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Numerical investigation of the weakly singular Volterra integro-differential equations using He′s Homotopy perturbation method

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Abstract
In this paper, we present a reliable algorithm for solving a system of Volterra integro-differential equations (VIDE) using He's Homotopy Perturbation Method (HHPM) [9, 10] and spectral methods [12]. This method converts a system of Volterra integro-differential equations to the system of linear algebraic equations. Some illustrative examples have been presented to illustrate the implementation of the algorithm and efficiency of the method.

Keywords
Integro-Differential Equations, Weakly singular Volterra integro-differential equations, Single-term Haar wavelet series, He's Homotopy Perturbation Method.

AMS Subject Classification
41A45, 41A46, 41A58.

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Contents

1. Introduction

Volterra integro-differential equations (VIDEs) arise widely in mathematical models of certain biological and physical phenomena. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. Brunner [3] introduced polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels. Brunner et al. [4] introduced piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels.

So far, very few works have touched the numerical approximations to weakly singular VIDEs. Approximate solutions are a class of techniques used in applied mathematics and scientific computing to numerically solve certain partial differential equations. In practice, Baratella et al. [1] obtained numerical solution of weakly singular linear Volterra intergo-differential equations. Bougoffa et al. [2] obtained an approximate method for solving a class of weakly-singular Volterra intergo-differential equations. Marek Kolk and Arvet Pedas [6] obtained numerical solution of weakly singular Volterra integro-differential equations with change of variables. Parts and Pedas [7] introduced spline collocation methods for weakly singular Volterra integro-differential equations.

Parts and Pedas [8] introduced collocation approximations for weakly singular Volterra integro-differential equations. Xueqin Lv and Sixing Shi [11] introduced the combined RKM and ADM for solving nonlinear weakly singular Volterra integro-differential equations. Yunxia Wei and Yanping Chen [12] derived the convergence analysis of the spectral methods for weakly singular Volterra integro-differential equations with smooth solutions. Recently Diogo et al. [5] introduced smoothing transformation and spline collocation for weakly singular Volterra integro-differential equations. Sekar and Thirumurugan [9, 10] obtained numerical investigation of integro-differential equations and Nonlinear Integro-Differential Equations using He's Homotopy Perturbation Method.

In this article we developed numerical methods for weakly singular Volterra integro-differential equations to get discrete solutions via He's Homotopy Perturbation method which was studied by Sekar et al. [9, 10]. The subject of this paper is to try to find numerical solutions of weakly singular Volterra integro-differential equations using He's Homotopy Perturbation method and compare the discrete results with the spectral methods which is presented previously by Yunxia Wei and Yanping Chen [12]. Finally, we show the method to achieve the desired accuracy. Details of the structure of the present method are explained in sections. We apply He's Homotopy Perturbation method and spectral methods for weakly singular Volterra integro-differential equations. In Section 4, it's proved the efficiency of the He's Homotopy Perturbation method. Finally, Section 5 contains some conclusions and directions for future expectations and researches.

2. He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [10]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u)-f(t)=0, u(0)=u_0, t\in\Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u)+N(u)-f(t)=0$$

By the homotopy technique, we construct a homotopy $U(t,p):\Omega\times[0,1]\rightarrow\Re$, which satisfies:

$$H(U,p)=(1-p)[LU(t)-Lu_0(t)]+p[AU(t)-f(t)]=0, \tag{2.2}$$

where $p\in[0,1], t\in\Omega$

or

$$H(U,p)=LU(t)-Lu_0(t)+pLu_0(t)+p[NU(t)-f(t)]=0, \tag{2.3}$$

where $p\in[0,1], t\in\Omega$. where $p\in[0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U,0)=LU(t)-Lu_0(t)=0, H(U,1)=AU(t)-f(t)=0$.

