



General solution and generalized Ulam - Hyers stability of r_i - type n dimensional quadratic-cubic functional equation in random normed spaces: Direct and fixed point methods

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Abstract

In this paper, the authors introduce and establish the general solution and generalized Ulam- Hyers stability of a r_i type n - dimensional Quadratic-Cubic functional equation

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left(\sum_{u=0}^1 \left[\sum_{v=0}^1 \left(\frac{r_{2i}r_{2i+1}(-1)^{u+v} + r_{2i}r_{2i+1}^2(-1)^u + r_{2i}^2r_{2i+1}(-1)^v}{4} \right) (f((-1)^u x_{2i} + (-1)^v x_{2i+1})) \right] \right) \\ &+ \left(\frac{r_{2i}^3 + r_{2i}^2 - r_{2i}r_{2i+1}^2}{4} \right) f(x_{2i}) + \left(\frac{r_{2i}^2 - r_{2i}^3 + r_{2i}r_{2i+1}^2}{4} \right) f(-x_{2i}) \\ &+ \left(\frac{r_{2i+1}^3 + r_{2i+1}^2 - r_{2i}^2r_{2i+1}}{4} \right) f(x_{2i+1}) + \left(\frac{r_{2i+1}^2 - r_{2i+1}^3 + r_{2i}r_{2i+1}^2}{4} \right) f(-x_{2i+1}) \end{aligned}$$

where $r_{2i}, r_{2i+1} \in R - \{0\}$, ($i = 0, 1, 2 \dots n$) and n is a positive integer in Random normed spaces.

Keywords

Quadratic functional equation, Cubic functional equation, Mixed functional equation, Generalized Ulam - Hyers stability, fixed point, Random normed spaces.

AMS Subject Classification

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1. Introduction

In 1940, Ulam [30] at the University of Wisconsin proposed the following stability problem:

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an

exact solution of the equation?

In 1941, D. H. Hyers [12] gave an affirmative answer to the question of S.M. Ulam for Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [24] succeeded in extending Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0, 1)$, to be unbounded. In 1982, J.M. Rassias [23] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^q$ for $p, q \in \mathbb{R}$. A generalization of all the above stability results was obtained by P. Gavruta [8] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y)$. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by K.Ravi et al., [26] by considering the summation of both the sum and the product of two p - norms. These terminologies are also applied to the folder of other functional equations and it has been extensively investigated by a number of authors and there are countless remarkable results pertaining to this problem together with mixed type functional equations (see[1, 13, 17, 19, 25]) and references cited there in.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping (see[1, 19]). It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. K.W.Jun and H.M.Kim [15] considered the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.2}$$

which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping. G.H.Kim, H.Y.Shin [20] introduced and investigated the generalized Ulam-Hyers Stability of the following more generalized cubic functional equation

$$f(rx+sy) + f(rx-sy) = rs^2f(x+y) + rs^2f(x-y) + 2r(r^2 - s^2)f(x) \tag{1.3}$$

where $r \neq \pm 1, 0$, s are real numbers. In 2003, I.S.Chang and Y.S.Jung [5], investigated the solution and stability of the functional equation

$$6f(x+y) + 6f(x-y) + 4f(3y) = 3f(x+2y) - 3f(x-2y) + 9f(2x) \tag{1.4}$$

deriving from cubic and quadratic functions. Recently the fuzzy stability of (1.3) was discussed by Z.H.Wang and W.X.Zhang [31]. Yeol Je Cho et al., [7] established the general solution and stability of generalized mixed type quadratic-cubic func-

tional equations

$$f(x+ky) + f(x-ky) = k^2f(x+y) + k^2f(x-y) + \frac{2(k^2-1)}{k^2(k-2)}f(kx) - \frac{(k^3-k^2-k+1)}{2(k-1)}f(2x) + f(\tilde{2}y) - 8f(\tilde{y}) \tag{1.5}$$

where $f(\tilde{y}) = f(y) - f(-y)$ for fixed integers k with $k \neq 0, \pm 1, 2$ in Random Normed Spaces.

Recently M. Arunkumar and P. Agilan [3] introduced and investigated the solution and stability of generalized Ulam-Hyers Stability of a r_i type n -dimensional Additive Quadratic functional equation

$$f\left(\sum_{i=1}^n r_i x_i\right) = \sum_{i=1}^n \left(\sum_{j=1}^2 \frac{r_i^j}{2} [f(x_i) + (-1)^j f(x_i)]\right) + \sum_{1 \leq i < j \leq n} \frac{r_i r_j}{4} \left(\sum_{p=0}^1 \left(\sum_{q=0}^1 (-1)^{p+q} f[(-1)^p x_i + (-1)^q x_j]\right)\right) \tag{1.6}$$

where r_i and n are positive integers with $n \geq 2$ in quasi beta normed spaces.

In this paper, the authors establish the general solution and generalized Ulam- Hyers stability of a r_i type n - dimensional Quadratic-Cubic functional equation

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left(\sum_{u=0}^1 \left[\sum_{v=0}^1 \left(\frac{1}{4} \left(\begin{aligned} & r_{2i}r_{2i+1}(-1)^{u+v} \\ & + r_{2i}r_{2i+1}^2(-1)^u \\ & + r_{2i}^2r_{2i+1}(-1)^v \end{aligned} \right) \right) \right. \right. \\ & \quad \left. \left. f((-1)^u x_{2i} + (-1)^v x_{2i+1}) \right] \right) \\ &+ \left(\frac{r_{2i}^3 + r_{2i}^2 - r_{2i}r_{2i+1}^2}{4} \right) f(x_{2i}) \\ &+ \left(\frac{r_{2i}^2 - r_{2i}^3 + r_{2i}r_{2i+1}^2}{4} \right) f(-x_{2i}) \\ &+ \left(\frac{r_{2i+1}^3 + r_{2i+1}^2 - r_{2i}^2r_{2i+1}}{4} \right) f(x_{2i+1}) \\ &+ \left(\frac{r_{2i+1}^2 - r_{2i+1}^3 + r_{2i}^2r_{2i+1}}{4} \right) f(-x_{2i+1}) \end{aligned} \tag{1.7}$$

where $r_{2i}, r_{2i+1} \in \mathbb{R} - \{0\}$, $(i = 0, 1, 2, \dots, n)$ and n is a positive integer in Random normed spaces .

In Section 2, the general solution of the functional equation (1.7) is given, In Section 3, basic definition and preliminaries of Random normed space is present, In Section 4, the generalized Ulam - Hyers stability of the functional equation (1.7) is proved via Hyers method.



2. General Solution

In this section, we present the general solution of the functional equation (1.7). Through out this section let X and Y be real vector spaces.

Lemma 2.1. *An even function $f : X \rightarrow Y$ satisfies the quadratic functional equation (1.1) if and only if $f : X \rightarrow Y$ satisfies the functional equation (1.7) for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$.*

Proof. Assume $f : X \rightarrow Y$ satisfies the functional equation (1.7). Using evenness of f in (1.7), we arrive

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n [r_{2i}^2 f(x_{2i}) + r_{2i+1}^2 f(x_{2i+1}) + \\ & \quad \frac{r_{2i}r_{2i+1}}{2} (f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1}))] \end{aligned} \quad (2.1)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$. Substituting $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(0, 0, \dots, 0, 0)$ in (2.1), we get $f(0) = 0$.

Replacing $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(x, y, 0, \dots, 0, 0)$ in (2.1), we have

$$\begin{aligned} f(r_0x + r_1y) &= r_0^2 f(x) + r_1^2 f(y) \\ &+ \frac{r_0r_1}{2} [f(x+y) - f(x-y)] \end{aligned} \quad (2.2)$$

for all $x, y \in X$. If we put y by 0 in (2.2), we obtain

$$f(r_0x) = r_0^2 f(x) \quad (2.3)$$

for all $x \in X$. Again, if we put x by 0 in (2.2) and using evenness of f , we get

$$f(r_1y) = r_1^2 f(y) \quad (2.4)$$

for all $y \in X$. Setting y by $-y$ and using evenness of f in (2.2), we reach

$$\begin{aligned} f(r_0x - r_1y) &= r_0^2 f(x) + r_1^2 f(y) \\ &+ \frac{r_0r_1}{2} [f(x-y) - f(x+y)] \end{aligned} \quad (2.5)$$

for all $x, y \in X$. Adding (2.2) and (2.5), we arrive

$$\begin{aligned} f(r_0x + r_1y) + f(r_0x - r_1y) &= 2r_0^2 f(x) \\ &+ 2r_1^2 f(y) \end{aligned} \quad (2.6)$$

for all $x, y \in X$. Using (2.3) and (2.4) in (2.6), we have

$$\begin{aligned} f(r_0x + r_1y) + f(r_0x - r_1y) &= 2f(r_0x) \\ &+ 2f(r_1y) \end{aligned} \quad (2.7)$$

for all $x, y \in X$. Finally, replacing (x, y) by $(\frac{x}{r_0}, \frac{y}{r_1})$ in (2.7), we arrive (1.1). Conversely, Let $f : X \rightarrow Y$ satisfies (1.1).

