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Hamiltonian property of intersection graph of zero divisors of the ring Z_n

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Abstract
The intersection graph $G_Z^{'}(Z_n)$ of zero-divisors of the ring Z_n, the ring of integers modulo n is a simple undirected graph with the vertex set is $Z(Z_n)^* = Z(Z_n) \setminus \{0\}$, the set of all nonzero zero-divisors of the ring Z_n and for any two distinct vertices are adjacent if and only if their corresponding principal ideals have a nonzero intersection. We determine some results concerning the necessary and sufficient condition for the graph $G_Z^{'}(Z_n)$ is Hamiltonian. Also, we investigate for all values of for which the graph $G_Z^{'}(Z_n)$ is Hamiltonian and as an example we show that how the results give as easy proof of the existence of a Hamilton cycle.

Keywords
Finite commutative ring, Zero-divisors, Principal ideals, Intersection graph, Hamilton Cycle.

AMS Subject Classification
05C25, 05C38, 05C45, 06B10, 11A07.

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Contents

1. Introduction

In 1736, Leonard Euler [10] starts the journey of Graph theory with the famous problem Konigsberg bridge problem. A graph is said to be Hamiltonian, if it posses a Hamilton cycle. The Hamilton cycle problem is a NP-compete problem which place a central role in graph theory and which has various applications in computational theory, see [3, 11]. This Hamilton problem traces its origin to the 1850's, named for Sir William Rowan Hamilton. Generally, the Hamilton problem is considered to be determining the conditions under which a graph contains a spanning cycle. Many authors have studied the Hamilton cycles for several types of graphs and in which those are refer [6, 13, 14].

The intersection graph is a graph that represents the pattern of intersection of a family of sets. Let $F = \{A_j : j \in J\}$ be a family of nonempty sets, then the intersection graph $G(F)$ defined on F as, for two distinct vertices A_i and A_j are adjacent whenever $A_i \cap A_j \neq \phi$. Intersection graph was first introduced by Bosak in 1965 for semigroup see [7], defined as vertices are the sub semigroups of that semigroup and in which two distinct vertices are adjacent if they have non trivial intersection. Many researchers worked on these intersection graphs by considering the members of F have different algebraic structures and in which those see [8, 9, 17]. In [16], the intersection graph $G_Z^{'}(R)$ of zero-divisors of a finite commutative ring R is a simple undirected graph whose vertices are the nonzero zero-divisors of R and in which two distinct vertices x and y are adjacent if and only if their corresponding principal ideals having nonzero intersection. i.e., x is adjacent to y if and only if $(x) \cap (y) \neq \{0\}$, $\forall x, y \in V(G_Z^{'}(R))$. In this paper, we illustrate some results that shows the necessary and sufficient condition for the intersection graph $G_Z^{'}(Z_n)$ is Hamiltonian. Also, we investigate the problem of existence of Hamilton cycles in the intersection graph $G_Z^{'}(Z_n)$ for all characterizations of n.

2. Definitions and Notations

In this section, we consider the ring theoretic definitions and notations from [1, 4]. The set of all elements in the ring of integers modulo n, Z_n can be partitioned into the disjoint union of zero-divisors and regular elements of Z_n and which are denoted by $Z(Z_n)$ and $Reg(Z_n) = Z_n \setminus Z(Z_n)$ respectively. The set of all nonzero zero-divisors in Z_n is denoted by $Z(Z_n)^* = Z(Z_n) \setminus \{0\}$. For an element x in Z_n, the principal ideal generated by x is $(x) = \{xr : r \in Z_n\}$. For further definitions of ring theory, the reader may refer [12].

In [15], for every positive integer $n > 1$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$, $p_1 < p_2 < ... < p_m$ are primes, α_i is a positive integer for every $i = 1,2,...,m$ and $m \geqslant 1$. The subset D of Z_n be the set of all non trivial proper divisors of n. i.e., $D = \{d : d|n$ and $1 < d < n\}$. The divisor function $d(n)$ is the cardinality of the set of all divisors of n. i.e., $|D| = d(n) - 2$ and $d(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_m + 1)$. For any positive integer m is called the least common multiple of a and b, if m is a common multiple of a and b, and also $m|m_0$ for any common multiple m_0 of a and b. We write $m = lcm(a,b)$. For the integers a,b and $n > 0$, if n divides the difference of a and b, we denote that a is congruent to b modulo n and defined as $a \equiv b(mod\ n)$. Otherwise, we denote that a is incongruent to b modulo n and defined as $a \not\equiv b(mod\ n)$. For any positive integer n is called a square free integer, if a positive integer d with $d^2|n$ implies that $d = 1$. In particular, $n > 1$ is a square free integer if and only if $n = p_1 p_2 ... p_m, p_1 < p_2 < ... < p_m$ are primes. For further definitions of number theory, see [2]. We consider the graph theoretic definitions and notations from [5, 18]. For the graph G, the two distinct vertices x and y are adjacent, write $x - y$. The graph G is called complete if there exist an edge between every pair of two distinct vertices. A complete graph with n vertices is denoted by K_n. A vertex induced subgraph is a subgraph that can be obtained by deleting a set of vertices. i.e., for the graph G the subgraph induced with the vertex set T is denoted by $< T >$, $< T > = G - T$, where $T = V(G) - T$. A walk in a graph is an alternating sequence of vertices and edges, which begins and ends with a vertex. A trail is a walk in which all edges are distinct, and also a path is a trail in which all vertices are distinct. The graph G is said to be connected whenever there exist a path between every pair of two distinct vertices, otherwise disconnected. A cycle is a 2-regular connected subgraph of a graph. i.e., a closed path said to be a cycle. A collection of disjoint cycles that includes all the vertices of the graph G is said to be cycle factor of G. We denote cycle factor as the union of cycles, i.e., $C_1 \cup C_2 \cup ... \cup C_n$ where all cycle are disjoint and each vertex of G belongs to some cycle $C_i, \forall 1 \leqslant i \leqslant n$. If $t = 1$, then C_1 is called Hamilton cycle of G. i.e., the cycle which visits each vertex of the graph exactly once is called Hamilton cycle of the graph. Also a graph is said to be Hamiltonian if it has a Hamilton cycle. The following definition and results are taken from [16].

