Some lower bound for holomorphic functions at the boundary

Bülent Nafi Örnek[1]*

Abstract
In this paper, a boundary version of the Schwarz lemma for classes $\mathscr{H}(\alpha)$ is investigated. For the function $f(z) = 1 + c_1 z + c_2 z^2 + ...$ defined in the unit disc such that $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point b with $f(b) = e^{\frac{\pi\alpha}{2}}$. The sharpness of these inequalities is also proved.

Keywords
Holomorphic function, Jack's lemma, Angular derivative.

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[1]*Department of Computer Engineering, Amasya University, Merkez-Amasya 05100, Turkey.*
Corresponding author: nafiornek@gmail.com

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1. Introduction

The most classical version of the Schwarz lemma involves the behavior at the origin of a bounded, holomorphic function on the unit disc $D = \{z : |z| < 1\}$. In its must basic form, the familiar Schwarz lemma says this ([5], p.329):

Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be a holomorphic function with $f(0) = 0$. Under these circumstances $|f(z)| \leq |z|$ for all $z \in D$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\theta}$, θ real.

In order to show our main results, we need the following lemma due to Jack's Lemma [6].

Lemma 1.1 (Jack's Lemma). *Let $f(z)$ be a non-constant and holomorphic function in the unit disc D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k,$$

where $k \geq 1$ is a real number.

Let $\mathscr{A}$ denote the class of functions $f(z)$ that are holomorphic in the unit disc D, so that $f(0) = 1$. That is,

$$f(z) = 1 + c_1 z + c_2 z^2 + ...$$

Also, let $\mathscr{H}(\alpha)$ be the subclass of $\mathscr{A}$ consisting of all functions $f(z)$ which satisfy

$$\left| \frac{z f'(z)}{f(z)} \right| < \frac{\pi\alpha}{2} \quad (z \in D),$$

where $0 < \alpha \leq 1$.

Let $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$ and consider the function

$$\Phi(z) = \frac{2}{\pi\alpha} \ln f(z),$$

where $0 < \alpha \leq 1$.

Clearly, $\Phi(z)$ is holomorphic function in D and $\Phi(0) = 0$.

Now, let us show that the function $|\Phi(z)|$ is less than 1 in the unit disc D. From the definition of $\Phi(z)$, we take

$$\frac{z f'(z)}{f(z)} = \frac{\pi\alpha z}{2} \Phi'(z).$$

We suppose that there exists a point $z_0 \in D$ such that

$$\max_{|z| \leq |z_0|} |\Phi(z)| = |\Phi(z_0)| = 1.$$

From the Jack's lemma, we obtain

$$\Phi(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0 \Phi'(z_0)}{\Phi(z_0)} = k.$$

Therefore, we have

$$\left| \frac{z_0 f'(z_0)}{f(z_0)} \right| = \frac{\pi \alpha k}{2} |\Phi(z_0)| \geq \frac{\pi \alpha}{2}.$$

This contradicts the condition $f(z) \in \mathscr{H}(\alpha)$. This means that there is no point $z_0 \in D$ such that $|\Phi(z_0)| = 1$ for all $z \in D$. Therefore, $|\Phi(z)| < 1$ for $|z| < 1$. By the Schwarz lemma, we obtain

$$|f'(0)| \leq \frac{\pi \alpha}{2}. \tag{1.1}$$

The equality in (1.1) holds if and only if $\Phi(z) = z e^{i\theta}$ (see, [16]) that is,

$$f(z) = e^{\frac{\pi \alpha}{2} z e^{i\theta}}.$$

That proves

Lemma 1.2. *If $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$, then we have*

$$|f'(0)| \leq \frac{\pi \alpha}{2}. \tag{1.2}$$

The equality in (1.2) *holds if and only if*

$$f(z) = e^{\frac{\pi \alpha}{2} z e^{i\theta}},$$

where θ is a real number.

The boundary version of Schwarz lemma is known as simple:

Let $f(z)$ be a holomorphic function in the unit disc D, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. Assume that, there is a $b \in \partial D$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Therefore, the inequality $|f'(b)| \geq 1$, that is, known as Schwarz lemma at the boundary from the classic Schwarz lemma, is obtained. The equality in $|f'(b)| \geq 1$ holds if and only if $f(z) = z e^{i\theta}$, θ real. This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature. This improvement was obtained in [22] by Helmut Unkelbach, and rediscovered by R. Osserman in [16] 60 years later.

