It follows from property (FPC2) that there exists a fixed point $\mathscr{Q}_2$ of J in $\mathscr{B}$ such that

$$\mathscr{Q}_2(x)=\lim_{q\to\infty}\frac{1}{\alpha_b^{2q}}f_2(\alpha_b^q x) \tag{4.14}$$

for all $x\in\mathscr{U}$. In order to show that $\mathscr{Q}_2$ satisfies (1.2), replacing (x,y) by $(\alpha_b^q x,\alpha_b^q y)$ and dividing by α_b^{2q} in (4.3), we have

$$\begin{aligned}\|\mathscr{Q}_2(x,y)\|_{\mathscr{V}}&=\lim_{q\to\infty}\frac{1}{\alpha_b^{2q}}\left\|\mathscr{F}_{QCQ}(\alpha_b^q x,\alpha_b^q y)\right\|_{\mathscr{V}}\\ &\le\lim_{q\to\infty}\frac{1}{\alpha_b^{2q}}\rho\left(\alpha_b^q x,\alpha_b^q y\right)=0\end{aligned}$$

for all $x,y\in\mathscr{U}$. That is $\mathscr{Q}_2$ satisfies the functional equation (1.2). By property (FPC3), $\mathscr{Q}_2$ is the unique fixed point of J in the set

$$\mathscr{D}=\{\mathscr{Q}_2\in\mathscr{B}:d(f,\mathscr{Q}_2)<\infty\},$$

such that

$$\inf\{\eta\in(0,\infty):\|f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}\le\eta\Delta(x,x),x\in\mathscr{U}\}.$$

Finally, by property (FPC4), we arrive

$$\begin{aligned}&\|f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}\le\|f(x)-Jf(x)\|_{\mathscr{V}},\qquad x\in\mathscr{U};\\ &\|f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}\le\frac{L^{1-i}}{1-L},\qquad x\in\mathscr{U};\\ &\|f(x)-\mathscr{Q}_2(x)\|_{\mathscr{V}}\le\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x),\qquad x\in\mathscr{U}.\end{aligned}$$

This finishes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.2).

Corollary 4.4. *Let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping. If there exist real numbers r and t such that*

$$\|\mathscr{F}_{QCQ}(x,y)\| \leq \begin{cases} r, & \\ r\left\{||x||^t + ||y||^t\right\} & t \neq 2; \\ r||x||^t||y||^t & 2t \neq 2; \\ r\left\{||x||^t||y||^t + \left\{||x||^{2t} + ||y||^{2t}\right\}\right\} & 2t \neq 2; \end{cases} \tag{4.15}$$

for all $x,y \in \mathscr{U}$, then there exists a unique quadratic mapping $\mathscr{Q}_2 : \mathscr{U} \longrightarrow \mathscr{V}$ such that

$$\|f(x) - \mathscr{Q}_2(x)\| \leq \begin{cases} \dfrac{r_{2c}}{|3|}, \\ \dfrac{r_{2s}2^t||x||^t}{|4-2^{\beta t}|}, \\ \dfrac{r_{2r}2^{2t}||x||^{2t}}{|4-2^{2\beta t}|}, \\ \dfrac{r_{2sr}2^{2t}||x||^{2t}}{|4-2^{2\beta t}|} \end{cases} \tag{4.16}$$

where

$$\begin{aligned} r_{2c} &= \frac{4K^3(24m^2+6)r}{[4\, m^2(m^2-1)]^\beta}; \\ r_{2s} &= \frac{4K^3(36m^2+6\cdot 2^t+12m^t-12)r}{[4\, m^2(m^2-1)]^\beta}; \\ r_{2p} &= \frac{4K^3(12m^2+12m^t-12)r}{[4\, m^2(m^2-1)]^\beta}; \\ r_{2sp} &= \frac{4K^3(48m^2+6\cdot 2^{2t}+12(m^{2t}+m^t)-12)r}{[4\, m^2(m^2-1)]^\beta} \end{aligned} \tag{4.17}$$

for all $x \in \mathscr{U}$.