Definition 2.1. *The intersection graph $G_Z'(R)$ of a finite commutative ring R with unity is a simple undirected graph whose vertices are all the nonzero zero-divisors of R and in which two distinct vertices are joined by an edge if and only if their corresponding principal ideals having nonzero intersection. i.e., x is adjacent to y if and only if $(x) \bigcap (y) \neq 0$, $\forall x, y \in V(G_Z'(R))$.*

Theorem 2.2. *For the ring Z_n, order of the intersection graph $G_Z'(Z_n)$ is $n - \varphi(n) - 1$.*

Theorem 2.3. *Let x and y be any two distinct nonzero zero-divisors of a finite commutative ring Z_n. Then the least common multiple of x and y is congruent to zero modulo n if and only if x is not adjacent to y in $G_Z'(Z_n)$.*

Theorem 2.4. *If $n = p$, p is prime, then the graph intersection graph $G_Z'(Z_n)$ does not exist.*

Theorem 2.5. *The graph $G_Z'(Z_n)$ is complete if and only if $n = p^m$, p is prime and $m > 1$.*

Theorem 2.6. *If n can be written as a product of two distinct primes p_1 and p_2, then the graph $G_Z'(Z_n)$ is disconnected with two components, which are complete with $p_2 - 1$ and $p_1 - 1$ vertices respectively.*

Theorem 2.7. *The graph $G_Z'(Z_n)$ is connected, not complete if and only if either of the following conditions is hold*
(i) n can be written as a product of more than two primes.
(ii) n can be written as a product of at least two prime powers.

3. Hamilton cycle in the intersection graph of zero divisors of the ring Z_n

In this section, we show that the Intersection graph $G_Z'(Z_n)$ of zero-divisors of a finite commutative ring Z_n is Hamiltonian for those characterizations of n and not Hamiltonian for those characterizations of n, for all $n \in N$. Let p be a prime. Then, in view of Theorem 2.4, the graph $G_Z'(Z_p)$ does not exist.

Theorem 3.1. *The intersection graph $G_Z'(Z_{p^m})$ is Hamiltonian if and only if $m > 2$.*

Proof. **Necessity.** Suppose the graph $G_Z'(Z_{p^m})$ is Hamiltonian. But by the Theorem 2.5 the graph $G_Z'(Z_{p^m})$ is complete. So, there exist a Hamilton cycle $C = (p, 2p, 3p, ..., p^m - p)$ whose length is $p^{m-1} - 1 > 2$. This shows $m > 2$.

Sufficiency. Let $m > 2$. Then, by Theorem 2.5, the graph is complete with at least 3 vertices. Hence the graph $G_Z'(Z_{p^m})$ is Hamiltonian. □

Example 3.2. *For $m = 3$, the intersection graph $G_Z'(Z_{2^3}) = G_Z'(Z_8)$ having a Hamilton cycle $C = (2,4,6)$ and is shown in Fig. 1.*

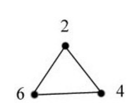

Figure 1. The graph $G_Z'(Z_8)$.

Remark 3.3. *The intersection graph $G_Z'(Z_{p^m})$ is Hamiltonian if and only if $m = 2$ for all p, except $p \in \{2,3\}$, since the graphs $G_Z'(Z_4)$ and $G_Z'(Z_9)$ are not Hamiltonian.*

Example 3.4. *For $m = 2$, the Hamilton cycle C in the intersection graph $G_Z'(Z_{5^2}) = G_Z'(Z_{25})$ is $C = (5, 10, 15, 20)$ and is shown in Fig. 2.*

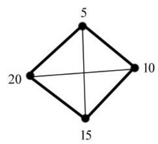

Figure 2. The graph $G'_Z(Z_{25})$.