In the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [3], [4], [7], [8], [10], [11], [16], [17], [18], [20] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(b)| = 1$ condition of the boundary of the unit circle.

In [16], R. Osserman offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result.

Lemma 1.3. *Let $f : D \to D$ be holomorphic function with $f(0) = 0$. Assume that there is a $b \in \partial D$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{1.3}$$

Inequality (1.3) *is sharp, with equality possible for each value of $|f'(0)|$.*

Corollary 1.4. *Under the hypotheses of Lemma 1.3, we have*

$$|f'(b)| \geq 1 \tag{1.4}$$

and

$$|f'(b)| > 1 \quad \text{unless} \quad f(z) = z e^{i\theta}, \, \theta \text{ real.}$$

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [19]).

Lemma 1.5 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

D. M. Burns and S. G. Krantz [9] and D. Chelst [2] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12], [13], [14] and [15]).

X. Tang, T. Liu and J. Lu [11] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Taishun Liu, Jianfei Wang, Xiaomin Tang [21] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$. They then applied their new Schwarz lemma to study problems from the geometric function theory in several complex variables.

Also, M. Jeong [7] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [6] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [1], [20]).

2. Main Results

In this section, for holomorphic function $f(z)$ belong to the class of $\mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$, it has been estimated from below the modulus of the angular derivative of the function on the boundary point of the unit disc. It has been proved that these result are sharp. Also, we derive an improvement of the above Lemma 3 and Corollary 1 as the special cases of our main result.

Theorem 2.1. *Let $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = e^{\frac{\pi\alpha}{2}}$. Then we have the inequality*

$$\left|f'(b)\right| \geq \frac{\pi\alpha}{2} e^{\frac{\pi\alpha}{2}}. \qquad (2.1)$$

The equality in (2.1) holds if and only if

$$f(z) = e^{\frac{\pi\alpha}{2} z e^{i\theta}},$$

where θ is a real number.

Proof. Let

$$\Phi(z) = \frac{2}{\pi\alpha} \ln f(z).$$

Then $\Phi(z)$ is holomorphic function in the unit disc D and $\Phi(0) = 0$. By the Jack's lemma and since $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$, we take $|\Phi(z)| < 1$ for $|z| < 1$. Also, we have $|\Phi(b)| = 1$ for $b \in \partial D$. From (1.4), we obtain

$$1 \leq \left|\Phi'(b)\right| = \frac{2}{\pi\alpha} \left|\frac{f'(b)}{f(b)}\right| = \frac{2}{\pi\alpha} \frac{1}{e^{\frac{\pi\alpha}{2}}} \left|f'(b)\right|$$

Thus, we get

$$\left|f'(b)\right| \geq \frac{\pi\alpha}{2} e^{\frac{\pi\alpha}{2}}.$$

If $|f'(b)| = \frac{\pi\alpha}{2} e^{\frac{\pi\alpha}{2}}$, then $|\Phi'(b)| = 1$ and so by Osserman [16], $\Phi(z) = z e^{i\theta}$ for some real θ. It means that

$$f(z) = e^{\frac{\pi\alpha}{2} z e^{i\theta}}.$$

$\square$

Theorem 2.2. *Let $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = e^{\frac{\pi\alpha}{2}}$. Then we have the inequality*

$$\left|f'(b)\right| \geq \frac{(\pi\alpha)^2 e^{\frac{\pi\alpha}{2}}}{\pi\alpha + 2|f'(0)|}. \qquad (2.2)$$

The inequality (2.2) is sharp with equality for the function

$$f(z) = e^{\frac{\pi\alpha}{2} \frac{z^2 + az}{1 + az}},$$

where $a = \frac{2|f'(0)|}{\pi\alpha}$, $a \in [0,1]$ and $0 < \alpha \leq 1$ (see (1.2)).