Letting $x = y = 0$ in (1.1), we get $f(0) = 0$. Replacing y by x in (1.1), we get

$$f(2x) = 4f(x) \quad (2.8)$$

for all $x \in X$. In general for any positive integer a , we have

$$f(ax) = a^2 f(x) \quad (2.9)$$

for all $x \in X$. By [[1, 19]], there exists a unique symmetric bi-additive mapping $B : X \times X \rightarrow Y$ such that $f(x) = B(x, x)$ for all $x \in X$ and

$$B(x, y) = \frac{1}{4} [f(x+y) - f(x-y)] \quad (2.10)$$

for all $x, y \in X$. Hence, for $(i = 0, 1, 2 \dots n)$, we have

$$\begin{aligned} & f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1}) \\ &= B(r_{2i}x_{2i} + r_{2i+1}x_{2i+1}, r_{2i}x_{2i} + r_{2i+1}x_{2i+1}) \\ &= B(r_{2i}x_{2i}, r_{2i}x_{2i} + r_{2i+1}x_{2i+1}) \\ & \quad + B(r_{2i+1}x_{2i+1}, r_{2i}x_{2i} + r_{2i+1}x_{2i+1}) \\ &= B(r_{2i}x_{2i}, r_{2i}x_{2i}) + B(r_{2i}x_{2i}, r_{2i+1}x_{2i+1}) \\ & \quad + B(r_{2i+1}x_{2i+1}, r_{2i}x_{2i}) + B(r_{2i+1}x_{2i+1}, r_{2i+1}x_{2i+1}) \\ &= r_{2i}r_{2i}B(x_{2i}, x_{2i}) + r_{2i}r_{2i+1}B(x_{2i}, x_{2i+1}) \\ & \quad + r_{2i+1}r_{2i}B(x_{2i+1}, x_{2i}) + r_{2i+1}r_{2i+1}B(x_{2i+1}, x_{2i+1}) \\ &= r_{2i}^2 B(x_{2i}, x_{2i}) + r_{2i}r_{2i+1}B(x_{2i}, x_{2i+1}) \\ & \quad + r_{2i+1}r_{2i}B(x_{2i+1}, x_{2i}) + r_{2i+1}^2 B(x_{2i+1}, x_{2i+1}) \\ &= r_{2i}^2 B(x_{2i}, x_{2i}) + r_{2i}r_{2i+1}B(x_{2i}, x_{2i+1}) \\ & \quad + r_{2i}r_{2i+1}B(x_{2i}, x_{2i+1}) + r_{2i+1}^2 B(x_{2i+1}, x_{2i+1}) \\ &= r_{2i}^2 B(x_{2i}, x_{2i}) + 2r_{2i}r_{2i+1}B(x_{2i}, x_{2i+1}) \\ & \quad + r_{2i+1}^2 B(x_{2i+1}, x_{2i+1}) \\ &= r_{2i}^2 f(x_{2i}) + \frac{r_{2i}r_{2i+1}}{2} (f(x_{2i} + x_{2i+1}) \\ & \quad - f(x_{2i} - x_{2i+1})) + r_{2i+1}^2 f(x_{2i+1}) \end{aligned} \quad (2.11)$$

for all $x_{2i}, x_{2i+1} \in X$. Thus, from (2.11) that

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n [r_{2i}^2 f(x_{2i}) + r_{2i+1}^2 f(x_{2i+1}) \\ & \quad + \frac{r_{2i}r_{2i+1}}{2} (f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1}))] \end{aligned} \quad (2.12)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$. Since f is an even function, it can be written as

$$f(x) = \frac{1}{2} (f(x) + f(-x)) \quad (2.13)$$



for all $x \in X$. With the help of (2.13), (2.12) can be written as $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(x_0, x_1, 0, \dots, 0, 0)$ in (2.16), we have

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left[\frac{r_{2i}^2}{2} [f(x_{2i}) + f(-x_{2i})] \right. \\ & \quad + \frac{r_{2i+1}^2}{2} [f(x_{2i+1}) + f(-x_{2i+1})] \\ & \quad + \frac{r_{2i}r_{2i+1}}{4} [f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1}) \\ & \quad \left. + f(-(x_{2i} + x_{2i+1})) - f(-(x_{2i} - x_{2i+1}))] \right] \end{aligned} \quad (2.14)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$. Adding the following terms

$$\begin{aligned} & \sum_{i=0}^n f(r_{2i}r_{2i+1}^2 - r_{2i}^3)f(x_{2i}), \\ & \sum_{i=0}^n f(r_{2i}^2r_{2i+1} - r_{2i+1}^3)f(x_{2i+1}), \\ & \sum_{i=0}^n \frac{r_{2i}r_{2i+1}^2}{2} [f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1})], \\ & \sum_{i=0}^n \frac{r_{2i}^2r_{2i+1}}{2} [f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1})] \end{aligned} \quad (2.15)$$

on both sides of (2.14), and using evenness of f , we reach (1.7) for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$. Hence the proof is complete. \square

Lemma 2.2. An odd function $f : X \rightarrow Y$ satisfies the cubic functional equation (1.3) where $r \neq \pm 1, 0$, s are real numbers if and only if $f : X \rightarrow Y$ satisfies the functional equation (1.7) for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$.

Proof. Assume $f : X \rightarrow Y$ satisfies the functional equation (1.7). Using oddness of f in (1.7), we arrive

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left[\frac{r_{2i}r_{2i+1}^2}{2} (f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1})) \right. \\ & \quad + \frac{r_{2i}^2r_{2i+1}}{2} (f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1})) \\ & \quad \left. - (r_{2i}r_{2i+1}^2 - r_{2i}^3)f(x_{2i}) - (r_{2i}^2r_{2i+1} - r_{2i+1}^3)f(x_{2i+1}) \right] \end{aligned} \quad (2.16)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in X$. Substituting $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(0, \dots, 0, 0)$ in (2.16), we get $f(0) = 0$. Replacing

$$\begin{aligned} f(r_0x_0 + r_1x_1) &= \frac{r_0r_1^2}{2} [f(x_0 + x_1) + f(x_0 - x_1)] \\ & \quad + \frac{r_0^2r_1}{2} [f(x_0 + x_1) - f(x_0 - x_1)] \\ & \quad - (r_0r_1^2 - r_0^3)f(x_0) - (r_0^2r_1 - r_1^3)f(x_1) \end{aligned} \quad (2.17)$$

for all $x_0, x_1 \in X$. If we put x_1 by 0 in (2.17), we obtain

$$f(r_0x_0) = r_0^3f(x_0) \quad (2.18)$$

for all $x_0 \in X$. Again, if we put x_0 by 0 in (2.17) and using oddness of f , we get

$$f(r_1x_1) = r_1^3f(x_1) \quad (2.19)$$

for all $x_1 \in X$. Setting x_1 by $-x_1$ and using oddness of f in (2.17), we reach

$$\begin{aligned} f(r_0x_0 - r_1x_1) &= \frac{r_0r_1^2}{2} [f(x_0 - x_1) + f(x_0 + x_1)] \\ & \quad + \frac{r_0^2r_1}{2} [f(x_0 - x_1) - f(x_0 + x_1)] \\ & \quad - (r_0r_1^2 - r_0^3)f(x_0) - (r_0^2r_1 + r_1^3)f(x_1) \end{aligned} \quad (2.20)$$

for all $x_0, x_1 \in X$. Adding (2.17) and (2.20), we arrive (1.3) in the form of

$$\begin{aligned} f(r_0x_0 + r_1x_1) + f(r_0x_0 - r_1x_1) &= r_0r_1^2f(x_0 + x_1) \\ & \quad + r_0r_1^2f(x_0 - x_1) + 2r_0(r_0^2 - r_1^2)f(x_0) \end{aligned} \quad (2.21)$$

for all $x_0, x_1 \in X$. where $r_0 \neq \pm 1, 0$, r_1 are real numbers. Hence f Satisfies (1.3).