Notation 3.5. *Let D be the set of all proper divisors of n. i.e., $D = \{d : d|n \text{ and } 1 < d < n\}$ and G_i be the set of all non unit divisors of $p_1^{\alpha_1} p_2^{\alpha_2} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_m^{\alpha_m}$, for all $i = 1,2,...,m$. i.e., $G_i = \{d : d$ is a non unit divisor of $p_1^{\alpha_1} p_2^{\alpha_2} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_m^{\alpha_m}\}, \forall 1 \leqslant i \leqslant m$. Also H_j, for all $j = 0,1,2,...,m$ are*

$$H_0 = \bigcap_{1 \leq i \leq m} G_i, H_j = G_j - \bigcup_{0 \leq k \leq j-1} H_k, \forall j = 1,2,...,m.$$

*This shows that the cardinality of each H_j, for all $j = 0,1,2,...,$
m are $|H_0| = \alpha_1 \alpha_2 ... \alpha_m - 1$, $|H_1| = \alpha_1(\alpha_2+1)(\alpha_3+1)...(\alpha_m+1) - \alpha_1\alpha_2...\alpha_m$, $|H_k| = \alpha_k(\alpha_{k+1}+1)(\alpha_{k+2}+1)...(\alpha_m+1)$, $\forall\, k = 2,3,...,m$.*

Theorem 3.6. *For all $i = 1,2,...,m$ and $j = 0,1,...,m$, then the intersection graphs induced with the vertex sets G_i and H_j are complete induced subgraphs of the graph $G'_Z(Z_n)$. In particular, D is the disjoint union of H_j, for all $j = 0,1,...,m$.*

Proof. From the notation of G_i, (d) contains an element $p_1^{\alpha_1} p_2^{\alpha_2} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_m^{\alpha_m} \neq 0$, for every $d \in G_i$, for all $i = 1,2,...,m$. By the Definition 2.1 there exist an edge between every pair of two distinct vertices in the intersection graph induced with the vertex set G_i, $\forall 1 \leq i \leq m$.

This shows that the intersection graph induced with the vertex set G_i and H_j is complete by Notations 3.5. To show that D is the disjoint union of H_j for all $0 \leq j \leq m$, first we prove that the set D can be written as the union of G_i for all $i = 1,2,...,m$.

We know that the set $D = \{d : d$ is a proper divisor of $p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}\}$ and $G_i = \{d : d$ is a non unit divisor of $p_1^{\alpha_1} p_2^{\alpha_2} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_m^{\alpha_m}\}$, $\forall 1 \leq i \leq m$. i.e.,
$G_1 = \{d : d$ is a non unit divisor of $p_1^{\alpha_1 - 1} p_2^{\alpha_2} ... p_m^{\alpha_m}\}$,
$G_2 = \{d : d$ is a non unit divisor of $p_1^{\alpha_1} p_2^{\alpha_2 - 1} p_m^{\alpha_m}\}$,...,
$G_m = \{d : d$ is a non unit divisor of $p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m - 1}\}$.
By the above construction of G_i's, we have

$$D = \bigcup_{1 \leq i \leq m} G_i.$$

From the notations of $H_0, H_1, H_2, ..., H_m$, we have

$$H_1 = G_1 - H_0, H_2 = G_2 - (H_0 \cup H_1), ..., H_j = G_j - \bigcup_{0 \leq k \leq j-1} H_k,$$

$$..., H_m = G_m - \bigcup_{0 \leq k \leq m-1} H_k.$$

Hence D can be written as the disjoint union of H_j, $\forall j = 0,1,...,m$. ☐

Example 3.7. *If $n = 12$, $12 = 2^2.3$. Then $D = \{2,3,4,6\}$, $G_1 = \{2,3,6\}, G_2 = \{2,4\}, H_0 = \{2\}, H_1 = \{3,6\}$ and $H_2 = \{4\}$. Therefore, D is the disjoint union of H_0, H_1 and H_2. The subgraphs induced with vertex sets G_1, G_2, H_0, H_1, H_2 and the graph $G'_Z(Z_{12})$ are shown in the following Figs. [3-4].*

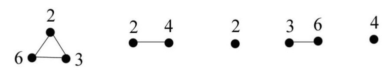

Figure 3. The subgraphs induced with vertex sets G_1, G_2, H_0, H_1 and H_2 respectively.

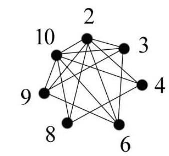

Figure 4. The graph $G'_Z(Z_{12})$.

Definition 3.8. *For every proper divisor d of n, D_d be the set of all elements in the ring Z_n such that whose principal ideal is equal to principal ideal of d. i.e., $D_d = \{x \in Z_n : (x) = (d)\}, \forall d \in D$, where D be the set of all proper divisors of n.*

Lemma 3.9. *The intersection graph induced with the vertex set D_d, for all d in D is a complete induced subgraph of the graph $G'_Z(Z_n)$.*

Proof. By the Definition 3.8, for $d \in D$ we have $(x) = (d)$ for every $x \in D_d$. Clearly, $d \neq 0$, since d is a proper divisor of n. Hence, there exists an edge between every pair of two distinct vertices in the intersection graph with the vertex set D_d,$\forall d \in D$ from the Definition 2.1. Hence the proof follows. ☐

Theorem 3.10. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$, $m > 2$. Then the intersection graph $G'_Z(Z_n)$ is Hamiltonian.*

Proof. The set D can be written as $D = H_0 \cup H_1 \cup H_2 \cup ... \cup H_m$, where $H_i \cap H_j = \emptyset$ for distinct i and j varying from 0 to m from Theorem 3.6. We shall prove that the graph $G'_Z(Z_n)$ is Hamiltonian. For this we construct a path P, which contains vertices are the set of all elements in D as follows.