Proof. If we take

$$\rho(x,y) = \begin{cases} r, \\ r\left\{||x||^t + ||y||^t\right\} \\ r||x||^t||y||^t \\ r\left\{||x||^t||y||^t + \left\{||x||^{2t} + ||y||^{2t}\right\}\right\} \end{cases} \tag{4.18}$$

for all $x,y \in \mathscr{U}$. Now

$$\frac{1}{\alpha_b^{2q}}\rho(\alpha_b^q x, \alpha_b^q y) = \begin{cases} \dfrac{r}{\alpha_b^{2q}}, \\ \dfrac{r}{\alpha_b^{2q}}\left\{||\alpha_b^n x||^t + ||\alpha_b^n y||^t\right\}, \\ \dfrac{r}{\alpha_b^{2q}}||\alpha_b^n x||^t\ ||\alpha_b^n y||^t \\ \dfrac{r}{\alpha_b^{2q}}\Big\{||\alpha_b^n x||^t\ ||\alpha_b^n y||^t \\ \qquad + \left\{||\alpha_b^n x||^{2t} + ||\alpha_b^n y||^{2t}\right\}\Big\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } q \to \infty, \\ \to 0 \text{ as } q \to \infty, \\ \to 0 \text{ as } q \to \infty, \\ \to 0 \text{ as } q \to \infty. \end{cases}$$

Thus, (4.1) holds. But from (4.4), (4.18) and (3.15), we have

$$\begin{aligned} \Delta(x,x) &= \frac{4}{[4\, m^2(m^2-1)]^\beta}\Delta\left(\frac{x}{2},\frac{x}{2}\right) \\ &= \frac{4}{[4\, m^2(m^2-1)]^\beta}\ K^3\Big[12m^2\rho(0,x) + 12(m^2-1)\rho(x,x) \\ &\qquad + 6\rho(0,2x) + 12\rho(mx,x)\Big] \\ &= \frac{4K^3}{[4\, m^2(m^2-1)]^\beta} \\ &\quad \times \begin{cases} (24m^2+6)r; \\ (36m^2+6\cdot 2^t+12m^t-12)r||x||^t; \\ (12m^2+12m^t-12)r||x||^{2t}; \\ (48m^2+6\cdot 2^{2t}+12(m^{2t}+m^t)-12)r||x||^{2t}; \end{cases} \\ &= \frac{4K^3}{[4\, m^2(m^2-1)]^\beta} \times \begin{cases} r_{2c}, \\ r_{2s}||x||^t, \\ r_{2p}||x||^{2t}, \\ r_{2sp}||x||^{2t} \end{cases} \end{aligned} \tag{4.19}$$

for all $x \in \mathscr{U}$. Now, similarly by (4.5) and (4.18), we prove

$$\frac{1}{\alpha_b^2}\Delta(\alpha_b x, \alpha_b x) = \begin{cases} \alpha_b^{-2}\, r_{2c}, \\ \alpha_b^{t-2} r_{2s}, \\ \alpha_b^{2t-2} r_{2p}, \\ \alpha_b^{2t-2} r_{2sp}. \end{cases} = L\,\Delta(x,x)$$

Case (i): $L = \alpha_b^{-2} = 2^{-2}$ if $b = 0$ and $L = \frac{1}{\alpha_b^{-2}} = \frac{1}{2^{-2}} = 2^2$ if