Again, replacing $(x_0, x_1, x_2, x_3, \dots, x_{2n}, x_{2n+1})$ by $(0, 0, x_2, x_3, \dots, 0, 0)$ in (2.16), we have

$$\begin{aligned} f(r_2x_2 + r_3x_3) &= \frac{r_2r_3^2}{2} [f(x_2 + x_3) + f(x_2 - x_3)] \\ & \quad + \frac{r_2^2r_3}{2} [f(x_2 + x_3) - f(x_2 - x_3)] \\ & \quad - (r_2r_3^2 - r_2^3)f(x_2) - (r_2^2r_3 - r_3^3)f(x_3) \end{aligned} \quad (2.22)$$

for all $x_2, x_3 \in X$. It follows from the steps (2.18) to (2.21), we have (1.3) in the form of

$$\begin{aligned} f(r_2x_2 + r_3x_3) + f(r_2x_2 - r_3x_3) &= r_2r_3^2f(x_2 + x_3) \\ & \quad + r_2r_3^2f(x_2 - x_3) + 2r_2(r_2^2 - r_3^2)f(x_2) \end{aligned} \quad (2.23)$$

for all $x_2, x_3 \in X$. where $r_2 \neq \pm 1, 0$ and r_3 are real numbers. By continuing this manner, finally replacing $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by



$(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x_{2n}, x_{2n+1})$
in (2.16), we have

$$\begin{aligned} & f(r_{2n}x_{2n} + r_{2n+1}x_{2n+1}) \\ &= \frac{r_{2n}r_{2n+1}^2}{2} [f(x_{2n} + x_{2n+1}) + f(x_{2n} - x_{2n+1})] \\ & \quad + \frac{r_{2n}^2r_{2n+1}}{2} [f(x_{2n} + x_{2n+1}) - f(x_{2n} - x_{2n+1})] \\ & \quad - (r_{2n}r_{2n+1}^2 - r_{2n}^3)f(x_{2n}) \\ & \quad - (r_{2n}^2r_{2n+1} - r_{2n+1}^3)f(x_{2n+1}) \end{aligned} \quad (2.24)$$

for all $x_{2n}, x_{2n+1} \in X$. Again, It follows from the steps (2.18) to (2.21), we have (1.3) in the mold of

$$\begin{aligned} & f(r_{2n}x + r_{2n+1}y) + f(r_{2n}x - r_{2n+1}y) \\ &= r_{2n}r_{2n}^2f(x+y) + r_{2n}r_{2n+1}^2f(x-y) \\ & \quad + 2r_{2n}(r_{2n}^2 - r_{2n+1}^2)f(x) \end{aligned} \quad (2.25)$$

for all $x_{2n}, x_{2n+1} \in X$.

where $r_{2n} \neq \pm 1, 0$ and r_{2n+1} are real numbers.

Conversely, assume $f : X \rightarrow Y$ satisfies the functional equation (1.3). Substituting (x_0, x_1) by $(0, 0)$ in (2.21), we get $f(0) = 0$. Replacing (x_0, x_1) by $(x_0, 0)$ in (2.21), we have

$$f(r_0x_0) = r_0^3f(x_0) \quad (2.26)$$

for all $x_0 \in X$. Replacing x_0 by r_1x_1 and x_1 by r_0x_0 in (2.21), and dividing the resultant by $r_0r_1^2$, we obtain

$$\begin{aligned} & f(r_1x_1 + r_0x_0) + f(r_1x_1 - r_0x_0) \\ &= r_0^2r_1 [f(x_1 + x_0) + f(x_1 - x_0)] \\ & \quad + 2r_1(r_0^2 - r_1^2)f(x_1) \end{aligned} \quad (2.27)$$

for all $x_0, x_1 \in X$. Again, replacing x_0 by x_1 and x_1 by x_0 and using oddness of f in (2.27), we get

$$\begin{aligned} & f(r_0x_0 + r_1x_1) = f(r_0x_0 + r_1x_1) \\ & \quad - r_0^2r_1 [f(x_0 + x_1) - f(x_0 - x_1)] \\ & \quad + 2r_1(r_0^2 - r_1^2)f(x_1) \end{aligned} \quad (2.28)$$

for all $x_0, x_1 \in X$. Substituting (2.28) in (2.27), we arrive

$$\begin{aligned} & f(r_0x_0 + r_1x_1) = \frac{r_0r_1^2}{2} [f(x_0 + x_1) + f(x_0 - x_1)] \\ & \quad + \frac{r_0^2r_1}{2} [f(x_0 + x_1) - f(x_0 - x_1)] \\ & \quad - (r_0r_1^2 - r_0^3)f(x_0) - (r_0^2r_1 - r_1^3)f(x_1) \end{aligned} \quad (2.29)$$

for all $x_0, x_1 \in X$. By applying the procedure from (2.26) to (2.29), in (2.23) and (2.25), we have the following equations

$$\begin{aligned} & f(r_2x_2 + r_3x_3) = \frac{r_2r_3^2}{2} [f(x_2 + x_3) + f(x_2 - x_3)] \\ & \quad + \frac{r_2^2r_3}{2} [f(x_2 + x_3) - f(x_2 - x_3)] \\ & \quad - (r_2r_3^2)f(x_2) - (r_2^2r_3 - r_3^3)f(x_3) \end{aligned} \quad (2.30)$$

for all $x_2, x_3 \in X$, Finally

$$\begin{aligned} & f(r_{2n}x_{2n} + r_{2n+1}x_{2n+1}) \\ &= \frac{r_{2n}r_{2n+1}^2}{2} [f(x_{2n} + x_{2n+1}) + f(x_{2n} - x_{2n+1})] \\ & \quad + \frac{r_{2n}^2r_{2n+1}}{2} [f(x_{2n} + x_{2n+1}) - f(x_{2n} - x_{2n+1})] \\ & \quad - (r_{2n}r_{2n+1}^2 - r_{2n}^3)f(x_{2n}) \\ & \quad - (r_{2n}^2r_{2n+1} - r_{2n+1}^3)f(x_{2n+1}) \end{aligned} \quad (2.31)$$

for all $x_{2n}, x_{2n+1} \in X$. Adding (2.29), (2.30) and (2.31), we reach

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}r_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left[\frac{r_{2i}r_{2i+1}^2}{2} (f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1})) \right. \\ & \quad + \frac{r_{2i}^2r_{2i+1}}{2} (f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1})) \\ & \quad \left. - (r_{2i}r_{2i+1}^2 - r_{2i}^3)f(x_{2i}) - (r_{2i}^2r_{2i+1} - r_{2i+1}^3)f(x_{2i+1}) \right] \end{aligned} \quad (2.32)$$

for all $x_0, \dots, x_{2n}, x_{2n+1} \in X$. Since f is an odd function, it can be written as

$$f(x) = \frac{1}{2}(f(x) - f(-x)) \quad (2.33)$$

for all $x \in X$. With the help of (2.33), (2.32) can be remodify as,

$$\begin{aligned} & \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1})] \\ &= \sum_{i=0}^n \left[\frac{r_{2i}r_{2i+1}^2}{4} [((f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1}))) \right. \\ & \quad \left. - (f(-(x_{2i} + x_{2i+1})) + f(-(x_{2i} - x_{2i+1}))) \right] \\ & \quad + \frac{r_{2i}^2r_{2i+1}}{4} [((f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1}))) \\ & \quad \left. - ((f(-(x_{2i} + x_{2i+1})) - f(-(x_{2i} - x_{2i+1}))) \right] \\ & \quad - \frac{r_{2i}r_{2i+1}^2 - r_{2i}^3}{2} [f(x_{2i}) - f(-x_{2i})] \\ & \quad - \frac{(r_{2i}^2r_{2i+1} - r_{2i+1}^3)}{2} [f(x_{2i+1}) - f(-x_{2i+1})] \end{aligned} \quad (2.34)$$

for all $x_0, \dots, x_{2n}, x_{2n+1} \in X$. Adding the followings terms

$$\begin{aligned} & \sum_{i=1}^n \frac{r_{2i}^2}{2} f(x_{2i}), \\ & \sum_{i=1}^n \frac{r_{2i+1}^2}{2} f(x_{2i+1}), \\ & \sum_{i=0}^n \frac{r_{2i}r_{2i+1}}{4} (f(x_{2i} + x_{2i+1}) - f(x_{2i} - x_{2i+1})) \end{aligned} \quad (2.35)$$



on both sides of (2.34), and using oddness of f , we reach (1.7) for all $x_0, \dots, x_{2n}, x_{2n+1} \in X$. Hence the proof is completed. \square

3. Preliminaries of Random Normed Spaces

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [6, 28, 29].

Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1]$, such that F is leftcontinuous and nondecreasing on \mathbb{R} , $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $F(+\infty) = 1$, where $F^- f(x)$ denotes the left limit of the function f at the point x , that is, $F^- f(x) = \lim_{t \rightarrow x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (3.1)$$

Definition 3.1. [28] A mapping $\tau : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous triangular norm (briefly, a continuous t -norm) if τ satisfies the following conditions:

- (a) τ is commutative and associative;
- (b) τ is continuous;
- (c) $\tau(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $\tau(a, b) \leq \tau(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t -norms are $\tau_P(a, b) = ab$, $\tau_M(a, b) = \min(a, b)$ and $\tau_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t -norm). Recall (see [10, 11]) that if τ is a t -norm and x_n is a given sequence of numbers in $[0, 1]$, then $\tau_{i=1}^n x_{n+i}$ is defined recurrently by

$$\tau_{i=1}^1 x_i = x_1 \quad \text{and} \quad \tau_{i=1}^n x_i = \tau(\tau_{i=1}^{n-1} x_i, x_n) \quad \text{for } n \geq 2.$$

$\tau_{i=n}^\infty x_i$ is defined as $\tau_{i=1}^\infty x_{n+i}$. It is known [11] that, for the Lukasiewicz t -norm, the following implication holds:

$$\lim_{n \rightarrow \infty} (\tau_L)_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^\infty (1 - x_n) < \infty \quad (3.2)$$

Definition 3.2. [29] A random normed space (briefly, RN-space) is a triple (X, μ, τ) , where X is a vector space, τ is a continuous t -norm and μ is a mapping from X into D^+ satisfying the following conditions:

- (RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq \tau(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Example 3.3. Every normed spaces $(X, \|\cdot\|)$ defines a random normed space (X, μ, τ_M) , where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

and τ_M is the minimum t -norm. This space is called the induced random normed space.

Definition 3.4. Let (X, μ, τ) be a RN-space.

- (1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.
- (2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
- (3) A RN-space (X, μ, τ) is said to be complete if every Cauchy sequence in X is convergent to a point in X .

Theorem 3.5. If (X, μ, τ) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Hereafter, throughout this paper, let us consider U be a linear space and (V, μ, τ) is a complete RN-space. Define a mapping $f : U \rightarrow V$ by

$$\begin{aligned} f(x_0, x_1, \dots, x_{2n}, x_{2n+1}) &= \sum_{i=0}^n [f(r_{2i}x_{2i} + r_{2i+1}x_{2i+1}) \\ &- \sum_{i=0}^n \left(\sum_{u=0}^1 \left[\sum_{v=0}^1 \left(\frac{1}{4} \left(\begin{matrix} r_{2i}r_{2i+1}(-1)^{u+v} \\ +r_{2i}r_{2i+1}^2(-1)^u \\ +r_{2i}^2r_{2i+1}(-1)^v \end{matrix} \right) \right) \right] \right. \\ &\quad \left. f((-1)^u x_{2i} + (-1)^v x_{2i+1}) \right) \\ &- \left(\frac{r_{2i}^3 + r_{2i}^2 - r_{2i}r_{2i+1}^2}{4} \right) f(x_{2i}) \\ &- \left(\frac{r_{2i}^2 - r_{2i}^3 + r_{2i}r_{2i+1}^2}{4} \right) f(-x_{2i}) \\ &- \left(\frac{r_{2i+1}^3 + r_{2i+1}^2 - r_{2i}^2r_{2i+1}}{4} \right) f(x_{2i+1}) \\ &- \left(\frac{r_{2i+1}^2 - r_{2i+1}^3 + r_{2i}^2r_{2i+1}}{4} \right) f(-x_{2i+1}) \end{aligned}$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$.

4. Stability Results : Direct Method

In this section, the generalized Ulam -Hyers stability of the functional equation (1.7) using direct method is provided.



Theorem 4.1. Let $j = \pm 1$. Let $f : U \rightarrow V$ be an even mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ with the condition

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau_{i=0}^{\infty} \Psi \underbrace{(0, \dots, 0, \Omega^{nj}x, \Omega^{nj}x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) &= 1 \\ &= \lim_{n \rightarrow \infty} \Psi_{\Omega^{nj}x_0, \Omega^{nj}x_1, \dots, \Omega^{nj}x_{2n}, \Omega^{nj}x_{2n+1}} \left(\Omega^{2nj} t \right) \end{aligned} \quad (4.1)$$

such that the functional inequality such that

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (4.2)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : U \rightarrow V$ satisfying the functional equation (1.7) and

$$\Lambda_{Q(x)-f(x)}(t) \geq \tau_{i=0}^{\infty} \Psi \underbrace{(0, \dots, 0, \Omega^{nj}x, \Omega^{nj}x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) \quad (4.3)$$

for all $x \in U$ and all $t > 0$. The mapping $Q(x)$ is defined by

$$\Lambda_{Q(x)}(t) = \lim_{n \rightarrow \infty} \Lambda_{\frac{f(\Omega^{nj}x)}{\Omega^{2nj}}}(t) \quad (4.4)$$

for all $x \in U$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$

by $(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)$ and using evenness

of f in (4.2), we get

$$\begin{aligned} \Lambda_{f[(r_{2n}+r_{2n+1})x]-r_{2n}^2f(x)-r_{2n+1}^2f(x)-\frac{r_{2n}r_{2n+1}}{2}f(2x)}(t) \\ \geq \Psi \underbrace{(0, \dots, 0, x, x)}_{2n-1 \text{ times}}(t) \end{aligned} \quad (4.5)$$

for all $x \in U$ and all $t > 0$. Using (2.8) in (4.5), we have

$$\begin{aligned} \Lambda_{f[(r_{2n}+r_{2n+1})x]-r_{2n}^2f(x)-r_{2n+1}^2f(x)-2r_{2n}r_{2n+1}f(x)}(t) \\ \geq \Psi \underbrace{(0, \dots, 0, x, x)}_{2n-1 \text{ times}}(t) \end{aligned} \quad (4.6)$$

for all $x \in U$ and all $t > 0$. The above inequality can be written as

$$\Lambda_{f[(r_{2n}+r_{2n+1})x]-(r_{2n}+r_{2n+1})^2f(x)}(t) \geq \Psi \underbrace{(0, \dots, 0, x, x)}_{2n-1 \text{ times}}(t) \quad (4.7)$$

for all $x \in U$ and all $t > 0$. Define $\Omega = r_{2n} + r_{2n+1}$ in (4.7), it can be written as

$$\Lambda_{f(\Omega x) - (\Omega)^2 f(x)}(t) \geq \Psi \underbrace{(0, \dots, 0, x, x)}_{2n-1 \text{ times}}(t) \quad (4.8)$$

for all $x \in U$ and all $t > 0$. It follows from (4.8) and (RN2), we have

$$\Lambda_{\frac{f(\Omega x)}{\Omega^2} - f(x)} \left(\frac{t}{\Omega^2} \right) \geq \Psi \underbrace{(0, \dots, 0, x, x)}_{2n-1 \text{ times}}(t) \quad (4.9)$$

for all $x \in U$ and all $t > 0$. Replacing x by $\Omega^k x$ in (4.9), we arrive

$$\Lambda_{\frac{f(\Omega^{k+1}x)}{\Omega^2} - f(\Omega^k x)} \left(\frac{t}{\Omega^2} \right) \geq \Psi \underbrace{(0, \dots, 0, \Omega^k x, \Omega^k x)}_{2n-1 \text{ times}}(t) \quad (4.10)$$

for all $x \in U$ and all $t > 0$. It follows from (4.10) that

$$\Lambda_{\frac{f(\Omega^{k+1}x)}{\Omega^{2(k+1)}} - \frac{f(\Omega^k x)}{\Omega^{2k}}} \left(\frac{t}{\Omega^{2(k+1)}} \right) \geq \Psi \underbrace{(0, \dots, 0, \Omega^k x, \Omega^k x)}_{2n-1 \text{ times}}(t) \quad (4.11)$$

for all $x \in U$ and all $t > 0$. Substitute t by $\Omega^{2(k+1)}$ in (4.11), we arrive

$$\Lambda_{\frac{f(\Omega^{k+1}x)}{\Omega^{2(k+1)}} - \frac{f(\Omega^k x)}{\Omega^{2k}}} (t) \geq \Psi \underbrace{(0, \dots, 0, \Omega^k x, \Omega^k x)}_{2n-1 \text{ times}} \left(\Omega^{2(k+1)} t \right) \quad (4.12)$$

for all $x \in U$ and all $t > 0$. It is easy to see that

$$\frac{f(\Omega^k x)}{\Omega^{2k}} - f(x) = \sum_{i=0}^{n-1} \frac{f(\Omega^{i+1}x)}{\Omega^{2(i+1)}} - \frac{f(\Omega^i x)}{\Omega^{2i}} \quad (4.13)$$