The changing process of p from zero to unity is just that of $U(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U=\sum_{n=0}^{\infty}p^nU_n=U_0+pU_1+p^2U_2+p^3U_3+\ldots \tag{2.4}$$

Setting $p=1$, results in the approximate solution of Eq.(2.1)

$$U(t)=\lim_{p\rightarrow 1}U=U_0+U_1+U_2+U_3+\ldots$$

Applying the inverse operator $L^{-1}=\int_0^t(.)dt$ to both sides of Eq. (2.3), we obtain

$$U(t)=U(0)+\int_0^t Lu_0(t)dt-p\int_0^t Lu_0(t)dt-p[\int_0^t(NU(t)-f(t))dt] \tag{2.5}$$

where $U(0)=u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t)=\sum_{n=0}^{\infty}\alpha_nP_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t),\ldots$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\left.\begin{array}{l} p^0: U_0(t)=u_0+\sum_{n=0}^{\infty}\alpha_n\int_0^tP_n(t)dt\\ p^1: U_1(t)=-\sum_{n=0}^{\infty}\alpha_n\int_0^tP_n(t)dt-\int_0^t(NU_0(t)-f(t))dt\\ p^2: U_2(t)=-\int_0^tNU_1(t)dt\\ \vdots\\ p^j: U_j(t)=-\int_0^tNU_{j-1}(t)dt \end{array}\right\} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t)=0$, then Eq. (2.7) results in $U_1(t)=U_2(t)=U_3(t)=\ldots=0$ and therefore the exact solution can be obtained by using

$$U(t)=U_0(t)=u_0+\sum_{n=0}^{\infty}\alpha_n\int_0^tP_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t=t_0$, then their Taylor series

$$U(t)=\sum_{n=0}^{\infty}a_n(t-t_0)^n$$

can be used in Eq. (2.8), where $a_0, a_1, a_2,\ldots$ are known coefficients and α_n are unknown ones, which must be computed.

3. General format for weakly singular Volterra integro-differential equations

The Volterra integro-differential equation that we shall study in details reads:

$$y'(t) = a(t)y(t) + b(t) + (v_\mu y)(t), t \in I := [0,T], y(0) = y_0, \tag{3.1}$$

where $v_\mu : C(I) \to C(I)$ is defined by

$$(v_\mu \phi)(t) := \int_0^t (t-s)^{-\mu} K(t,s)\phi(s)ds,$$

with $0 < \mu < 1$, the functions $a(t), b(t) \in C(I), y(t)$ is the unknown function and $K \in C(I \times I), K(t,t) \neq 0$ for $t \in I$. Equations of this type arise as model equations for describing turbulent diffusion problems. The numerical treatment of the Volterra integro-differential equation 3.1 is not simple, mainly due to the fact that the solutions of 3.1 usually have a weak singularity at $t = 0$. As discussed in [4], the second derivative of the solution $y(t)$ behaves like

$$y''(t) \sim t^{-\mu}.$$

We point out that for 3.1 without the singular kernel (i.e., $\mu = 0$).

4. Numerical Experiments

In this section, the following examples 4.1 and 4.2 has been solved numerically using the He's homotopy perturbation method and Spectral methods. The obtained results (with step size time = 0.1) along with exact solutions of the examples 4.1 to 4.2 and absolute errors between them are calculated and are presented in Table 1 to 3. A graphical representation is given for the weakly singular VIDEs in Figures 1 to 2, using three-dimensional effect to highlight the efficiency of the VIDE.

4.1 Example

Consider the weakly singular VIDE

$$\left.\begin{array}{l} u'(x) = xu(x) + (2-x)e^{2x} - \frac{4}{3}(x+1)^{\frac{3}{4}} \\ + \int_{-1}^{x} (x-\tau)^{-\frac{1}{4}} e^{-2\tau} u(\tau) d\tau, \\ x \in [-1,1], \\ u(-1) = e^{-2} \end{array}\right\}$$

The corresponding exact solution is given by $u(x) = e^{2x}$.

4.2 Example

Consider the weakly singular VIDE

$$\left.\begin{array}{l} w'(x) = 2xw(x) + cosx - 2xsinx - \frac{9}{4}(x+1)^{\frac{4}{3}} + 3(x+1)^{\frac{1}{3}} \\ + \int_{-1}^{x} (x-\tau)^{-\frac{2}{3}} \frac{\tau}{sin\tau} w(\tau) d\tau, \\ x \in [-1,1], \\ w(-1) = sin(-1) \end{array}\right\}$$

The corresponding exact solution is given by $w = sin(x)$.