$b = 1$. It follows from (4.6) that

$$\begin{aligned}\|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{-2})^{1-0}}{1-2^{-2}}\right)\Delta(x,x)\\ &= \left(\frac{1}{3}\right)\Delta(x,x);\\ \|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{2})^{1-1}}{1-2^{2}}\right)\Delta(x,x)\\ &= \left(\frac{1}{-3}\right)\Delta(x,x)\end{aligned}$$

for all $x \in \mathscr{U}$.
Case (ii): $L = \alpha_b^{t-2} = 2^{t-2}$ for $t < 2$ if $b = 0$ and $L = \frac{1}{\alpha_b^{t-2}} = 2^{2-t}$ for $t > 2$ if $b = 1$. It follows from (4.6) that

$$\begin{aligned}\|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{t-2})^{1-0}}{1-2^{t-2}}\right)\Delta(x,x)\\ &= \left(\frac{2^t}{4-2^t}\right)\Delta(x,x);\\ \|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{2-t})^{1-1}}{1-2^{2-t}}\right)\Delta(x,x)\\ &= \left(\frac{2^t}{2^t-4}\right)\Delta(x,x)\end{aligned}$$

for all $x \in \mathscr{U}$.
Case (iii): $L = \alpha_b^{2t-2} = 2^{2t-2}$ for $2t > 2$ if $b = 0$ and $L = \frac{1}{\alpha_b^{2t-2}} = 2^{2-2t}$ for $2t > 2$ if $b = 1$. It follows from (4.6) that

$$\begin{aligned}\|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{2t-2})^{1-0}}{1-2^{2t-2}}\right)\Delta(x,x)\\ &= \left(\frac{2^{2t}}{4-2^{2t}}\right)\Delta(x,x);\\ \|f_2(x) - \mathscr{Q}_2(x)\|_{\mathscr{V}} &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\\ &= \left(\frac{(2^{2-2t})^{1-1}}{1-2^{2-2t}}\right)\Delta(x,x)\\ &= \left(\frac{2^{2t}}{2^{2t}-4}\right)\Delta(x,x)\end{aligned}$$

for all $x \in \mathscr{U}$. Hence the proof is complete. □

Theorem 4.5. *Let $f : \mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping for which there exists a function $\rho, \Delta : \mathscr{U}^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{q\to\infty} \frac{1}{\alpha_b^{4Q}} rho(\alpha_b^q x, \alpha_b^q y) = 0 \tag{4.20}$$

where

$$\alpha_b = \begin{cases} 2 & if \quad b = 0,\\ \frac{1}{2} & if \quad b = 1\end{cases} \tag{4.21}$$

such that the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \le \rho(x,y) \tag{4.22}$$

holds for all $x, y \in \mathscr{U}$. Assume that there exists $L = L(i)$ such that the function

$$\Delta(x,x) = \frac{16}{[16\, m^2(m^2-1)]^{\beta}} \Delta\left(\frac{x}{2},\frac{x}{2}\right) \tag{4.23}$$

where $\Delta(x,x)$ is defined in (3.15) with the property

$$\frac{1}{\alpha_b^4}\Delta(\alpha_b x, \alpha_b x) = L\,\Delta(x,x) \tag{4.24}$$

for all $x \in \mathscr{U}$. Then there exists a unique quartic mapping $\mathscr{Q}_4 : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f_4(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} = \|f(2x) - 4f(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \le \left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x) \tag{4.25}$$

for all $x \in \mathscr{U}$.

Proof. In order to prove the stability result, let us consider the set

$$\mathscr{B} = \{g \,|\, g : \mathscr{U} \longrightarrow \mathscr{V},\ g(0) = 0\}$$

and introduce the generalized metric on $\mathscr{B}$,

$$d(f_1, f_4) = \inf\{\eta \in (0,\infty) : \|f_1(x) - f_4(x)\|_{\mathscr{V}} \le \eta\,\Delta(x,x), x \in \mathscr{U}\}. \tag{4.26}$$

Hence, (4.26) is complete with respect to the defined metric.
Now, define $J : \mathscr{B} \longrightarrow \mathscr{B}$ by

$$Jf(x) = \frac{1}{\alpha_b^4} f(\alpha_b x), \qquad for\ \ all\ \ x \in \mathscr{U}. \tag{4.27}$$

It is easy to prove that J is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.5 concerning the stability of (1.2).