In H_0 there is an edge between $x_{01} = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} ... p_m^{\alpha_m - 1}$ and $x_{02} = p_1^{\alpha_1 - 2} p_2^{\alpha_2 - 1} ... p_m^{\alpha_m - 1}$; x_{02} and $x_{03} = p_1^{\alpha_1 - 3} p_2^{\alpha_2 - 1} ... p_m^{\alpha_m - 1}$;...; $x_{0(|H_0|-1)} = p_2$ and $x_{0|H_0|} = p_1$ again using Theorem 3.6.

Let $x_{0|H_0|}$ in H_0 and $x_{11} = p_1^{\alpha_1 - 1} p_2^{\alpha_2} p_3^{\alpha_3 - 1} ... p_m^{\alpha_m - 1}$ in H_1. Since $lcm(x_{0|H_0|}, x_{11}) \not\equiv 0 (mod\ n)$, then there exist an edge between $x_{0|H_0|}$ and x_{11}. Similarly in H_0 there exist an edge between x_{11} and $x_{12} = p_1^{\alpha_1 - 2} p_2^{\alpha_2} p_3^{\alpha_3 - 1} ... p_m^{\alpha_m - 1}$; x_{12} and $x_{13} = p_1^{\alpha_1 - 3} p_2^{\alpha_2} p_3^{\alpha_3 - 1} ... p_m^{\alpha_m - 1}$;...; $x_{1(|H_1|-1)} = p_1 p_m^{\alpha_m}$ and $x_{1|H_1|} = p_m^{\alpha_m}$ in H_1.

We can find an edge between $x_{1|H_1|}$ in H_1 and $x_{21} = p_1^{\alpha_1} p_3^{\alpha_3} ... p_m^{\alpha_m}$ in H_2. Because $lcm(x_{1|H_1|}, x_{21}) \not\equiv 0 (mod\ n)$.

In H_2, there is an edge between x_{21} and $x_{22} = p_1^{\alpha_1} p_2 p_3^{\alpha_3} ... p_m^{\alpha_m}$; x_{22} and $x_{23} = p_1^{\alpha_1} p_2^2 p_3^{\alpha_3} ... p_m^{\alpha_m}$;...; $x_{2(|H_2|-1)} = p_1^{\alpha_1} p_2$ and $x_{2|H_2|} = p_1^{\alpha_1}$.

We have an edge between $x_{2|H_2|}$ in H_2 and $x_{31} = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3 - 1} p_4^{\alpha_4} ... p_m^{\alpha_m}$ in H_3 whereas $lcm(x_{2|H_2|}, x_{31}) \not\equiv 0(mod\ n)$.

Continuing in this way, we get an edge between $x_{(m-1)|H_{m-1}|} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-2}^{\alpha_{m-2}}$ in H_{m-1} and $x_{m1} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-1}^{\alpha_{m-1}} p_m^{\alpha_{m-1}}$ in H_m by considering $lcm(x_{m-1|H_{m-1}|}, x_{m1}) \not\equiv 0(mod\ n)$. In the same way in H_2 we can find an edge between x_{m1} and $x_{m2} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-1}^{\alpha_{m-1}} p_m^{\alpha_{m-2}}$; x_{m2} and $x_{m3} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-1}^{\alpha_{m-1}} p_m^{\alpha_m - 3}$;...; $x_{m(|H_m|-1)} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-1}^{\alpha_{m-1}} p_m$ and $x_{m|H_m|} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{m-1}^{\alpha_{m-1}}$ in H_m.

Thus we get a path P that joining all the elements D with the initial vertex x_{01} and the terminal vertex $x_{m|H_m|}$ as follows and also shown in Fig. 5.

$P : x_{01}, x_{02}, ..., x_{0|H_0|}, x_{11}, x_{12}, ..., x_{1|H_1|}, x_{21}, x_{22}, ..., x_{2|H_2|}, ..., x_{m-1|H_{m-1}|}, x_{m1}, x_{m2}, x_{m|H_m|}$.

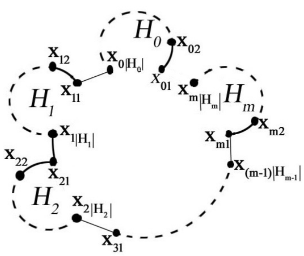

Figure 5. The path P.

The vertex set of the graph $G'_Z(Z_n)$ is $V(G'_Z(Z_n)) = \{x \in Z_n : (x) = (d), \forall d \in D\}$.