for all $x \in U$. From equations (4.12) and (4.13), we have

$$\begin{aligned} \Lambda_{\frac{f(\Omega^k x)}{\Omega^{2k}} - f(x)}(t) \\ &= \Lambda_{\sum_{i=0}^{k-1} \frac{f(\Omega^{i+1}x)}{\Omega^{2(i+1)}} - \frac{f(\Omega^i x)}{\Omega^{2i}}}(t) \\ &\geq \tau_{i=0}^{k-1} \Lambda_{\frac{f(\Omega^{i+1}x)}{\Omega^{2(i+1)}} - \frac{f(\Omega^i x)}{\Omega^{2i}}} \left(\frac{t}{\Omega^{2(i+1)}} \right) \\ &\geq \tau_{i=0}^{k-1} \Psi \underbrace{(0, \dots, 0, \Omega^i x, \Omega^i x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) \end{aligned} \quad (4.14)$$

for all $x \in U$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{ \frac{f(\Omega^k x)}{\Omega^{2k}} \right\}$, we replace x by $\Omega^m x$ in (4.14), we arrive

$$\begin{aligned} \Lambda_{\frac{f(\Omega^{k+m}x)}{\Omega^{2(k+m)}} - f(x)}(t) \\ &\geq \tau_{i=0}^{k-1} \Psi \underbrace{(0, \dots, 0, \Omega^{i+m}x, \Omega^{i+m}x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+m+1)j} t \right) \\ &= \tau_{i=m}^{m+k-1} \Psi \underbrace{(0, \dots, 0, \Omega^i x, \Omega^i x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) \\ &\rightarrow 1 \text{ as } m \rightarrow \infty \end{aligned} \quad (4.15)$$

for all $x \in U$ and all $t > 0$. Thus $\left\{ \frac{f(\Omega^k x)}{\Omega^{2k}} \right\}$ is a Cauchy sequence. Since V is complete there exists a mapping $Q : U \rightarrow V$, we define

$$\Lambda_{Q(x)}(t) = \lim_{n \rightarrow \infty} \Lambda_{\frac{f(\Omega^k x)}{\Omega^{2k}}}(t)$$

for all $x \in U$ and all $t > 0$. Letting $m = 0$ and $n \rightarrow \infty$ in (4.14), we arrive (4.3) for all $x \in U$ and all $t > 0$. Now, we have to



show that Q satisfies (1.7), replacing $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(\Omega^k x_0, \Omega^k x_1, \dots, \Omega^k x_{2n}, \Omega^k x_{2n+1})$, we have

$$\Lambda_{f(\Omega^k x_0, \Omega^k x_1, \dots, \Omega^k x_{2n}, \Omega^k x_{2n+1})}(t) \geq \Psi_{\Omega^k x_0, \Omega^k x_1, \dots, \Omega^k x_{2n}, \Omega^k x_{2n+1}}(t) \quad (4.16)$$

for all $x \in U$ and all $t > 0$. Taking $k \rightarrow \infty$ both sides, we find that Q satisfies (1.7) for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$. Therefore the mapping $Q : X \rightarrow Y$ is Quadratic.

Finally, to prove the uniqueness of the quadratic function Q subject to (4.4), let us assume that there exist a quadratic function Q' which satisfies (4.3) and (4.4). Since $Q(\Omega^k x) = \Omega^{2k} Q(x)$ and $Q'(\Omega^k x) = \Omega^{2k} Q'(x)$ for all $x \in U$ and all $n \in \mathbb{N}$, it follows from (4.4) that

$$\begin{aligned} & \Lambda_{Q(x)-Q'(x)}(2t) \\ &= \Lambda_{Q(\Omega^k x)-Q'(\Omega^k x)}(2\Omega^{2k}t) \\ &= \Lambda_{Q(\Omega^k x)-f(\Omega^k x)+f(\Omega^k x)-Q'(\Omega^k x)}(2\Omega^{2k}t) \\ &\geq \tau \left(\Lambda_{Q(\Omega^k x)-f(\Omega^k x)}(\Omega^{2k}t), \Lambda_{f(\Omega^k x)-Q'(\Omega^k x)}(\Omega^{2k}t) \right) \\ &= \tau \left(\underbrace{\tau_{i=0}^{\infty} \Psi(0, \dots, 0, \Omega^{i+1}x, \Omega^{i+1}x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right), \right. \\ & \quad \left. \underbrace{\tau_{i=0}^{\infty} \Psi(0, \dots, 0, \Omega^{i+1}x, \Omega^{i+1}x)}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) \right) \\ &\rightarrow 1 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x \in U$ and all $t > 0$. Hence Q is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. \square

Corollary 4.2. Let Φ and s be nonnegative real numbers. Let an even function $f : U \rightarrow V$ satisfies the inequality

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \begin{cases} \Psi_{\Phi}(t), & s \neq 2; \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & s \neq \frac{2}{(2n+1)}; \\ \Psi_{\Phi(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s})}(t), & s \neq \frac{2}{(2n+1)}; \end{cases} \quad (4.17)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$.

Then there exists a unique quadratic function $Q : U \rightarrow V$ such that

$$\Lambda_{f(x)-Q(x)}(t) \leq \begin{cases} \Psi_{\frac{\Phi}{\Omega^2-1}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{\Omega^2-\Omega^{ns}}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{\Omega^2-\Omega^{(2n+1)s}}}(t), \end{cases} \quad (4.18)$$

for all $x \in U$ and all $t > 0$.

Theorem 4.3. Let $j = \pm 1$. Let $f : U \rightarrow V$ be an odd mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ with the condition

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tau_{i=0}^{\infty} \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, \Omega^{nj}x, \Omega^{nj}x) \left(\Omega^{3(i+1)j} t \right) = 1 \\ &= \lim_{n \rightarrow \infty} \Psi_{\Omega^{nj}x_0, \Omega^{nj}x_1, \dots, \Omega^{nj}x_{2n}, \Omega^{nj}x_{2n+1}} \left(\Omega^{3nj} t \right) \end{aligned} \quad (4.19)$$

such that the functional inequality such that

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (4.20)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique cubic mapping $C : U \rightarrow V$ satisfying the functional equation (1.7) and

$$\Lambda_{C(x)-f(x)}(t) \geq \tau_{i=0}^{\infty} \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, \Omega^{nj}x, \Omega^{nj}x) \left(\Omega^{3(i+1)j} t \right) \quad (4.21)$$

for all $x \in U$ and all $t > 0$. The mapping $Q(x)$ is defined by

$$\Lambda_{C(x)}(t) = \lim_{n \rightarrow \infty} \Lambda_{\frac{f(\Omega^{nj}x)}{\Omega^{3nj}}} (t) \quad (4.22)$$

for all $x \in U$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)$ and using oddness of f in (4.20), we get

$$\Lambda_{f[(r_{2n}+r_{2n+1})x] - \frac{r_{2n}r_{2n+1}^2}{2}f(2x) - \frac{r_{2n}^2r_{2n+1}}{2}f(2x)} \quad (4.23)$$

$$\begin{aligned} & - (r_{2n}r_{2n+1}^2 - r_{2n}^3)f(x) - (r_{2n}^2r_{2n+1} - r_{2n+1}^2)f(x) \left(t \right) \\ & \geq \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)(t) \end{aligned} \quad (4.24)$$

for all $x \in U$ and all $t > 0$. Using (2.18) in (4.23), we have

$$\begin{aligned} & \Lambda_{f[(r_{2n}+r_{2n+1})x] - r_{2n}^3f(x) - r_{2n+1}^3f(x) - 3r_{2n}^2r_{2n+1}f(x) - 3r_{2n}r_{2n+1}^2f(x)}(t) \\ & \geq \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)(t) \end{aligned} \quad (4.25)$$

for all $x \in U$ and all $t > 0$. The above inequality can be written as

$$\Lambda_{f[(r_{2n}+r_{2n+1})x] - (r_{2n}+r_{2n+1})^3f(x)}(t) \geq \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)(t) \quad (4.26)$$

for all $x \in U$ and all $t > 0$. Define $\Omega = r_{2n} + r_{2n+1}$ in (4.26), we get

$$\Lambda_{f(\Omega x) - (\Omega)^3 f(x)}(t) \geq \Psi(\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, x, x)(t) \quad (4.27)$$

for all $x \in U$ and all $t > 0$. \square

The rest of the proof is similar to that of Theorem 4.1. Hence the details of the proof are omitted.