Corollary 4.6. *Let $f:\mathscr{U}\longrightarrow\mathscr{V}$ be an even mapping. If there exist real numbers r and t such that*

$$\|\mathscr{F}_{QCQ}(x,y)\|\le\begin{cases} r, & \\ r\{||x||^t+||y||^t\} & t\neq 4;\\ r||x||^t||y||^t & 2t\neq 4;\\ r\{||x||^t||y||^t+\{||x||^{2t}+||y||^{2t}\}\} & 2t\neq 4;\end{cases}\tag{4.28}$$

for all $x,y\in\mathscr{U}$, then there exists a unique quartic mapping $\mathscr{Q}_4:\mathscr{U}\longrightarrow\mathscr{V}$ such that

$$\|f(x)-\mathscr{Q}_4(x)\|\le\begin{cases}\dfrac{r_{4c}}{|15|},\\ \dfrac{r_{4s}2^t||x||^t}{|16-2^{\beta t}|},\\ \dfrac{r_{4p}2^{2t}||x||^{2t}}{|16-2^{2\beta t}|},\\ \dfrac{r_{4sp}2^{2t}||x||^{2t}}{|16-2^{2\beta t}|}\end{cases}\tag{4.29}$$

where

$$\begin{aligned} r_{4c}&=\frac{16K^3(24m^2+6)r}{[16\,m^2(m^2-1)]^\beta};\\ r_{4s}&=\frac{16K^3(36m^2+6\cdot 2^t+12m^t-12)r}{[16\,m^2(m^2-1)]^\beta};\\ r_{4p}&=\frac{16K^3(12m^2+12m^t-12)r}{[16\,m^2(m^2-1)]^\beta};\\ r_{4sp}&=\frac{16K^3(48m^2+6\cdot 2^{2t}+12(m^{2t}+m^t)-12)r}{[16\,m^2(m^2-1)]^\beta}\end{aligned}\tag{4.30}$$

for all $x\in\mathscr{U}$.

Theorem 4.7. *Let $f:\mathscr{U}\longrightarrow\mathscr{V}$ be an even mapping for which there exists a function $\rho,\Delta:\mathscr{U}^2\longrightarrow[0,\infty)$ with the conditions (4.1), (4.20) and the satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}}\le\rho(x,y)\tag{4.31}$$

for all $x,y\in\mathscr{U}$. Assume that there exists $L=L(i)$ such that the function (4.4), (4.23) with properties (4.5), (4.24) for all $x\in\mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x):\mathscr{U}\longrightarrow\mathscr{V}$ and a unique quartic function $\mathscr{Q}_4(x):\mathscr{U}\longrightarrow\mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x)-\mathscr{Q}_2(x)-\mathscr{Q}_4(x)\|_{\mathscr{V}}\le\frac{2K}{12^\beta}\frac{L^{1-i}}{1-L}\Delta(x,x)\tag{4.32}$$

where $\Delta(x,x)$ is defined in (3.15) for all $x\in\mathscr{U}$.

Proof. By Theorem 4.3 there exists a unique quadratic mapping such that

$$\left\|f(2x)-16f(x)-\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\le\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\tag{4.33}$$

for all $x\in\mathscr{U}$. Also, by Theorem 4.5 there exists a unique quartic mapping such that

$$\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\le\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\tag{4.34}$$

for all $x\in\mathscr{U}$. Now,

$$\begin{aligned}&\left\|12f(x)+\mathscr{Q}_2'(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\\ &=\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)-f(2x)+16f(x)+\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\\ &\le K\left\{\left\|f(2x)-4f(x)-\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}+\left\|f(2x)-16f(x)-\mathscr{Q}_2'(x)\right\|_{\mathscr{V}}\right\}\\ &\le K\left\{\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)+\left(\frac{L^{1-i}}{1-L}\right)\Delta(x,x)\right\}\end{aligned}\tag{4.35}$$