Let P_d be the spanning path of the intersection graph induced with the vertex set $D_d, \forall d \in D$ by considering Lemma 3.9. Now, we replace P_d in place of d in the above path P. Thus, we get a spanning path of the graph $G'_Z(Z_n)$. Finally we join the initial vertex x_{01} of the spanning path $P_{x_{01}}$ corresponding to the element x_{01} in H_0 and the terminal vertex $n - x_{m|H_m|}$ of the spanning path $P_{x_{m|H_m|}}$ corresponding to the element $x_{m|H_m|}$ in H_m, because $lcm(x_{01}, n - x_{m|H_m|}) \not\equiv 0(mod/n)$ and hence the graph $G'_Z(Z_n)$ is Hamiltonian. □

Remark 3.11. *The set D can be written as the disjoint union of $H_1, H_2, ..., H_m$ from Theorem 3.6 when n is a square free integer. Since H_0 is empty from the notation of H_0.*

Theorem 3.12. *Let n be s a square free integer except $n \neq pq$. Then the intersection graph $G'_Z(Z_n)$ is Hamiltonian.*

Proof. Consider $n = p_1 p_2 ... p_m$ with $m > 2$. Then the set D can be written as $D = H_1 \cup H_2 \cup ... \cup H_m$, $H_i \cap H_j = \phi$ for distinct i and j varying from 1 to m. Also $|H_1| = 2^{m-1} - 1$, $|H_2| = 2^{m-2}, |H_3| = 2^{m-3}, ..., |H_{m-1}| = 2$ and $|H_m| = 1$.

Now, we construct a path P with the vertex set D follows.

In H_1 there is an edge between $x_{11} = p_2$ and $x_{12} = p_2 p_3$; x_{12} and $x_{13} = p_2 p_4$;...; $x_{1(2^{m-1}-2)} = p_{m-1} p_m$ and $x_{1(2^{m-1}-1)} = p_m$. There exist an edge between $x_{1(2^{m-1}-1)}$ in H_1 and $x_{21} = p_1 p_3 p_4 ... p_m$ in H_2, since $lcm(x_{1(2^{m-1}-1)}, x_{21}) \not\equiv 0(mod\ n)$.

Again, there exist an edge between x_{21} and $x_{22} = p_1 p_4 p_5 ... p_m$; x_{22} and $x_{23} = p_1 p_3 p_5 p_6 ... p_m$;...; $x_{2(2^{m-2}-1)} = p_1 p_3$ and $x_{2(2^{m-2})} = p_1$ in H_2. We have an edge between $x_{2(2^{m-2})}$ in H_2 and $x_{31} = p_1 p_2 p_4 ... p_m$ in H_3 whereas $lcm(x_{2(2^{m-2})}, x_{31}) \not\equiv 0(mod\ n)$.

Continuing in this way, we get an edge between $x_{(m-1)2} = p_1 p_2 p_3 ... p_{m-2}$ in H_{m-1} and $x_{m1} = p_1 p_2 p_3 ... p_{m-1}$ in H_m, because $lcm(x_{(m-1)2}, x_{m1}) \not\equiv 0(mod\ n)$.

Therefore, we get a path P whose vertices are the set of all elements D with the initial vertex x_{11} and the terminal vertex x_{m1} as follows and also shown in Fig. 6.

$P : x_{11}, x_{12}, ..., x_{1(2^{m-1}-1)}, x_{21}, x_{22}, ..., x_{2(2^{m-2})}, ..., x_{(m-1)2}, x_{m1}$.

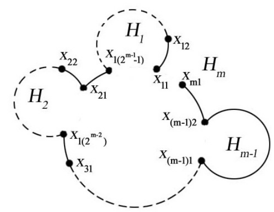

Figure 6. The path P.

Similarly in the Theorem 3.10, we replace the spanning path P_d of the intersection graph induced with the vertex set $D_d, \forall d \in D$ in place of d in the above path P.

Thus we obtain a spanning path of the graph $G'_Z(Z_n)$, and finally we join the initial vertex x_{11} of the spanning path $P_{x_{11}}$ corresponding to the element x_{11} in H_1 and the terminal vertex $n - x_{m1}$ of the spanning path $P_{x_{m1}}$ corresponding to the element x_{m1} in H_m, since $lcm(x_{11}, n - x_{m1}) \not\equiv 0(mod\ n)$. This completes the proof.

The following Example 3.13 illustrates an immediate consequence of Theorem 3.12. □

Example 3.13. *Consider the intersection graph $G'_Z(Z_{30})$, where $30 = 2.3.5$. Then D can be written as the disjoint union of H_1, H_2 and H_3, where $D = \{2, 3, 5, 6, 10, 15\}$, $H_1 = \{3, 5, 15\}, H_2 = \{2, 10\}$ and $H_3 = \{6\}$. We now construct a path P with vertices are all the elements in D as follows.*

In H_1, there is an edge between 3 and 15; 15 and 5. Let 5 in H_1 and 10 in H_2. Then there exist an edge between 5 and 10, since $lcm(5, 10) \not\equiv 0(mod\ 30)$. As same as in H_1, we can find an edge between 10 and 2 in H_2. Again, let 2 in H_2 and 6 in H_3. Then, we have an edge between 2 and 6, because $lcm(2, 6) \not\equiv 0(mod\ 30)$.