Proof. Let $\Phi(z)$ be the same as in the proof of Theorem 2.1. From (1.3), we obtain

$$\frac{2}{1 + |\Phi'(0)|} \leq \left|\Phi'(b)\right| = \frac{2}{\pi\alpha} \left|\frac{f'(b)}{f(b)}\right| = \frac{2}{\pi\alpha} \frac{1}{e^{\frac{\pi\alpha}{2}}} \left|f'(b)\right|.$$

Since

$$\left|\Phi'(0)\right| = \frac{2|f'(0)|}{\pi\alpha},$$

we take

$$\frac{2}{1 + \frac{2|f'(0)|}{\pi\alpha}} \leq \frac{2}{\pi\alpha} \frac{1}{e^{\frac{\pi\alpha}{2}}} \left|f'(b)\right|$$

and

$$\left|f'(b)\right| \geq \frac{(\pi\alpha)^2 e^{\frac{\pi\alpha}{2}}}{\pi\alpha + 2|f'(0)|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = e^{\frac{\pi\alpha}{2} \frac{z^2 + az}{1 + az}}.$$

Then,

$$f'(z) = \frac{\pi\alpha}{2} \frac{(2z+a)(1+az) - a(z^2+az)}{(1+az)^2} e^{\frac{\pi\alpha}{2} \frac{z^2+az}{1+az}}$$

and

$$\left|f'(1)\right| = \frac{\pi\alpha}{2} \frac{2}{1+a} e^{\frac{\pi\alpha}{2}}.$$

Since $a = \frac{2|f'(0)|}{\pi\alpha}$, we take

$$\left|f'(1)\right| = \frac{(\pi\alpha)^2 e^{\frac{\pi\alpha}{2}}}{\pi\alpha + 2|f'(0)|}.$$

$\square$

The inequality (2.2) can be strengthened as below by taking into account c_2 which is second coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = e^{\frac{\pi\alpha}{2}}$. Then we have the inequality*

$$\left|f'(b)\right| \geq \frac{\pi\alpha e^{\frac{\pi\alpha}{2}}}{2} \left(1 + \frac{2[\pi\alpha - 2|c_1|]^2}{\pi^2\alpha^2 - 4|c_1|^2 + \pi\alpha |2c_2 - c_1^2|}\right). \qquad (2.3)$$

The equality in (2.3) occurs for the function

$$f(z) = e^{\frac{\pi\alpha}{2} z}.$$

Proof. Let $\Phi(z)$ be the same as in the proof of Theorem 2.1 and $\kappa(z) = z$. By the maximum principle for each $z \in D$, we have

$$\left|\Phi(z)\right| \leq \left|\kappa(z)\right|.$$

Therefore,

$$\varphi(z) = \frac{\Phi(z)}{\kappa(z)} = \frac{\Phi(z)}{z}$$

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is holomorphic function in D and $|\kappa(z)| < 1$ for $|z| < 1$. In particular, we have

$$|\varphi(0)| = \frac{2|c_1|}{\pi\alpha} \leq 1 \qquad (2.4)$$

and

$$|\varphi'(0)| = \frac{|2c_2 - c_1^2|}{\pi\alpha}.$$

Furthermore, the geometric meaning of the derivative and the inequality $|\Phi(z)| \leq |\kappa(z)|$ imply the inequality

$$\frac{b\Phi'(b)}{\Phi(b)} \geq |\Phi'(b)| \geq |\kappa'(b)| = \frac{b\kappa'(b)}{\kappa(b)}.$$

The function

$$\Psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \varphi(z)\overline{\varphi(0)}}$$

is holomorphic function in D, $|\Psi(z)| < 1$ for $|z| < 1$, $\Psi(0) = 0$ and $|\Psi(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\frac{2}{1+|\Psi'(0)|} \leq |\Psi'(b)| \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left|\frac{\Phi'(b)}{\kappa(b)} - \frac{\Phi(b)\kappa'(b)}{\kappa^2(b)}\right|$$

$$= \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left\{|\Phi'(b)| - |\kappa'(b)|\right\}.$$

