Malaya Journal of Matematik, Vol. 6, No. 1, 140-144, 2018
https://doi.org/10.26637/MJM0601/0018

# Curvature and torsion of a legendre curve in $(\varepsilon,\delta)$ Trans-Sasakian manifolds

Bilal Eftal ACET[1]* and Selcen YÜKSEL PERKTAŞ[2]

**Abstract**
In present paper, we obtain curvature and torsion of Legendre curves in 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifolds. Also important theorems concerning about biharmonic Legendre curves of $(\varepsilon,\delta)$ trans-Sasakian manifolds have been given.

**Keywords**
Legendre curves, Biharmonic map.

**AMS Subject Classification**
53C25, 53C43.

[1,2] *Department of Mathematics, Adıyaman University, 02040, Adıyaman, TURKEY.*
***Corresponding author**: 1 eacet@adiyaman.edu.tr; 1 2sperktas@adiyaman.edu.tr
**Article History**: Received **23** June **2017**; Accepted **12** December **2017**


## Contents



## 1. Introduction

The concept of $(\varepsilon)-$Sasakian manifolds were introduced by A. Bejancu and K. L. Duggal [1] and X. Xufeng and C. Xiaoli [23] proved that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. After, curvature conditions of these manifolds were obtained in [18] and $(\varepsilon)-$almost paracontact manifolds were defined in [17]. Also U. C. De and A. Sarkar [22] established $(\varepsilon)-$Kenmotsu manifolds and studied curvature conditions on such manifolds. H. G. Nagaraja et. al. [12] have studied $(\varepsilon,\delta)$ trans-Sasakian structures which generalize both $(\varepsilon)-$Sasakian and $(\varepsilon)-$Kenmotsu manifolds.

Harmonic maps $\Psi:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(\Psi):\frac{1}{2}\int_M \mid d\Psi\mid^2 \vartheta_g,$$

and therefore the solutions of the corresponding Euler-Lagrange equation. Harmonic equation is given by the vanishing of the tension field given by

$$\tau(\Psi)=trace\nabla d\Psi.$$

As suggested by J. Eells and J. H. Sampson [14], one can define the bienergy of a map $\Psi:(M,g)\to(N,h)$ by

$$E_2(\Psi)=\frac{1}{2}\int_M \mid\tau(\Psi)\mid^2 \vartheta_g,$$

and say that $\Psi$ is biharmonic if it is critical point of the bienergy.

In [10], G. Y. Jiang derived the first and second variation formula for the bienergy, showing that the Euler-Lagrange equation associated to $E_2$ is

$$\begin{aligned}\tau_2(\Psi) &= -\triangle\tau(\Psi)-traceR^N(d\Psi,\tau(\Psi))d\Psi\\ &= 0.\end{aligned}$$

The equation $\tau_2(\Psi)=0$ is called biharmonic equation. Since any harmonic maps is biharmonic, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps. Biharmonic maps have been studied intensively in the last decade (see [7], [8], [9], [24], [25], [26], [2], [19], [20], [21]).

In the study of almost contact manifolds, Legendre curves play an important role, e. g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curve to Legendre curve. In [4], the notion of Legendre curves in almost contact metric manifolds was introduced.

J. Inoguchi [13] classified the proper biharmonic Legendre curves and Hopf cylinders in 3-dimensional Sasakian space form and in [6] the explicit formulas were obtained. C. Özgür and M. M. Tripathi [3] proved that a Legendre curve in an $\alpha-$Sasakian manifold is biharmonic if and only if its curvature is zero.

Moreover J. Welyczko have studied Legendre curve on quasi Sasakian manifolds [15] and almost paracontact metric manifolds [16]. Motivated by these works, in this paper we study Legendre curves on 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifolds.

Our paper is structured as follows. The first section is a very brief review of $(\varepsilon,\delta)$ trans-Sasakian manifolds and Frenet curves in Riemannian manifolds. The next section is devoted to the examine of curvature and torsion of Legendre curves in 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifolds. Finally we give necessary and sufficient conditions for Legendre curve of 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifolds being biharmonic.