Corollary 4.4. Let Φ and s be nonnegative real numbers. Let an odd function $f : U \rightarrow V$ satisfies the inequality

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \begin{cases} \Psi_{\Phi}(t), & s \neq 3; \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & s \neq \frac{3}{(2n+1)}; \\ \Psi_{\Phi(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s})}(t), & s \neq \frac{3}{(2n+1)}; \end{cases} \quad (4.28)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique cubic function $C : U \rightarrow V$ such that

$$\Lambda_{f(x)-C(x)}(t) \leq \begin{cases} \Psi_{\frac{\Phi}{|\Omega^3-1|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{|\Omega^3-\Omega^{ns}|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^3-\Omega^{(2n+1)s}|}}(t), \end{cases} \quad (4.30)$$

for all $x \in U$ and all $t > 0$.

Theorem 4.5. Let $j = \pm 1$. Let $f : U \rightarrow V$ be a mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ satisfying the conditions (4.1) and (4.19) and the functional inequality such that

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (4.31)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique quadratic function $Q : U \rightarrow V$ and a unique cubic function $C : U \rightarrow V$ satisfying the functional equation (1.7) such that

$$\begin{aligned} \mu_{Q(x)-C(x)-f(x)}(t) &\geq \tau \left\{ \tau_{i=0}^{\infty} \left(\tau \left(\Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{2(i+1)j} t), \right. \right. \\ &\quad \left. \left. \Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{2(i+1)j} t) \right) \right) \\ &\quad \left. \tau_{i=0}^{\infty} \left(\tau \left(\Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{3(i+1)j} t), \right. \right. \right. \\ &\quad \left. \left. \left. \Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{3(i+1)j} t) \right) \right) \right) \end{aligned} \quad (4.32)$$

for all $x \in U$ and all $t > 0$. The mapping $\Lambda_{Q(x)}$ and $\Lambda_{C(x)}$ are defined in (4.4) and (4.22) respectively for all $x \in U$ and all $t > 0$.

Proof. Let $f_e(x) = \frac{1}{2} \{f(x) + f(-x)\}$ for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = -f_e(x)$ for all $x \in U$. Hence

$$\Lambda_{f_e(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(2t) \geq \tau \left[\Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t), \Psi_{-x_0, -x_1, \dots, -x_{2n}, -x_{2n+1}}(t) \right] \quad (4.34)$$

all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. By Theorem 4.1, we have

$$\Lambda_{Q(x)-f_e(x)}(t) \geq \tau_{i=0}^{\infty} \left(\tau \left(\Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{2(i+1)j} t), \Psi_{\underbrace{(0, \dots, 0, -\Omega^{nj_x}, -\Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{2(i+1)j} t) \right) \right) \quad (4.35)$$

for all $x \in U$.

Also, Let $f_o(x) = \frac{1}{2} \{f(x) - f(-x)\}$ for all $x \in U$. Then $f_o(0) = 0, f_o(-x) = f_o(x)$ for all $x \in U$. Hence

$$\Lambda_{f_o(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(2t) \geq \tau \left[\Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t), \Psi_{-x_0, -x_1, \dots, -x_{2n}, -x_{2n+1}}(t) \right]$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. By Theorem 4.3, we have

$$\Lambda_{C(x)-f_o(x)}(t) \geq \tau_{i=0}^{\infty} \left(\tau \left(\Psi_{\underbrace{(0, \dots, 0, \Omega^{nj_x}, \Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{3(i+1)j} t), \Psi_{\underbrace{(0, \dots, 0, -\Omega^{nj_x}, -\Omega^{nj_x})}_{2n-1 \text{ times}}}(\Omega^{3(i+1)j} t) \right) \right) \quad (4.36)$$

for all $x \in U$ and all $t > 0$. Define

$$f(x) = f_e(x) + f_o(x) \quad (4.37)$$



for all $x \in U$. From 4.35,4.36 and 4.37, we arrive

$$\begin{aligned} & \Lambda_{Q(x)-C(x)-f(x)}(t) \\ &= \Lambda_{Q(x)-C(x)-f_o(x)-f_e(x)}(t) \\ &\geq \tau \left(\Lambda_{Q(x)-f_e(x)}(t), \Lambda_{C(x)-f_o(x)}(t) \right) \\ &\geq \tau \left\{ \tau_{i=0}^{\infty} \left(\tau \left(\underbrace{\Psi(0, \dots, 0, \Omega^{nj_x, \Omega^{nj_x}})}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right), \right. \right. \right. \\ &, \left. \left. \left. \underbrace{\Psi(0, \dots, 0, \Omega^{nj_x, \Omega^{nj_x}})}_{2n-1 \text{ times}} \left(\Omega^{2(i+1)j} t \right) \right) \right) \right\} \\ &, \tau_{i=0}^{\infty} \left(\tau \left(\underbrace{\Psi(0, \dots, 0, \Omega^{nj_x, \Omega^{nj_x}})}_{2n-1 \text{ times}} \left(\Omega^{3(i+1)j} t \right), \right. \right. \\ &, \left. \left. \left. \underbrace{\Psi(0, \dots, 0, \Omega^{nj_x, \Omega^{nj_x}})}_{2n-1 \text{ times}} \left(\Omega^{3(i+1)j} t \right) \right) \right) \right) \end{aligned}$$

for all $x \in U$ and all $t > 0$. Hence the theorem is proved. \square

Corollary 4.6. Let Φ and s be nonnegative real numbers. Let a function $f : U \rightarrow V$ satisfies the inequality

$$\begin{aligned} & \Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \\ &\geq \begin{cases} \Psi_{\Phi}(t), & s \neq 2, 3; \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & s \neq \frac{2}{(2n+1)}, \frac{3}{(2n+1)}; \\ \Psi_{\Phi(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s})}(t), & s \neq \frac{2}{(2n+1)}, \frac{3}{(2n+1)}; \end{cases} \Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \end{aligned} \quad (4.38)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique quadratic function $Q : U \rightarrow V$ and a unique cubic function $C : U \rightarrow V$ such that

$$\Lambda_{f(x)-Q(x)-C(x)}(t) \leq \begin{cases} \Psi_{\frac{\Phi}{|\Omega^2-1|} + \frac{\Phi}{|\Omega^3-1|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{|\Omega^2-\Omega^{2s}|} + \frac{2\Phi\|x\|^s}{|\Omega^3-\Omega^{3s}|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^2-\Omega^{(2n+1)s}|} + \frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^3-\Omega^{(2n+1)s}|}}(t), \end{cases} \quad (4.39)$$

for all $x \in U$ and all $t > 0$.

5. Stability Results: Fixed point Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.7) in Random normed space using fixed point method.

Through out this section, Let us consider U be a vector space and (V, Λ, T) is a complete RN-space.

Now we will recall the fundamental results in fixed point theory.

Theorem 5.1. [21](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \rightarrow X$ with Lipschitz constant L . Then, for each given element $x \in X$, either

$$(B1) \quad d(T^n x, T^{n+1} x) = \infty \quad \forall n \geq 0,$$

or

(B2) there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^n_0 x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

For to prove the stability result we define the following: δ_i is a constant such that

$$\delta_i = \begin{cases} \Omega & \text{if } i = 0, \\ \frac{1}{\Omega} & \text{if } i = 1 \end{cases}$$

and Γ is the set such that

$$\Gamma = \{g \mid g : X \rightarrow Y, g(0) = 0\}.$$

Theorem 5.2. Let $f : U \rightarrow V$ be an even mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ with the condition

$$\lim_{k \rightarrow \infty} \Psi_{\delta_i^k x_0, \delta_i^k x_1, \dots, \delta_i^k x_{2n}, \delta_i^k x_{2n+1}}(\delta_i^{2k} t) = 1 \quad (5.1)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$ and satisfying the functional inequality

$$\Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (5.2)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\Psi_D(x, t) = \Psi_{\underbrace{(0, \dots, 0, \frac{x}{\Omega}, \frac{x}{\Omega})}_{2n-1 \text{ times}}}(t) \quad (5.3)$$

has the property

$$\Psi_D(x, t) = L \frac{1}{\delta_i^2} \Psi_D(\delta_i x, t), \quad \forall x \in X, t > 0. \quad (5.4)$$

Then there exists a unique quadratic function $Q : U \rightarrow V$ satisfying the functional equation (1.7) and

$$\Lambda_{Q(x)-f(x)}\left(\frac{L^{1-i}}{1-L} t\right) \geq \Psi_D(x, t), \quad \forall x \in X, t > 0. \quad (5.5)$$

Proof. Let d be a general metric on Γ , such that

$$d(g, h) = \inf \{K \in (0, \infty) \mid \Lambda_{g(x)-h(x)}(Kt) \geq \Psi_D(x, t), x \in X, t > 0\}.$$

It is easy to see that (Γ, d) is complete. Define $T : \Gamma \rightarrow \Gamma$ by

$$Tg(x) = \frac{1}{\delta_i^2} g(\delta_i x), \quad \text{for all } x \in X. \text{ Now for } g, h \in \Gamma, \text{ we have}$$



$$d(g, h) \leq K$$

$$\begin{aligned} \Rightarrow & \Lambda_{g(x)-h(x)}(Kt) \geq \Psi_D(x, t) \\ \Rightarrow & \Lambda_{\frac{g(\delta_i x)}{\delta_i^2} - \frac{h(\delta_i x)}{\delta_i^2}}\left(\frac{Kt}{\delta_i^2}\right) \geq \Psi_D(\delta_i x, t) \\ \Rightarrow & \Lambda_{Tg(x)-Th(x)}\left(\frac{Kt}{\delta_i^2}\right) \geq \Psi_D(x, t) \\ \Rightarrow & d(Tg(x), Th(x)) \leq KL \\ \Rightarrow & d(Tg, Th) \leq Ld(g, h) \end{aligned} \quad (5.6)$$

for all $g, h \in \Gamma$. There fore T is strictly contractive mapping on Γ with Lipschitz constant L .