Since

$$\Psi'(z) = \frac{1-|\varphi(0)|^2}{\left(1-\overline{\varphi(0)}\varphi(z)\right)^2}\varphi'(z)$$

and

$$|\Psi'(0)| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2} = \frac{\frac{|2c_2-c_1^2|}{\pi\alpha}}{1-\left(\frac{2|c_1|}{\pi\alpha}\right)^2} = \frac{\pi\alpha|2c_2-c_1^2|}{\pi^2\alpha^2 - 4|c_1|^2}$$

we take

$$\frac{2}{1+\frac{\pi\alpha|2c_2-c_1^2|}{\pi^2\alpha^2-4|c_1|^2}} \leq \frac{1+\frac{2|c_1|}{\pi\alpha}}{1-\frac{2|c_1|}{\pi\alpha}}\left\{\frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}|f'(b)| - 1\right\},$$

$$\frac{2\left[\pi^2\alpha^2 - 4|c_1|^2\right]}{\pi^2\alpha^2 - 4|c_1|^2 + \pi\alpha|2c_2-c_1^2|}\frac{\pi\alpha - 2|c_1|}{\pi\alpha + 2|c_1|}$$

$$\leq \frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}|f'(b)| - 1,$$

$$\frac{2\left[\pi\alpha - 2|c_1|\right]^2}{\pi^2\alpha^2 - 4|c_1|^2 + \pi\alpha|2c_2-c_1^2|} \leq \frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}|f'(b)| - 1,$$

$$1 + \frac{2\left[\pi\alpha - 2|c_1|\right]^2}{\pi^2\alpha^2 - 4|c_1|^2 + \pi\alpha|2c_2-c_1^2|} \leq \frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}|f'(b)|$$

and

$$|f'(b)| \geq \frac{\pi\alpha e^{\frac{\pi\alpha}{2}}}{2}\left(1 + \frac{2\left[\pi\alpha - 2|c_1|\right]^2}{\pi^2\alpha^2 - 4|c_1|^2 + \pi\alpha|2c_2-c_1^2|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = e^{\frac{\pi\alpha}{2}z}.$$

Then

$$f'(1) = \frac{\pi\alpha}{2}e^{\frac{\pi\alpha}{2}}.$$

Since $|c_1| = \frac{\pi\alpha}{2}$, (2.3) is satisfied with equality. □

If $f(z) - 1$ has no zeros different from $z = 0$ in Theorem 2.2, the inequality (2.3) can be further strengthened. This is given by the following Theorem.

Theorem 2.4. *Let* $f(z) \in \mathscr{H}(\alpha)$ $(0 < \alpha \leq 1)$, $f(z) - 1$ *has no zeros in D except $z = 0$ and $c_1 > 0$. Assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = e^{\frac{\pi\alpha}{2}}$. Then we have*

$$|f'(b)| \geq \frac{\pi\alpha e^{\frac{\pi\alpha}{2}}}{2}\left(1 - \frac{4|c_1|\left(\ln\left(\frac{2c_1}{\pi\alpha}\right)\right)^2}{4|c_1|\ln\left(\frac{2c_1}{\pi\alpha}\right) - \frac{|2c_2-c_1^2|}{2|c_1|}}\right). \quad (2.5)$$

The equality in (2.5) occurs for the function

$$f(z) = e^{\frac{\pi\alpha}{2}z}.$$

Proof. Let $c_1 > 0$ in the expression of the function $f(z)$. Besides, let $\Phi(z)$, $\kappa(z)$ and $\varphi(z)$ be as in the proof of Theorem 2.3 and the function $f(z) - 1$ has no zeros point in D except $D - \{0\}$. Having in the mind inequality (2.4), we denote by $\ln\varphi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln\varphi(0) = \ln\left(\frac{2c_1}{\pi\alpha}\right) < 0.$$

The composite function

$$\Theta(z) = \frac{\ln\varphi(z) - \ln\varphi(0)}{\ln\varphi(z) + \ln\varphi(0)}$$

is holomorphic in the unit disc D, $|\Theta(z)| < 1$ for $|z| < 1$, $\Theta(0) = 0$ and $|\Theta(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(b)| = \frac{|2\ln\varphi(0)|}{|\ln\varphi(b) + \ln\varphi(0)|^2}\left|\frac{\varphi'(b)}{\varphi(b)}\right|$$

$$= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b) + \ln\varphi(0)|^2}|\varphi'(b)|$$

$$= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b) + \ln\varphi(0)|^2}\left|\frac{\Phi'(b)}{\kappa(b)} - \frac{\Phi(b)\kappa'(b)}{\kappa^2(b)}\right|$$

$$= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0) + \arg^2\varphi(b)}\left\{|\Phi'(b)| - |\kappa'(b)|\right\}.$$