## 2. Preliminaries

### 2.1 $(\varepsilon,\delta)$ trans-Sasakian manifolds

Let $(M,g)$ be a $n-$dimensional almost contact metric manifold equipped with an almost contact metric structure $(\phi,\xi,\eta,g)$ consisting of a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a $1-$form $\eta$ and a Riemannian metric $g$ satisfying the following conditions

$$\phi^2 = I - \eta\otimes\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi\xi = 0, \quad \eta\circ\phi = 0. \tag{2.3}$$

An almost contact metric manifold $M$ is called an $(\varepsilon)-$almost contact metric manifold if

$$g(\xi,\xi) = \varepsilon, \tag{2.4}$$

$$\eta(X) = \varepsilon g(X,\xi), \tag{2.5}$$

$$g(\phi X,\phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{2.6}$$

where $\varepsilon = g(\xi,\xi) = \pm 1$.

An $(\varepsilon)-$almost contact metric manifold is called an $(\varepsilon,\delta)$ trans-Sasakian if

$$\begin{aligned}(\nabla_X\phi)Y &= \alpha(g(X,Y)\xi - \varepsilon\eta(Y)X) \\ &\quad +\beta(g(\phi X,Y)\xi - \delta\eta(Y)\phi X),\end{aligned} \tag{2.7}$$

holds for some smooth functions $\alpha$ and $\beta$ on $M$ and $\varepsilon = \pm 1 = \delta$. For $\beta = 0,\ \alpha = 1$ $(resp., \alpha = 0,\ \beta = 1)$ an $(\varepsilon,\delta)$ trans-Sasakian reduces to an $(\varepsilon)-$Sasakian (resp., a $(\delta)-$Kenmotsu) manifold.

From (2.7), it can be easily seen that

$$\nabla_X\xi = -\alpha\varepsilon\phi X - \beta\delta\phi^2 X. \tag{2.8}$$

For a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold, the curvature tensor is given by [11]

$$\begin{aligned}R(X,Y)Z &= \left(\frac{r}{2} - 2(\alpha^2-\beta^2)\right)(g(Y,Z)X - g(X,Z)Y) \\ &\quad -\left(\varepsilon\frac{r}{2} - 3(\alpha^2-\beta^2)\right)\begin{pmatrix} g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi\end{pmatrix} \\ &\quad -\left(\varepsilon\frac{r}{2} - 3(\alpha^2-\beta^2)\right)\begin{pmatrix}\eta(Y)\eta(Z)X \\ -\eta(X)\eta(Z)Y\end{pmatrix},\end{aligned} \tag{2.9}$$

where $r$ is the scalar curvature.

### 2.2 Frenet Curve

Let $(M,g)$ be a $3-$dimensional semi-Riemannian manifold. Let $\gamma : I \to M$ be a curve in $M$ such that $g(\dot\gamma,\dot\gamma) = \varepsilon_1 = \pm 1$, and let $\nabla_{\dot\gamma}$ denotes the covariant differentiation. We say that $\gamma$ is a Frenet curve if one of the following three cases hold

$a)$ $\gamma$ is osculating order 1, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$,

$b)$ $\gamma$ is osculating order 2, i.e., there exist two ortonormal vector fields $\{E_1 = \dot\gamma, E_2\}$, $(g(E_2,E_2) = \varepsilon_2 = \pm 1$ and a positive function $\kappa_1$ along $\gamma$ such that

$$\begin{aligned}\nabla_{\dot\gamma}E_1 &= \kappa_1\varepsilon_2E_2, \\ \nabla_{\dot\gamma}E_2 &= -\kappa_1\varepsilon_1E_1,\end{aligned}$$

$c)$ $\gamma$ is osculating order 3, i.e., there exist three ortonormal vector fields $\{E_1 = \dot\gamma, E_2, E_3,\}$, $g(E_2,E_2) = \varepsilon_2 = \pm 1$, $(g(E_3,E_3) = \varepsilon_3 = \pm 1$ and two positive functions $\kappa_1$ and $\kappa_2$ along $\gamma$ such that

$$\begin{aligned}\nabla_{\dot\gamma}E_1 &= \kappa_1\varepsilon_2E_2, \\ \nabla_{\dot\gamma}E_2 &= -\kappa_1\varepsilon_1E_1 + \kappa_2\varepsilon_3E_3, \\ \nabla_{\dot\gamma}E_3 &= -\kappa_2\varepsilon_2E_2.\end{aligned} \tag{2.10}$$

## 3. Curvature and Torsion of Legendre Curves

Let $\gamma : I \to M$ be a curve parametrized by arclength in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold $M$ with Frenet frame $(E_1 = \dot\gamma, E_2, E_3)$.