for all $x\in\mathscr{U}$. Thus, it follows from the above inequality that

$$\left\|f(x)+\frac{1}{12}\mathscr{Q}_2'(x)-\frac{1}{12}\mathscr{Q}_4'(x)\right\|_{\mathscr{V}}\le\frac{2K}{12^\beta}\frac{L^{1-i}}{1-L}\Delta(x,x)\tag{4.36}$$

for all $x\in\mathscr{U}$. Hence, we obtain (4.32) by defining

$$\mathscr{Q}_2(x)=-\frac{1}{12}\mathscr{Q}_2'(x);\qquad\mathscr{Q}_4(x)=\frac{1}{12}\mathscr{Q}_4'(x);$$

for all $x\in\mathscr{U}$. □

Corollary 4.8. *Let r and t be positive integers and $f:\mathscr{U}\longrightarrow\mathscr{V}$ be an even mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}}\le\begin{cases} r; & \\ r(||x||^t+||y||^t) & t\neq 2,4;\\ r||x||^t||y||^t & 2t\neq 2,4;\\ r(||x||^t||y||^t+||x||^{2t}+||y||^{2t}) & 2t\neq 2,4;\end{cases}\tag{4.37}$$

for all $x,y\in\mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x):\mathscr{U}\longrightarrow\mathscr{V}$ and a unique quartic function $\mathscr{Q}_4(x):\mathscr{U}\longrightarrow\mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x)-\mathscr{Q}_2(x)-\mathscr{Q}_4(x)\|_{\mathscr{V}}\le\begin{cases}\dfrac{2K}{12^\beta}\left\{\dfrac{r_{2c}}{|3|}+\dfrac{r_{4c}}{|15|}\right\},\\ \dfrac{2K}{12^\beta}\left\{\dfrac{r_{2s}2^t||x||^t}{|4-2^{\beta t}|}+\dfrac{r_{4s}2^t||x||^t}{|16-2^{\beta t}|}\right\},\\ \dfrac{2K}{12^\beta}\left\{\dfrac{r_{2p}2^{2t}||x||^{2t}}{|4-2^{2\beta t}|}+\dfrac{r_{4p}2^{2t}||x||^{2t}}{|16-2^{2\beta t}|}\right\},\\ \dfrac{2K}{12^\beta}\left\{\dfrac{r_{2sp}2^{2t}||x||^{2t}}{|4-2^{2\beta t}|}+\dfrac{r_{4sp}2^{2t}||x||^{2t}}{|16-2^{2\beta t}|}\right\}\end{cases}\tag{4.38}$$

where $r_{2c},r_{4c},r_{2s},r_{4s},r_{2p},r_{4p},r_{2sp},r_{4sp}$ are respectively defined in (4.17), (4.30) for all $x\in\mathscr{U}$.

4.2 Case: 2 f is Odd

Theorem 4.9. *Let $f:\mathscr{U}\longrightarrow\mathscr{V}$ be an odd mapping for which there exists a function $\rho:\mathscr{U}^2\longrightarrow[0,\infty)$ with the condition*

$$\lim_{q\to\infty}\frac{1}{\alpha_b^{3q}}rho(\alpha_b^q x,\alpha_b^q y)=0\tag{4.39}$$

where

$$\alpha_b=\begin{cases}2 & if\quad b=0,\\ \frac{1}{2} & if\quad b=1\end{cases}\tag{4.40}$$

such that the functional inequality

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \leq \rho(x,y) \tag{4.41}$$

holds for all $x,y \in \mathscr{U}$. Assume that there exists $L = L(i)$ such that the function

$$\rho(0,x) = 8\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta} \rho\left(0,\frac{x}{2}\right) \tag{4.42}$$

with the property

$$\frac{1}{\alpha_b^3}\rho(0,\alpha_b x) = L\,\rho(0,x) \tag{4.43}$$

for all $x \in \mathscr{U}$. Then there exists a unique cubic mapping $\mathscr{C}:\mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x) - \mathscr{C}(x)\|_{\mathscr{V}} \leq \left(\frac{L^{1-i}}{1-L}\right)\rho(x,x) \tag{4.44}$$

for all $x \in \mathscr{U}$.