In addition, it can be seen that

$$\Theta'(z) = \frac{2\ln\varphi(0)}{(\ln\varphi(z)+\ln\varphi(0))^2}\frac{\varphi'(z)}{\varphi(z)},$$

$$\Theta'(0) = \frac{1}{2\ln\varphi(0)}\frac{\varphi'(0)}{\varphi(0)}$$

and

$$
\begin{aligned}
\left|\Theta'(0)\right| &= \frac{1}{|2\ln\varphi(0)|}\left|\frac{\varphi'(0)}{\varphi(0)}\right| \\
&= \frac{1}{-2\ln\left(\frac{2c_1}{\pi\alpha}\right)}\frac{\frac{|2c_2-c_1^2|}{\pi\alpha}}{\frac{2|c_1|}{\pi\alpha}} \\
&= \frac{1}{-2\ln\left(\frac{2c_1}{\pi\alpha}\right)}\frac{|2c_2-c_1^2|}{2|c_1|}
\end{aligned}
$$

Therefore, replacing $\arg^2\varphi(b)$ by zero, we take

$$\frac{2}{1-\frac{1}{2\ln\left(\frac{2c_1}{\pi\alpha}\right)}\frac{|2c_2-c_1^2|}{2|c_1|}} \le \frac{-2}{\ln\left(\frac{2c_1}{\pi\alpha}\right)}\left\{\frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}\left|f'(b)\right|-1\right\},$$

$$\frac{-4|c_1|\left(\ln\left(\frac{2c_1}{\pi\alpha}\right)\right)^2}{4|c_1|\ln\left(\frac{2c_1}{\pi\alpha}\right)-\frac{|2c_2-c_1^2|}{2|c_1|}} \le \frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}\left|f'(b)\right|-1,$$

$$1-\frac{4|c_1|\left(\ln\left(\frac{2c_1}{\pi\alpha}\right)\right)^2}{4|c_1|\ln\left(\frac{2c_1}{\pi\alpha}\right)-\frac{|2c_2-c_1^2|}{2|c_1|}} \le \frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}\left|f'(b)\right|.$$

and we obtain (2.5) with an obvious equality case. □

We note that the inequality (1.3) has been used in the proofs of Theorem 2.3 and Theorem 2.4. So, there are both c_1 and c_2 in the right side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker but more simpler inequality (not including c_2). It is formulated in the following Theorem.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, we have*

$$\left|f'(b)\right| \ge \frac{\pi\alpha e^{\frac{\pi\alpha}{2}}}{2}\left[1-\frac{1}{2}\ln\left(\frac{2c_1}{\pi\alpha}\right)\right]. \tag{2.6}$$

The equality in (2.6) *holds if and only if*

$$f(z) = e^{\frac{\pi\alpha}{2}z\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{2c_1}{\pi\alpha}\right)},$$

where θ is a real number and $c_1 > 0$.

Proof. From Theorem 2.4, using the inequality (1.4) for the function $\Upsilon(z)$, we obtain

$$
\begin{aligned}
1 &\le \left|\Theta'(b)\right| = \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{\varphi'(b)}{\varphi(b)}\right| \\
&= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\varphi'(b)\right| \\
&= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b)+\ln\varphi(0)|^2}\left|\frac{\Phi'(b)}{\kappa(b)}-\frac{\Phi(b)\kappa'(b)}{\kappa^2(b)}\right| \\
&= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0)+\arg^2\varphi(b)}\left\{\left|\Phi'(b)\right|-\left|\kappa'(b)\right|\right\}.
\end{aligned}
$$

Replacing $\arg^2\varphi(b)$ by zero, we take

$$1 \le \frac{-2}{\ln\left(\frac{2c_1}{\pi\alpha}\right)}\left\{\frac{2}{\pi\alpha}\frac{1}{e^{\frac{\pi\alpha}{2}}}\left|f'(b)\right|-1\right\}$$

and we obtain (2.6) with an obvious equality case. □

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