**Definition 3.1.** *A Frenet curve $\gamma$ in 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold $M$ is called to be Legendre if it is an integral curve of the contact distribution $D = \ker\eta$, equivalently*

$$\eta(\dot\gamma) = 0. \tag{3.1}$$

Assume that $\gamma$ is Legendre curve on a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold $M$. Then $\eta(\dot\gamma)=0$. Hence $\dot\gamma,\phi\dot\gamma$ and $\xi$ are orthonormal vector fields along $\gamma$.

It is well-known that the Levi-Civita connection $\nabla$ is a metric connection. So differentiating (3.1) and using (2.8), we have

$$\begin{aligned} 0 &= g(\nabla_{\dot\gamma}\dot\gamma,\xi)+g(\dot\gamma,\nabla_{\dot\gamma}\xi) \\ &= g(\nabla_{\dot\gamma}\dot\gamma,\xi)+g(\dot\gamma,-\alpha\varepsilon\phi\dot\gamma-\beta\delta\phi^2\dot\gamma) \\ &= g(\nabla_{\dot\gamma}\dot\gamma,\xi)+\beta\delta g(\dot\gamma,\dot\gamma), \end{aligned}$$

which yields

$$g(\nabla_{\dot\gamma}\dot\gamma,\xi)=-\beta\delta\varepsilon_1. \tag{3.2}$$

From the last equation one can see that $\nabla_{\dot\gamma}\dot\gamma$ is not orthogonal to $\xi$.

Since $\gamma$ is a curve parametrized by arclenght, then we write

$$g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma)=0, \tag{3.3}$$

which implies that $\nabla_{\dot\gamma}\dot\gamma$ orthogonal to $\dot\gamma$ Therefore, $\nabla_{\dot\gamma}\dot\gamma$ lies in a plane spanned by $\xi$ and $\phi\dot\gamma$.

By use of (2.4) and (3.2) we arrive at

$$\nabla_{\dot\gamma}\dot\gamma=-\frac{\beta\delta\varepsilon_1}{\varepsilon}\xi+f\phi\dot\gamma, \tag{3.4}$$

where $f$ is a scalar valued function.

In view of (3.4), we have

$$g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma)=\beta^2\varepsilon+f^2. \tag{3.5}$$

Equation (3.5) gives

$$\kappa=\sqrt{\beta^2\varepsilon+f^2.} \tag{3.6}$$

Thus we have the following.

**Theorem 3.2.** *The curvature of a Legendre curve $\gamma$ in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold is given by (3.6).*

Now using (2.10) with (3.4), we can write

$$E_2=\frac{1}{\kappa}\nabla_{\dot\gamma}\dot\gamma=-\frac{\delta\varepsilon_1}{\varepsilon\varepsilon_2}\frac{\beta}{\kappa}\xi+\frac{f}{\kappa\varepsilon_2}\phi\dot\gamma.$$

Differentiating the equation above and using (2.7), (2.8) and (3.4), we have

$$\begin{aligned} \nabla_{E_1}E_2 &= -\frac{\delta\varepsilon_1}{\varepsilon\varepsilon_2}\Big(\frac{\dot\beta\kappa-\beta\dot\kappa}{\kappa^2}\xi+\frac{\beta}{\kappa}\nabla_{\dot\gamma}\xi\Big) \\ &\quad +\Big(\frac{\dot f\kappa\varepsilon_2-f\dot\kappa\varepsilon_2}{\kappa^2}\Big)\phi\dot\gamma \\ &\quad +\frac{f}{\kappa\varepsilon_2}((\nabla_{\dot\gamma}\phi)\dot\gamma+\phi\nabla_{\dot\gamma}\dot\gamma) \\ &= \Big(-\frac{\beta^2\varepsilon_1}{\kappa\varepsilon\varepsilon_2}-\frac{f^2}{\kappa\varepsilon_2}\Big)\dot\gamma \\ &\quad +\Big(-\frac{\delta\varepsilon_1}{\varepsilon\varepsilon_2}\Big(\frac{\dot\beta\kappa-\beta\dot\kappa}{\kappa^2}\Big)+\frac{\alpha f\varepsilon_1}{\kappa\varepsilon_2}\Big)\xi \\ &\quad +\Big(\frac{\alpha\beta\delta\varepsilon_1}{\kappa\varepsilon_2}+\frac{\dot f\kappa\varepsilon_2-f\dot\kappa\varepsilon_2}{\kappa^2}\Big)\phi\dot\gamma. \end{aligned} \tag{3.7}$$