Thus we get a path P whose vertices are all the elements in D with the initial vertex 3 and the terminal vertex 6 as P : 3, 15, 5, 10, 2, 6 and is shown in the Fig. 7.

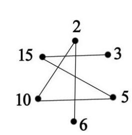

Figure 7. The path P.

The spanning path P_d, for all $d \in D$ are $P_2 : 2,4,8,14,16,22,26,28$, $P_3 : 3,9,21,27$, $P_5 : 5,25$, $P_6 : 6,12,18,24$, $P_{10} : 10,20$, $P_{15} : 15$.

Now, we replace P_d in place of d in the above path P, we get a spanning path of the graph $G_Z^{'}(Z_{30})$ with the initial vertex $x_{11} = 3$ and the terminal vertex $30 - x_{31} = 24$ as follows and also shown in Fig. 8.

Spanning path of the graph $G_Z^{'}(Z_{30})$,
$P : 3,9,21,27,15,5,25,10,20,2,4,8,14,16,22,26,28,6,12,18,24$.

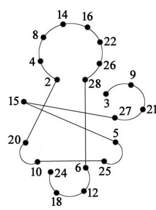

Figure 8. The spanning path of $G_Z^{'}(Z_{30})$.

Finally, we join the initial vertex 3 and the terminal vertex 24, since $lcm(3,24) \not\equiv 0(mod\ 30)$. Thus we get a Hamilton cycle in the graph $G_Z^{'}(Z_{30})$. Also the intersection graph $G_Z^{'}(Z_{30})$ including its Hamilton cycle with thick lines is shown in Fig. 9.

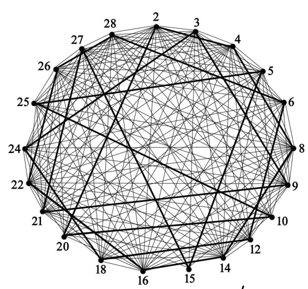

Figure 9. The graph $G_Z^{'}(Z_{30})$.

Remark 3.14. *Theorem 3.10 is not sufficient for $m = 2$, it is true for $m > 2$ only. Since every vertex in H_1 is not adjacent to every vertex in H_2, when $m = 2$.*

Now, we shall study the Hamilton property of graph $G_Z^{'}(Z_n)$ for $m = 2$ and $n \neq pq$. Since the graph $G_Z^{'}(Z_{pq})$ is never Hamilton. Now, we prove the Hamiltonian property for $n = p^2q$ and $n \neq p^2q$, $p < q$ are primes separately, since when $n = p^2q$ and $n \neq p^2q$ then H consists of only one element and more than one element respectively.

Theorem 3.15. *If $n = p_1^{\alpha_1}p_2^{\alpha_2}$ and $n \neq p_1p_2$, p^2q. Then the intersection graph $G_Z^{'}(Z_n)$ Hamiltonian.*

Proof. The set D can be written as the disjoint union of H_0, H_1 and H_2 such that $H_0 = \{x_{01}, x_{02}, ..., x_{0k}\}$ (assume), where $k = \alpha_1\alpha_2 - 2 > 1$, $H_1 = \{p_2^{\alpha_2}, p_1p_2^{\alpha_2}, p_1^2p_2^{\alpha_2}, ..., p_1^{\alpha_1-1}p_2^{\alpha_2}\}$ and $H_2 = \{p_1^{\alpha_1}, p_1^{\alpha_1}p_2, p_1^{\alpha_1}p_2^2, ..., p_1^{\alpha_1}p_2^{\alpha_2-1}\}$.

We now construct a path P with vertices are all the elements in D as follows.

Let x_{01} in H_0 and $x_{11} = p_2^{\alpha_2}$ in H_1. Then there exist an edge between x_{01} and x_{11}, since $lcm(x_{01}, x_{11}) \not\equiv 0(mod\ n)$.

In H_1 there is an edge between x_{11} and $x_{12} = p_1p_2^{\alpha_2}$; x_{12} and $x_13 = p_1^2p_2^{\alpha_2}$;...; $x_{1(\alpha_1-1)} = p_1^{\alpha_1-2}p_2^{\alpha_2}$ and $x_{1\alpha_1} = p_1^{\alpha_1-1}p_2^{\alpha_2}$.

Let $x_{1\alpha_1}$ in H_1 and x_{02} in H_0, then $lcm(x_{1\alpha_1}, x_{02}) \not\equiv 0(mod\ n)$. So, there exist an edge between $x_{1\alpha_1}$ and x_{02}.

Let x_{02} in H_0 and $x_{21} = p_1^{\alpha_1}$ in H_2. There is an edge between x_{02} and x_{21}, whereas $lcm(x_{02}, x_{21}) \not\equiv 0(mod\ n)$.