From (4.8) that

$$\Lambda_{f(\Omega x) - (\Omega^2)^2 f(x)}(t) \geq \Psi_{\underbrace{(0, \dots, 0)_{2n-1 \text{ times}}}}(t) \quad (5.7)$$

for all $x \in U$ and all $t > 0$. It follows from (5.7), we have

$$\Lambda_{\frac{f(\Omega x)}{\Omega^2} - f(x)}\left(\frac{t}{\Omega^2}\right) \geq \Psi_{\underbrace{(0, \dots, 0)_{2n-1 \text{ times}}}}(t) \quad (5.8)$$

for all $x \in U, t > 0$, Using (5.4) for the case $i = 0$ it reduces to

$$\begin{aligned} \Rightarrow & \Lambda_{\frac{f(\Omega x)}{\Omega^2} - f(x)}\left(\frac{t}{\Omega^2}\right) \geq \Psi_D(x, t) \\ \Rightarrow & d(Tf, f) \leq L = L^{1-0} < \infty. \end{aligned} \quad (5.9)$$

Again replacing x by $\frac{x}{\Omega}$ in (5.7), we obtain

$$\Lambda_{f(x) - \Omega^2 f\left(\frac{x}{\Omega}\right)}(t) \geq \Psi_{\underbrace{(0, \dots, 0, \frac{x}{\Omega}, \frac{x}{\Omega})}_{2n-1 \text{ times}}}(t) \quad (5.10)$$

for all $x \in U, t > 0$ with the help of (5.4) when $i = 1$, it follows from (5.10), we get

$$\begin{aligned} \Rightarrow & \Lambda_{f(x) - \Omega^2 f\left(\frac{x}{\Omega}\right)}(t) \geq \Psi_D(x, t) \\ \Rightarrow & d(f, Tf) \leq 1 = L^0 = L^{1-i} \end{aligned} \quad (5.11)$$

Then from (5.9) and (5.11), we can conclude,

$$d(f, Tf) \leq L^0 < \infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Γ such that

$$\Lambda_{Q(x)}(t) = \lim_{k \rightarrow \infty} \frac{\Lambda_{f(\delta_i^k x)}}{\delta_i^{2k}}(t), \quad \forall x \in U, t > 0. \quad (5.12)$$

Replacing $(x_0, x_1, \dots, x_{2n}, x_{2n+1})$ by $(\delta_i^k x_0, \delta_i^k x_1, \dots, \delta_i^k x_{2n}, \delta_i^k x_{2n+1})$ in (5.2) and dividing by δ_i^{2k} , it follows from (5.1) and (5.12), Q satisfies (1.7) for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$.

By fixed point alternative, since Q is unique fixed point of T in the set

$$\Delta = \{f \in \Gamma | d(f, Q) < \infty\},$$

therefore Q is a unique function such that

$$\Lambda_{f(x)-Q(x)}(Kt) \geq \Psi_D(x, t) \quad (5.13)$$

for all $x \in U, t > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} d(f, Q) & \leq \frac{1}{1-L} d(f, Tf) \\ \Rightarrow d(f, Q) & \leq \frac{L^{1-i}}{1-L} \\ \Rightarrow \Lambda_{f(x)-Q(x)}\left(\frac{L^{1-i}}{1-L}t\right) & \geq \Psi_D(x, t) \end{aligned} \quad (5.14)$$

for all $x \in U$ and $t > 0$. This completes the proof of the theorem. \square

From Theorem 5.2, we obtain the following corollary concerning the stability for the functional equation (1.7).

Corollary 5.3. Let Φ and s be nonnegative real numbers. Let an even function $f : U \rightarrow V$ satisfies the inequality

$$\begin{aligned} \Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) & \geq \begin{cases} \Psi_{\Phi}(t), \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & s \neq 2; \\ \Psi_{\Phi(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s})}(t), & s \neq \frac{2}{(2n+1)}; \end{cases} \end{aligned} \quad (5.15)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique quadratic function $Q : U \rightarrow V$ such that

$$\Lambda_{f(x)-Q(x)}(t) \leq \begin{cases} \Psi_{\frac{\Phi}{|\Omega^2-1|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{|\Omega^2-\Omega^{2s}|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^2-\Omega^{(2n+1)s}|}}(t), \end{cases} \quad (5.16)$$

for all $x \in U$ and all $t > 0$.

Proof. Setting

$$\Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) = \begin{cases} \Psi_{\Phi}(t), \\ \Psi_{\Phi \sum_{i=1}^{2n+1} \|x_i\|^s}(t), \\ \Psi_{\Phi\left(\prod_{i=1}^{2n+1} \|x_i\|^s + \sum_{i=1}^{2n+1} \|x_i\|^{(2n+1)s}\right)}(t), \end{cases}$$



for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then,

$$\begin{aligned} & \Psi_{\frac{1}{\delta_i^{2k}}}(\delta_i^k x_0, \delta_i^k x_1, \dots, \delta_i^k x_{2n}, \delta_i^k x_{2n+1})(t) \\ &= \begin{cases} \Psi_{\frac{\Phi}{\delta_i^{2k}}}(t), \\ \Psi_{\frac{\Phi}{\delta_i^{2k}} \sum_{i=1}^{2n+1} \|\delta_i^k x_i\|^s}(t), \\ \Psi_{\frac{\Phi}{\delta_i^{2k}} \left(\prod_{i=1}^{2n+1} \|\delta_i^k x_i\|^s + \sum_{i=1}^{2n+1} \|\delta_i^k x_i\|^{(2n+1)s} \right)}(t), \end{cases} \\ &= \begin{cases} \Psi_{\Phi \delta_i^{-2k}}(t), \\ \Psi_{\Phi \delta_i^{k(s-2)} \sum_{i=1}^n \|x_i\|^s}(t), \\ \Psi_{\Phi \delta_i^{k((2n+1)s-2)} \left(\prod_{i=1}^{2n+1} \|x_i\|^s + \sum_{i=1}^{2n+1} \|x_i\|^{(2n+1)s} \right)}(t), \end{cases} \\ &= \begin{cases} \rightarrow 1 \text{ as } k \rightarrow \infty, \\ \rightarrow 1 \text{ as } k \rightarrow \infty, \\ \rightarrow 1 \text{ as } k \rightarrow \infty. \end{cases} \end{aligned}$$

Thus, (5.1) is holds. From (5.3), we have $\forall x \in U, t > 0$. Also From (5.4), we arrive

$$\begin{aligned} & \frac{1}{\delta_i^2} \beta(\delta_i x, t) \\ &= \begin{cases} \Psi_{\Phi \delta_i^{-2k}}(t), \\ \Psi_{\Phi \delta_i^{s-2} \left(\frac{2}{\Omega^{2s}} \right) \|x\|^s}(t), \\ \Psi_{\Phi \delta_i^{(2n+1)s-2} \left(\frac{2}{\Omega^{2(2n+1)s}} \right) \|x\|^{(2n+1)s}}(t). \end{cases} \\ &= \begin{cases} \delta_i^{-2k} \beta(x, t), \\ \delta_i^{s-2} \beta(x, t), \\ \delta_i^{(2n+1)s-2} \beta(x, t). \end{cases} \end{aligned}$$

Now from (5.5), we prove the following cases for conditions (i) and (ii).