By use of (2.10), (3.6) and (3.7), we obtain

$$\begin{aligned} \tau E_3 &= \Big(-\frac{\delta\varepsilon_1}{\varepsilon\varepsilon_2}\Big(\frac{\dot\beta\kappa-\beta\dot\kappa}{\kappa^2}\Big)+\frac{\alpha f\varepsilon_1}{\kappa\varepsilon_2}\Big)\xi \\ &\quad +\Big(\frac{\alpha\beta\delta\varepsilon_1}{\kappa\varepsilon_2}+\frac{\dot f\kappa\varepsilon_2-f\dot\kappa\varepsilon_2}{\kappa^2}\Big)\phi\dot\gamma. \end{aligned}$$

So, we get

$$\tau=\sqrt{\begin{array}{c}\big(\frac{\alpha f\varepsilon_1}{\kappa\varepsilon_2}-\frac{\delta\varepsilon_1}{\varepsilon\varepsilon_2}\big(\frac{\dot\beta\kappa-\beta\dot\kappa}{\kappa^2}\big)\big)^2 \\ +\big(\frac{\alpha\beta\delta\varepsilon_1}{\kappa\varepsilon_2}+\frac{\dot f\kappa\varepsilon_2-f\dot\kappa\varepsilon_2}{\kappa^2}\big)^2\end{array}}. \tag{3.8}$$

**Theorem 3.3.** *The torsion of a Legendre curve $\gamma$ in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold is given by (3.8).*

## 4. Biharmonic Curves

In this section we investigate biharmonic curves on 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifolds.

Suppose that $\gamma: I\to M$ be a Frenet Legendre curve in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold. Then from (3.4), (2.7) and (2.8), we get

$$\begin{aligned} \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\Big(-\frac{\beta\delta\varepsilon_1}{\varepsilon}\xi+f\phi\dot\gamma\Big) \\ &= -\frac{\delta\varepsilon_1}{\varepsilon}(\dot\beta\xi+\beta\nabla_{\dot\gamma}\xi)+\dot f\phi\dot\gamma \\ &\quad +f((\nabla_{\dot\gamma}\phi)\dot\gamma+\phi\nabla_{\dot\gamma}\dot\gamma) \\ &= \Big(-\frac{\beta^2\varepsilon_1}{\varepsilon}-f^2\Big)\dot\gamma \\ &\quad +\Big(-\dot\beta\frac{\delta\varepsilon_1}{\varepsilon}+\alpha f\varepsilon_1\Big)\xi \\ &\quad +(\alpha\beta\delta\varepsilon_1+\dot f)\phi\dot\gamma. \end{aligned} \tag{4.1}$$