Again, in H_2, there exist an edge between x_{21} and $x_{22} = p_1^{\alpha_1}p_2$; x_{22} and $x_{23} = p_1^{\alpha_1}p_2^2$;...; $x_{2(\alpha_2-1)} = p_1^{\alpha_1}p_2^{\alpha_2-2}$ and $x_{2\alpha_2} = p_1^{\alpha_1}p_2^{\alpha_2-1}$.

If H_0 consists of more than two elements, then consider $x_{2\alpha_2}$ in H_2 and x_{03} in H_0. So that there exists an edge between $x_{2\alpha_2}$ and x_{03}, because $lcm(x_{2\alpha_2}, x_{03}) \not\equiv 0(mod\ n)$. Also there exist an edge between x_{03} and x_{04}; x_{04} and x_{05};...; $x_{0(k-1)}$ and x_{0k}.

Thus, we get a path P with the vertex set D having the initial vertex x_{01} and the terminal vertex $x_{2\alpha_2}$, if H_0 consists of only two elements or x_{0k}, if H_0 consists of more than two elements as follows and also shown in Figs. 10 and 11 respectively.

$P : x_{01}, x_{11}, x_{12}, ..., x_{1\alpha_1}, x_{02}, x_{21}, x_{22}, ..., x_{2\alpha_2}$, if H_0 consists of only two elements or

$P : x_{01}, x_{11}, x_{12}, ..., x_{1\alpha_1}, x_{02}, x_{21}, x_{22}, ..., x_{2\alpha_2}, x_{03}, x_{04}, ..., x_{0(k-1)}, x_{0k}$, if H_0 consists of more than two elements.

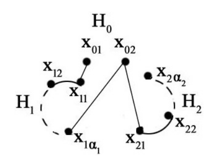

Figure 10. The path P if H_0 consists of only two elements.

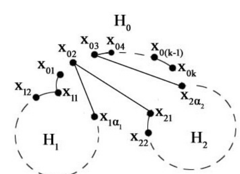

Figure 11. The path P if H_0 consists of more than two elements.

Similarly in the Theorem 3.10, we replace the spanning path P_d of the intersection graph induced with the vertex set D_d, $\forall d \in D$ in place of d in the above path P. Therefore we get a spanning path of the intersection graph $G_Z^{'}(Z_n)$.

Finally we draw an edge between the initial vertex x_{01} of the spanning path $P_{x_{01}}$ corresponding to the element x_{01} in H_0 and the terminal vertex $n - x_{2\alpha_2}$ or $n - x_{0k}$ of the spanning path $P_{x_{2\alpha_2}}$ or $P_{x_{0k}}$ corresponding to the element $x_{2\alpha_2}$ or x_{0k}, if H_0 consists of only two vertices or more than two vertices respectively. Because $lcm(x_{01}, n - x_{2\alpha_2}) \not\equiv 0(mod\ n)$ or $lcm(x_{01}, n - x_{0k}) \not\equiv 0(mod\ n)$ and hence the proof follows. ☐

Example 3.16. *Consider the intersection graph $G_Z^{'}(Z_{24})$, where $24 = 2^3 3$. The set D can be written as the disjoint union of H_0, H_1 and H_2, where $D = \{2,3,4,6,8,12\}$, $H_0 = \{2,4\}$, $H_1 = \{3,6,12\}$ and $H_2 = \{8\}$. Similarly in Example 3.13, we construct a path P with the vertex set D as follows.*

Let $x_{01} = 2$ in H_0 and $x_{11} = 3$ in H_1. Then there exist an edge between 2 and 3, since $lcm(2,3) \not\equiv 0(mod\ 24)$. In H_1, there is an edge between 3 and 6; 6 and 12 from Corollary 3.10. Let 12 in H_1 and 4 in H_0. So there is an edge between 12 and 4, since $lcm(12,4) \not\equiv 0(mod\ 24)$. Also we can find an edge between 4 in H_0 and 8 in H_2, because $lcm(4,8) \not\equiv 0(mod\ 24)$.

Thus we get path P with vertices are set of elements in D and whose initial vertex 2 and the terminal vertex 8 as $P: 2,3,6,12,4,8$, also shown in the Fig. 12. The spanning

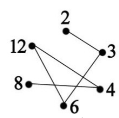

Figure 12. The path P.

path P_d, for all $d \in D$ are $P_2: 2,10,14,22$, $P_3 = 3,9,15,21$, $P_4 = 4,20$, $P_6 = 6,18$, $P_8 = 8,16$, $P_{12} = 12$. We replace P_d in place of d in the path P, we find a spanning path of the graph $G_Z^{'}(Z_{24})$ with the initial vertex 2 and the terminal vertex $24-8 = 16$ as follows and also shown in Fig. 13.

Spanning path of the graph $G_Z^{'}(Z_{24}): 2,10,14,22,3,9,15,21,6,18,12,4,20,8,16$.

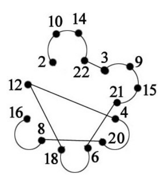

Figure 13. The spanning path of $G_Z^{'}(Z_{24})$.