Proof. In order to prove the stability result, let us consider the set

$$\mathscr{B} = \{g|\, g:\mathscr{U} \longrightarrow \mathscr{V},\; g(0)=0\}$$

and introduce the generalized metric on $\mathscr{B}$,

$$d(f_1,f_3) = \inf\{\eta \in (0,\infty) : \|f_1(x) - f_3(x)\|_{\mathscr{V}} \leq \eta\,\rho(x,x), x \in \mathscr{U}\}. \tag{4.45}$$

Hence, (4.45) is complete with respect to the defined metric.

Now, define $J:\mathscr{B} \longrightarrow \mathscr{B}$ by

$$Jf(x) = \frac{1}{\alpha_b^3} f(\alpha_b x), \qquad for\;\; all\;\; x \in \mathscr{U}. \tag{4.46}$$

It is easy to prove that J is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.9 concerning the stability of (1.2).

Corollary 4.10. *Let $f:\mathscr{U} \longrightarrow \mathscr{V}$ be an oddmapping. If there exist real numbers r and t such that*

$$\|\mathscr{F}_{QCQ}(x,y)\| \leq \begin{cases} r, & \\ r\{||x||^t + ||y||^t\} & t \neq 3; \\ r\{||x||^t||y||^t + \{||x||^{2t} + ||y||^{2t}\}\} & 2t \neq 3; \end{cases} \tag{4.47}$$

for all $x,y \in \mathscr{U}$, then there exists a unique cubic mapping $\mathscr{C}:\mathscr{U} \longrightarrow \mathscr{V}$ such that

$$\|f(x) - \mathscr{C}(x)\| \leq \begin{cases} \dfrac{r_{3c}}{|7|}, \\ \dfrac{r_{3s}2^t||x||^t}{|8-2^{\beta t}|}, \\ \dfrac{r_{3sp}2^{2t}||x||^{2t}}{|8-2^{2\beta t}|} \end{cases} \tag{4.48}$$

where

$$\begin{aligned} r_{3c} &= 8r\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}; \\ r_{3s} &= 8r\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta}; \\ r_{3sp} &= 8r\left[\frac{3}{4\,m^2(m^2-1)}\right]^{\beta} \end{aligned} \tag{4.49}$$

for all $x \in \mathscr{U}$.

4.3 Case 3: f is Odd-Even

Theorem 4.11. *Let $f:\mathscr{U} \longrightarrow \mathscr{V}$ be an even mapping for which there exists a function $\rho,\Delta:\mathscr{U}^2 \longrightarrow [0,\infty)$ with the conditions (4.1), (4.20), (4.39) and the satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \leq \rho(x,y) \tag{4.50}$$

for all $x,y \in \mathscr{U}$. Assume that there exists $L = L(i)$ such that the function (4.4), (4.23), (4.42) with properties (4.5), (4.24), (4.43) for all $x \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x):\mathscr{U} \longrightarrow \mathscr{V}$, a unique quartic function $\mathscr{Q}_4(x):\mathscr{U} \longrightarrow \mathscr{V}$ and a unique cubic function $\mathscr{C}(x):\mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\begin{aligned} &\|f(x) - \mathscr{Q}_2(x) - \mathscr{Q}_4(x) - \mathscr{C}(x)\|_{\mathscr{V}} \\ &\leq \frac{K}{2^{\beta}}\left(\frac{L^{1-i}}{1-L}\right)\left\{\frac{2K}{12^{\beta}}(\Delta(x,x)+\Delta(-x,-x)) \right. \\ &\qquad \left. +(\rho(0,-x)+\rho(0,-x))\right\} \end{aligned} \tag{4.51}$$

where $\Delta(x,x)$ is defined in (3.15) for all $x \in \mathscr{U}$.