Differentiating (4.1) along $\gamma$ and again using (2.7) and (2.8), we have

$$\begin{aligned} \nabla_{\dot\gamma}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\Big(\Big(-\frac{\beta^2\varepsilon_1}{\varepsilon}-f^2\Big)\dot\gamma+\Big(-\dot\beta\frac{\delta\varepsilon_1}{\varepsilon}+\alpha f\varepsilon_1\Big)\xi \\ &\quad +(\alpha\beta\delta\varepsilon_1+\dot f)\phi\dot\gamma\Big) \\ &= \Big(-\frac{2\beta\dot\beta\varepsilon_1}{\varepsilon}-2f\dot f\Big)\dot\gamma-\Big(\frac{\beta^2\varepsilon_1}{\varepsilon}+f^2\Big)\nabla_{\dot\gamma}\dot\gamma \\ &\quad +\Big(-\ddot\beta\frac{\delta\varepsilon_1}{\varepsilon}+\dot\alpha f\varepsilon_1+\alpha\dot f\varepsilon_1\Big)\xi \\ &\quad +\Big(-\dot\beta\frac{\delta\varepsilon_1}{\varepsilon}+\alpha f\varepsilon_1\Big)\nabla_{\dot\gamma}\xi \\ &\quad +(\dot\alpha\beta\delta\varepsilon_1+\alpha\dot\beta\delta\varepsilon_1+\ddot f)\phi\dot\gamma \\ &\quad +(\alpha\beta\delta\varepsilon_1+\dot f)\nabla_{\dot\gamma}\phi\dot\gamma \\ &= -3\Big(\frac{\beta\dot\beta\varepsilon_1}{\varepsilon}+f\dot f\Big)\dot\gamma \\ &\quad +\begin{pmatrix} \beta^3\delta+\beta f^2\frac{\delta\varepsilon_1}{\varepsilon}-\ddot\beta\frac{\delta\varepsilon_1}{\varepsilon} \\ +\dot\alpha f\varepsilon_1+2\alpha\dot f\varepsilon_1+\alpha^2\beta\delta \end{pmatrix}\xi \\ &\quad +\begin{pmatrix} -\frac{\beta^2 f\varepsilon_1}{\varepsilon}-f^3+2\alpha\dot\beta\delta\varepsilon_1 \\ -\alpha^2 f\varepsilon_1\varepsilon+\dot\alpha\beta\delta\varepsilon_1+\ddot f \end{pmatrix}\phi\dot\gamma. \end{aligned} \tag{4.2}$$

Hence, we get biharmonic equation for a Frenet Legendre curve in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold $M$ as follows:

$$\begin{aligned} 0 &= -3\left(\frac{\beta\dot{\beta}\varepsilon_1}{\varepsilon}+f\dot{f}\right)\dot{\gamma} \\ &+\begin{pmatrix} \beta^3\delta+\beta f^2\frac{\delta\varepsilon_1}{\varepsilon}-\ddot{\beta}\frac{\delta\varepsilon_1}{\varepsilon} \\ +\dot{\alpha}f+2\alpha\dot{f}\varepsilon_1+\alpha^2\beta\delta \end{pmatrix}\xi \\ &+\begin{pmatrix} -\frac{\beta^2 f}{\varepsilon}\varepsilon_1-f^3+2\alpha\dot{\beta}\delta\varepsilon_1 \\ -\alpha^2 f\varepsilon_1\varepsilon+\dot{\alpha}\beta\delta\varepsilon_1+\ddot{f} \end{pmatrix}\phi\dot{\gamma} \\ &+\beta\frac{\delta\varepsilon_1}{\varepsilon}R(\dot{\gamma},\xi)\dot{\gamma}-fR(\dot{\gamma},\phi\dot{\gamma})\dot{\gamma}, \end{aligned} \tag{4.3}$$

where $R$ is the curvature tensor of $M$.

By use of (2.9), we get

$$R(\dot{\gamma},\xi)\dot{\gamma}=\left(\frac{r}{2}(\varepsilon-1)-(\alpha^2-\beta^2)\right)\xi, \tag{4.4}$$

and

$$R(\dot{\gamma},\phi\dot{\gamma})\dot{\gamma}=-\left(\frac{r}{2}-2(\alpha^2-\beta^2)\right)\phi\dot{\gamma}. \tag{4.5}$$

Then replacing (4.4) with (4.5) in (4.3), bitension field of $\gamma$ is in the following:

$$\begin{aligned} \tau_2(\gamma) &= -3\left(\frac{\beta\dot{\beta}\varepsilon_1}{\varepsilon}+f\dot{f}\right)\dot{\gamma} \\ &+\begin{pmatrix} \beta^3\delta+\beta f^2\frac{\delta\varepsilon_1}{\varepsilon}-\ddot{\beta}\frac{\delta\varepsilon_1}{\varepsilon} \\ +\dot{\alpha}f\varepsilon_1+2\alpha\dot{f}\varepsilon_1+\alpha^2\beta\delta \\ +\frac{r}{2}(\varepsilon-1)\beta\frac{\delta\varepsilon_1}{\varepsilon}-(\alpha^2-\beta^2)\beta\frac{\delta\varepsilon_1}{\varepsilon} \end{pmatrix}\xi \\ &+\begin{pmatrix} -\frac{\beta^2 f\varepsilon_1}{\varepsilon}-f^3+2\alpha\dot{\beta}\delta\varepsilon_1 \\ -\alpha^2 f\varepsilon_1\varepsilon+\dot{\alpha}\beta\delta\varepsilon_1+\ddot{f} \\ +\frac{r}{2}f-2f(\alpha^2-\beta^2) \end{pmatrix}\phi\dot{\gamma}. \end{aligned} \tag{4.6}$$

So we can state the following.

**Theorem 4.1.** *Let $\gamma: I\to M$ be a Frenet Legendre curve in a 3-dimensional $(\varepsilon,\delta)$ trans-Sasakian manifold with $\alpha,\beta$ =constant. Then $\gamma$ is biharmonic if and only if*

$$\begin{aligned} f\dot{f} &= 0 \\ \begin{pmatrix} \beta^3\delta+\beta f^2\frac{\delta\varepsilon_1}{\xi}+2\alpha\dot{f}\varepsilon_1+\alpha^2\beta\delta \\ +\frac{r}{2}(\varepsilon-1)\beta\frac{\delta\varepsilon_1}{\varepsilon}-(\alpha^2-\beta^2)\beta\frac{\delta\varepsilon_1}{\varepsilon} \end{pmatrix} &= 0 \\ \begin{pmatrix} \frac{\beta^2 f\varepsilon_1}{\varepsilon}+f^3+\alpha^2 f\varepsilon_1\varepsilon-\ddot{f} \\ -\frac{r}{2}f-2f(\alpha^2-\beta^2) \end{pmatrix} &= 0. \end{aligned}$$

We know that, an $(\varepsilon,\delta)$ trans-Sasakian manifold reduces
*i)* an $(\delta)-$Kenmotsu manifold for $\alpha=0$, $\beta=1$,
*ii)* an $(\varepsilon)-$Sasakian manifold for $\beta=0$, $\alpha=1$.
Hence, we can give the followings:

**Proposition 4.2.** *Let $\gamma: I\to M$ be a Frenet Legendre curve in a 3-dimensional $(\delta)-$Kenmotsu manifold. Then $\gamma$ is proper biharmonic if and only if*

$$\begin{aligned} f &= constant\neq 0 \\ \begin{pmatrix} \beta^3\delta+\beta f^2\frac{\delta\varepsilon_1}{\varepsilon}+\frac{r}{2}(\varepsilon-1)\beta\frac{\delta\varepsilon_1}{\varepsilon} \\ +\beta^3\frac{\delta\varepsilon_1}{\varepsilon} \end{pmatrix} &= 0 \\ \left(\frac{\beta^2 f\varepsilon_1}{\varepsilon}+f^3-\frac{r}{2}f+2f\beta^2\right) &= 0. \end{aligned}$$

**Proposition 4.3.** *Let $\gamma: I\to M$ be a Frenet Legendre curve in a 3-dimensional $(\varepsilon)-$Sasakian manifold. Then $\gamma$ is proper biharmonic if and only if*

$$\begin{aligned} f &= constant\neq 0 \\ \left(f^3+\alpha^2 f\varepsilon_1\varepsilon-\frac{r}{2}f-2f\alpha^2\right) &= 0. \end{aligned}$$

## Acknowledgments

The authors would like to express sincere gratitude to the reviewers for his/her valuable suggestions.

## References

[1] A. Bejancu, K. L. Duggal, Real hypersurfaces of indefinite Kahler manifolds, *Int. J. Math. Math. Sci.*, 16(1993), 545–556.

[2] A. A. Balmus, S. Montaldo, C. Onicuic, Biharmonic hypersurfaces in 4-dimensional space forms, *Math. Nachr.*, 283(2010), 1696–1705.

[3] C. Özgür, M. M. Tripathi, On Legendre curves in $\alpha-$Sasakian manifolds, *Bull. Malays. Math. Sci. Soc.*, 31(2008), 91–96.