Finally, we join the initial vertex 2 and the terminal vertex 16 in the spanning path, since $lcm(2,16) \not\equiv 0(mod\ 24)$. Also the intersection graph $G_Z^{'}(Z_{24})$ including its Hamilton cycle with thick lines is shown in Fig. 14.

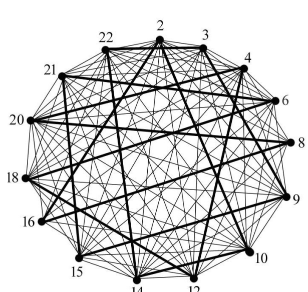

Figure 14. The graph $G_Z^{'}(Z_{24})$.

Remark 3.17. *The proof of Theorem 3.15 is not sufficient for $n = p^2 q$. Because in this case it is not possible to draw a closed path with the vertex set D, since H_0 consists of only one element p.*

Theorem 3.18. *If $n = p^2 q$, $p < q$ are primes. Then the intersection graph $G_Z^{'}(Z_n)$ is Hamiltonian.*

Proof. We have the set D can be written as the disjoint union of H_0, H_1 and H_2 such that $H_0 = \{p\}$, $H_1 = \{q, pq\}$ and $H_2 = \{p^2\}$. We now construct a trail P with the vertex D as follows.

Consider p in H_0 and q in H_1. Then there exist an edge between p and q, because $lcm(p,q) \not\equiv 0(mod\ n)$. There exist an edge between q and qp in H_1. Let qp in H_1 and p in H_0. So there exist an edge between qp and p whereas $lcm(qp,p) \not\equiv 0(mod\ n)$. Let p in H_0 and p^2 in H_2, then $lcm(p,p^2) \not\equiv 0(mod\ n)$ and thus there exist an edge between p and p^2.

Thus, we get a trail P whose vertex set D with the initial vertex p and the terminal vertex p^2 as $P: p,q,pq,p,p^2$ and also shown in Fig. 15. We construct the spanning path of the

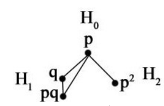

Figure 15. The trail P.

graph $G_Z^{'}(Z_n)$ by replacing q, qp and p^2 by P_q, P_{qp} and P_{p^2}, also repetition of p by $P_p - \{p\}$ in the above trail P, here P_d is the spanning path of the intersection graph induced with the vertex set D_d, $\forall d \in D$. Finally, we join the initial vertex p and the terminal vertex $n - p^2$ of the spanning path P_{p^2} corresponding to the element p^2 in H_2, because $lcm(p, n - p^2) \not\equiv 0(mod\ n)$. Thus we a Hamilton cycle of the graph $G_Z^{'}(Z_n)$. □

Example 3.19. *Consider the graph $G_Z^{'}(Z_{12})$, where $12 = 2^2 3$. Then D can be written as the disjoint union of H_0, H_1 and H_2 such that $D = \{2,3,4,6\}$, $H_0 = \{2\}$, $H_1 = \{3,6\}$ and $H_2 = \{4\}$. We construct a trail P with the vertex set D as follows.*

Let 2 in H_0 and 3 in H_1. Then there exist an edge between 2 and 3, since $lcm(2,3) \not\equiv 0(mod\ 12)$. In H_1, there exist an edge between 3 and 6. Let 6 in H_1 and 2 in H_0. So there is an edge between 6 and 2, because $lcm(6,2) \not\equiv 0(mod\ 12)$. Also we can find an edge between 2 in H_0 and 4 in H_2, whereas $lcm(2,4) \not\equiv 0(mod\ 12)$.

Now, we get the trail P with vertices are all the elements in D whose initial vertex 2 and terminal vertex 4 as $P: 2,3,6,2,4$ and also shown in the Fig. 16. The spanning path P_d, for all

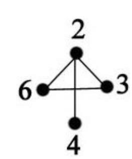

Figure 16. The trail P.

$d \in D$ are $P_2 : 2, 10$, $P_3 : 3, 9$, $P_4 : 4, 8$, $P_6 : 6$. Now we replace P_3, P_4, P_6 in place of 3, 4, 6 and also $P_2 - \{2\}$ in place of repetition of 2 in the above trail P. Thus, we find a spanning path of the graph $G_Z^{'}(Z_{12})$ with the initial vertex 2 and the terminal vertex $12 - 4 = 8$ as follows. Also shown in Fig. 17.

Spanning path of the graph $G_Z^{'}(Z_{12}) : 2, 3, 9, 6, 10, 4, 8$.

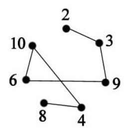

Figure 17. The spanning path of $G_Z^{'}(Z_{12})$.

Finally, we join the initial vertex 2 and the terminal vertex 8 in the spanning path, since $lcm(2, 8) \not\equiv 0(mod\ 12)$. The intersection graph $G_Z^{'}(Z_{12})$ including its Hamilton cycle with thick lines is shown in Fig. 18.

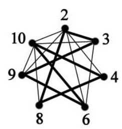

Figure 18. The graph $G_Z^{'}(Z_{12})$.

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