Case:1 $L = \Omega^{-2}$ for $s = 0$ if $i = 0$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{-2(1-0)}}{1 - \Omega^{-2}} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{1}{\Omega^2 - 1} t \right). \end{aligned}$$

Case:2 $L = \Omega^2$ for $s = 0$ if $i = 1$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{2(1-1)}}{1 - \Omega^2} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{1}{1 - \Omega^2} t \right). \end{aligned}$$

Case:3 $L = \Omega^{s-2}$ for $s > 2$ if $i = 0$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{s-2}}{1 - \Omega^{s-2}} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^s}{\Omega^2 - \Omega^s} t \right). \end{aligned}$$

Case:4 $L = \Omega^{2-s}$ for $s < 2$ if $i = 1$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{1}{1 - \Omega^{2-s}} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^s}{\Omega^s - \Omega^2} t \right). \end{aligned}$$

Case:5 $L = \Omega^{(2n+1)s-2}$ for $s > \frac{2}{(2n+1)}$ if $i = 0$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{(2n+1)s-2}}{1 - \Omega^{(2n+1)s-2}} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{(2n+1)s}}{\Omega^2 - \Omega^{(2n+1)s}} t \right). \end{aligned}$$

Case:6 $L = \Omega^{2-(2n+1)s}$ for $s < \frac{2}{(2n+1)}$ if $i = 1$,

$$\begin{aligned} \Psi_D(x, t) &\leq \Lambda_{Q(x)-f(x)} \left(\frac{1}{1 - \Omega^{2-(2n+1)s}} t \right) \\ &= \Lambda_{Q(x)-f(x)} \left(\frac{\Omega^{(2n+1)s}}{\Omega^{(2n+1)s} - \Omega^2} t \right). \end{aligned}$$

Hence the proof is complete. \square

Theorem 5.4. Let $f : U \rightarrow V$ be an odd mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ with the condition

$$\lim_{k \rightarrow \infty} \Psi_{\delta_i^k x_0, \delta_i^k x_1, \dots, \delta_i^k x_{2n}, \delta_i^k x_{2n+1}}(\delta_i^{3k} t) = 1 \quad (5.17)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$ and satisfying the functional inequality

$$\Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (5.18)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\Psi_D(x, t) = \Psi_{\underbrace{0, \dots, 0}_{2n-1 \text{ times}}, \frac{x}{\Omega}, \frac{x}{\Omega}}(t)$$

has the property

$$\Psi_D(x, t) \leq L \frac{1}{\delta_i^3} \Psi_D(\delta_i x, t), \quad \forall x \in U, t > 0. \quad (5.19)$$

Then there exists a unique cubic function $C : X \rightarrow Y$ satisfying the functional equation (1.7) and

$$\Lambda_{C(x)-f(x)} \left(\frac{L^{1-i}}{1-L} t \right) \geq \Psi_D(x, t), \quad \forall x \in U, t > 0. \quad (5.20)$$

Corollary 5.5. Let Φ and s be nonnegative real numbers. Let an odd function $f : U \rightarrow V$ satisfies the inequality

$$\begin{aligned} & \Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \\ & \geq \begin{cases} \Psi_{\Phi}(t), & s \neq 3; \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & \\ \Psi_{\Phi \left(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s} \right)}(t), & s \neq \frac{3}{(2n+1)}; \end{cases} \end{aligned} \quad (5.21)$$



for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique cubic function $C : U \rightarrow V$ such that

$$\Lambda_{f(x)-C(x)}(t) \leq \begin{cases} \Psi_{\frac{\Phi}{|\Omega^3-1|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{|\Omega^3-\Omega^{3s}|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^3-\Omega^{(2n+1)s}|}}(t), \end{cases} \quad (5.22)$$

for all $x \in U$ and all $t > 0$.

Theorem 5.6. Let $f : U \rightarrow V$ be a mapping for which there exist a function $\Psi : U^{2n+1} \rightarrow D^+$ with the condition (5.1) and (5.17) satisfying the functional inequality with

$$\Lambda_{Df(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t) \quad (5.23)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\Psi_D(x, t) = \Psi_{\underbrace{(0, \dots, 0)}_{2n-1 \text{ times}}, \frac{x}{\Omega}, \frac{x}{\Omega}}(t)$$

has the property (5.4) and (5.19) for all $x \in U$. Then there exists a unique quadratic function $Q : U \rightarrow V$ and a unique cubic function $C : U \rightarrow V$ satisfying the functional equation (1.7) and

$$\Lambda_{f(x)-Q(x)-C(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \tau \left[\tau \left[\Psi_D\left(x, \frac{t}{2}\right), \Psi_D\left(-x, \frac{t}{2}\right) \right], \tau \left[\Psi_D\left(x, \frac{t}{2}\right), \Psi_D\left(-x, \frac{t}{2}\right) \right] \right] \quad (5.24)$$

for all $x \in U$ and all $t > 0$.

Proof. Let $f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$ for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$ for all $x \in U$. Hence

$$\Lambda_{Df_e(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(2t) \geq \tau \left[\Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t), \Psi_{-x_0, -x_1, \dots, -x_{2n}, -x_{2n+1}}(t) \right] \quad (5.25)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. By Theorem 5.2, we have

$$\Lambda_{Q(x)-f_e(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \tau \left[\Psi_D(x, t), \Psi_D(-x, t) \right], \forall x \in X, t > 0. \quad (5.26)$$

Also, Let $f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$ for all $x \in X$. Then $f_o(0) = 0, f_o(-x) = -f_o(x)$ for all $x \in U$. Hence

$$\Lambda_{Df_o(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(2t) \geq \tau \left[\Psi_{x_0, x_1, \dots, x_{2n}, x_{2n+1}}(t), \Psi_{-x_0, -x_1, \dots, -x_{2n}, -x_{2n+1}}(t) \right] \quad (5.27)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. By Theorem 5.4, we have

$$\Lambda_{C(x)-f_o(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \tau \left[\Psi_D(x, t), \Psi_D(-x, t) \right], \forall x \in X, t > 0. \quad (5.28)$$

Define

$$f(x) = f_e(x) + f_o(x) \quad (5.29)$$

for all $x \in U$. From (5.26), (5.28) and (5.29), we arrive

$$\begin{aligned} \Lambda_{Q(x)-C(x)-f(x)}\left(\frac{L^{1-i}}{1-L}2t\right) &= \Lambda_{Q(x)-C(x)-f_e(x)-f_o(x)}\left(\frac{L^{1-i}}{1-L}2t\right) \\ &\geq \tau \left(\Lambda_{Q(x)-f_e(x)}\left(\frac{L^{1-i}}{1-L}t\right), \Lambda_{C(x)-f_o(x)}\left(\frac{L^{1-i}}{1-L}t\right) \right) \\ &\geq \tau \left[\tau \left[\Psi_D\left(x, \frac{t}{2}\right), \Psi_D\left(-x, \frac{t}{2}\right) \right], \tau \left[\Psi_D(x, t), \Psi_D(-x, t) \right] \right] \end{aligned}$$

for all $x \in U$ and all $t > 0$. Hence the theorem is proved. \square

Corollary 5.7. Let Φ and s be nonnegative real numbers. If a function $f : U \rightarrow V$ satisfies the inequality

$$\Lambda_{f(x_0, x_1, \dots, x_{2n}, x_{2n+1})}(t) \geq \begin{cases} \Psi_{\Phi}(t), \\ \Psi_{\Phi \sum_{i=0}^{2n+1} \|x_i\|^s}(t), & s \neq 2, 3; \\ \Psi_{\Phi \left(\prod_{i=0}^{2n+1} \|x_i\|^s + \sum_{i=0}^{2n+1} \|x_i\|^{(2n+1)s} \right)}(t), & s \neq \frac{2}{(2n+1)}, \frac{3}{(2n+1)}; \end{cases} \quad (5.30)$$

for all $x_0, x_1, \dots, x_{2n}, x_{2n+1} \in U$ and all $t > 0$. Then there exists a unique quadratic function $Q : U \rightarrow V$ and a unique cubic function $C : U \rightarrow V$ such that

$$\Lambda_{f(x)-Q(x)-C(x)}(t) \geq \begin{cases} \Psi_{\frac{\Phi}{|\Omega^2-1|} + \frac{\Phi}{|\Omega^3-1|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^s}{|\Omega^2-\Omega^{2s}|} + \frac{2\Phi\|x\|^s}{|\Omega^3-\Omega^{3s}|}}(t), \\ \Psi_{\frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^2-\Omega^{(2n+1)s}|} + \frac{2\Phi\|x\|^{(2n+1)s}}{|\Omega^3-\Omega^{(2n+1)s}|}}(t), \end{cases} \quad (5.31)$$

for all $x \in U$ and all $t > 0$.

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