[4] D. E. Blair, C. Baikoussis, On Legendre curves in contact 3-manifolds, *Geom. Dedicata*, 49(1994), 135–142.

[5] D. E. Blair, *Riemannian Geometry of Contact and Sympletic Manifolds*, Progress in Mathematics 203, Birkhauser Boston, Inc, 2002.

[6] D. Fetcu, C. Onicuic, Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres, *Abh. Math. Sem. Univ. Hamburg*, 77(2007), 179–190.

[7] D. Fetcu, Biharmonic Legendre curves in Sasakian space form, *J. Korean Math. Soc.*, 45(2008), 393–404.

[8] D. Fetcu, C. Onicuic, Biharmonic hypersurfaces in Sasakian space form, *Diff. Geo. Appl.*, 27(2009), 713–722.

[9] D. Fetcu, C. Onicuic, Explicit formulas for biharmonic submanifolds in Sasakian space form, *Pacific J. Math.*, 240(2009), 85–107.

[10] G. Y. Jiang, 2- harmonic maps and their first and second variation formulas, *Chinese Ann. Math. Ser. A.*, 7(1986), 389–402.

[11] H. G. Nagaraja, G. Somashekara, Special trans-Sasakian manifolds and curvature conditions, *Int. J. Pure and Appl. Math.*, 81(2012), 411–420.

[12] H. G. Nagaraja, R. C. Premalatha, G. Somashekara, On $(\varepsilon,\delta)$-trans-Sasakian structure, *Proceedings of the Estonian Academy of Sciences*, 61(2012), 20–28.

[13] I. Inoguchi, Submanifolds with harmonic mean curvature in contact 3-manifolds, *Colleq. Math.*, 100(2004), 163–179.

[14] J. Eells, J. H. Sampson, Harmonic mapping of the Riemannian manifold, *American J. Math.*, 86(1964), 109–160.

[15] J. Welyczko, On Legendre curves in 3-dimensional normal almost contact metric manifolds, *Soochow J. Math.*, 33(2007), 929–937.

[16] J. Welyczko, On Legendre curves in 3-dimensional normal almost paracontact metric manifolds, *Result. Math.*, 54(2009), 377–387.

[17] M. M. Tripathi, E. Kılıç, S. Yüksel Perktaş, S. Keleş, Indefinite almost paracontact metric manifods, *Int. J. Math. Math. Sci.*, (2010), Article ID 848195.

[18] R. Kumar, R. Rani, R. K. Nagaich, On sectional curvature of $(\varepsilon)$-Sasakian manifolds, *Int. J. Math. Math. Sci.*, (2007), Article ID 93562.

[19] S. Keleş, S. Yüksel Perktaş, E. Kılıç, Biharmonic curves in LP-Sasakian manifolds, *Bull. Malays. Math. Sci. Soc.*, 33(2010), 325–344.

[20] S. Yüksel Perktaş, E. Kılıç, Biharmonic maps between doubly warped product manifolds, *Balkan J. Geom. and Its Appl.*, 15(2010), 159–170.

[21] S. Yüksel Perktaş, E. Kılıç, S. Keleş, Biharmonic hypersurfaces of Lorentzian para-Sasakian manifolds, *An. Stiint. Univ. Al. I. Cuza Iasi.*, Tomul LVII, f.2 DOI: 10.2478/v10157-011-0034-z, 2011.

[22] U. C. De, A. Sarkar, On $(\varepsilon)$-Kenmotsu manifolds, *Hadronic Journal*, 32(2009), 231–242.

[23] X. Xufeng, C. Xiaoli, Two theorems on $(\varepsilon)$-Sasakian manifolds, *Int. J. Math. Math. Sci.*, 21(1998), 249–254.

[24] Y. L. Ou, p-Harmonic morphisms, biharmonic morphisms and non-harmonic biharmonic maps, *J. Geom. Phys.*, 56(2006), 358–374.

[25] Y. L. Ou, Biharmonic hypersurfaces in Riemannian manifolds,, *Pacific J. Math.*, 240(2010), 217–232.

[26] Y. L. Ou, Some constructions of biharmonic maps and Chen's conjectures on biharmonic hypersurfaces, *J. Geom. Phys.*, 62(2012), 751–762.


⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