Table 1. Numerical results for the Examples 4.1 and 4.2

	Exact Solution	
t	Example 4.1	Example 4.2
0.1	1.221402758	0.099833417
0.2	1.491824698	0.198669331
0.3	1.8221188	0.295520207
0.4	2.225540928	0.389418342
0.5	2.718281828	0.479425539
0.6	3.320116923	0.564642473
0.7	4.055199967	0.644217687
0.8	4.953032424	0.717356091
0.9	6.049647464	0.78332691
1	7.389056099	0.841470985

Table 2. Numerical results for the Examples 4.1 and 4.2

	Spectral Error	
t	Example 4.1	Example 4.2
0.1	1.63E-04	1.00E-07
0.2	2.77E-04	2.62E-07
0.3	3.43E-04	3.48E-07
0.4	4.63E-04	4.18E-07
0.5	5.48E-05	5.42E-07
0.6	6.11E-05	6.62E-07
0.7	7.11E-05	7.62E-07
0.8	8.11E-05	8.62E-07
0.9	9.11E-05	9.62E-07
1	9.11E-05	9.62E-07

Table 3. Numerical results for the Examples 4.1 and 4.2

	HHPM Error	
t	Example 4.1	Example 4.2
0.1	1.63E-06	1.00E-09
0.2	2.77E-06	2.62E-09
0.3	3.43E-06	3.48E-09
0.4	4.63E-06	4.18E-09
0.5	5.48E-07	5.42E-09
0.6	6.11E-07	6.62E-09
0.7	7.11E-07	7.62E-09
0.8	8.11E-07	8.62E-09
0.9	9.11E-07	9.62E-09
1	9.11E-07	9.62E-09

5. Conclusion

The applicability and effectiveness of the He's Homotopy Perturbation method [10] in determining discrete solutions for the weakly singular Volterra integro-differential equations has been studied by comparing with the discrete solutions obtained using Spectral methods. To demonstrate the effectiveness of the He's Homotopy Perturbation method, two exam-

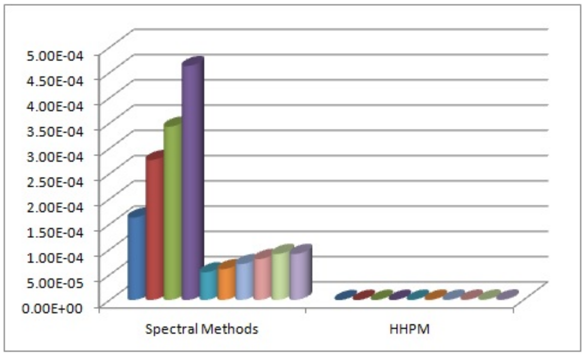

Figure 1. Error estimation of the Example 4.1

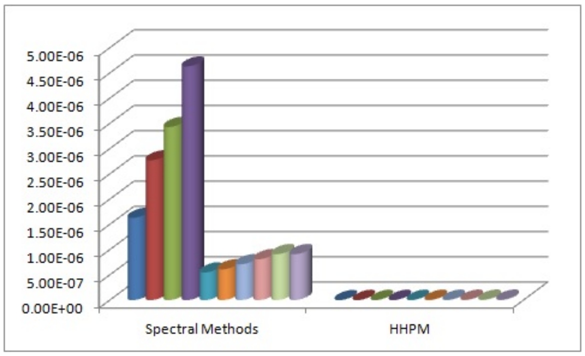

Figure 2. Error estimation of the Exaample 4.2

ples of weakly singular Volterra integro-differential equations have been considered. The ratios have been determined for the systems under discussion. From the Tables and Figures, it is evident that the absolute errors in He's Homotopy Perturbation method solutions of the weakly singular Volterra integro-differential equations given in Examples 4.1 - 4.2 are lesser than the absolute errors in the solutions of the Spectral methods [12]. This proves that the He's Homotopy Perturbation method has an edge over Spectral methods in terms of accuracy.

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