Proof. By definition of $f_e(x)$ and Theorem 4.7, we have

$$\begin{aligned} &\|f_e(x) - \mathscr{Q}_2(x) - \mathscr{Q}_4(x)\|_{\mathscr{V}} \\ &\leq \frac{2K^2}{24^{\beta}}\left(\frac{L^{1-i}}{1-L}\right)(\Delta(x,x)+\Delta(-x,-x)) \end{aligned} \tag{4.52}$$

for all $x \in \mathscr{U}$. Also, by the definition of $f_o(x)$ and Theorem 4.9, we have

$$\|f_o(x) - \mathscr{C}(x)\|_{\mathscr{V}} \leq \frac{K}{2^{\beta}}\left(\frac{L^{1-i}}{1-L}\right)(\rho(0,x)+\rho(0,-x)) \tag{4.53}$$

for all $x \in \mathscr{U}$. Suppose, if we define a function $f(x)$ by

$$f(x) = f_e(x) + f_o(x) \tag{4.54}$$

for all $x \in \mathscr{U}$. It follows from (4.52), (4.53) and (4.54), we arrive our desired result. □

Corollary 4.12. *Let r and t be positive integers and $f:\mathscr{U} \longrightarrow \mathscr{V}$ be a mapping satisfying the functional inequality*

$$\|\mathscr{F}_{QCQ}(x,y)\|_{\mathscr{V}} \leq \begin{cases} r; & \\ r(||x||^t + ||y||^t) & t \neq 2,3,4; \\ r(||x||^t||y||^t + ||x||^{2t} + ||y||^{2t}) & 2t \neq 2,3,4; \end{cases}$$

(4.55)

for all $x,y \in \mathscr{U}$. Then there exists a unique quadratic function $\mathscr{Q}_2(x) : \mathscr{U} \longrightarrow \mathscr{V}$, a unique quartic function $\mathscr{Q}_4(x) : \mathscr{U} \longrightarrow \mathscr{V}$ and a unique cubic function $\mathscr{C}(x) : \mathscr{U} \longrightarrow \mathscr{V}$ satisfying the functional equation (1.2) and

$$\|f(x) - \mathscr{Q}_2(x) - \mathscr{Q}_4(x) - \mathscr{C}(x)\|_{\mathscr{V}} \leq \begin{cases} \frac{K}{2^\beta}\left\{\frac{2K}{12^\beta}\left[\frac{r_{2c}}{|3|} + \frac{r_{4c}}{|15|}\right] + \frac{r_{3c}}{|7|}\right\}, \\ \frac{K}{2^\beta}\left\{\frac{2K}{12^\beta}\left[\frac{r_{2s}2^t||x||^t}{|4-2^{\beta t}|} + \frac{r_{4s}2^t||x||^t}{|16-2^{\beta t}|}\right] + \frac{r_{3s}2^t||x||^t}{|8-2^{\beta t}|}\right\}, \\ \frac{K}{2^\beta}\left\{\frac{2K}{12^\beta}\left[\frac{r_{2sp}2^{2t}||x||^{2t}}{|4-2^{2\beta t}|} + \frac{r_{4sp}2^{2t}||x||^{2t}}{|16-2^{2\beta t}|}\right] + \frac{r_{3sp}2^{2t}||x||^{2t}}{|8-2^{2\beta t}|}\right\} \end{cases} \tag{4.56}$$

where $r_{2c}, r_{3c}, r_{4c}, r_{2s}, r_{3s}, r_{4s}, r_{2sp}, r_{3sp}, r_{4sp}$ are respectively defined in (4.17), (4.30), (4.49) for all $x \in \mathscr{U}$.

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ISSN(P):2319 − 3786
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ISSN(O):2321 − 5666
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