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AQ and CQ functional equations

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Abstract

In this paper, the authors test the generalized Ulam - Hyers stability of the additive-quadratic and cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x); \qquad g(2x) = 12g(x) + 4g(-x),$$

via Quasi-Beta Banach space and Intuitionistic fuzzy Banach space using direct and fixed point methods.

Keywords

Additive functional equation, quadratic functional equation, cubic functional equation, quartic functional equation, mixed additive-quadratic functional equations, mixed cubic-quartic functional equations, generalized Ulam - Hyers stability, Quasi-Beta Banach space, Intuitionistic fuzzy Banach space, fixed point.

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Contents

1. Introduction

The stability problem of functional equations originated from the question of Ulam [53] in 1940, relating to the stability of group homomorphisms. In 1941, D. H. Hyers [28] gave the first positive answer to the question of Ulam for Banach spaces. It was further generalized and interesting results obtained by number of mathematicians [2, 23, 41, 45, 48].

During the last seven decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings via various spaces and mixed type equations (see [1, 5–14, 14, 15, 17, 22, 24–27, 29–33, 36, 38, 39, 42, 46, 47, 49, 54–56, 58]).

M.Arunkumar et. al., [13] introduced and established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x), \tag{1.1}$$

and

$$g(2x) = 12g(x) + 4g(-x), \tag{1.2}$$

having solutions

$$f(x) = ax + bx^2 \qquad \text{and} \qquad g(x) = cx^3 + dx^4, \tag{1.3}$$

respectively in via Banach spaces using direct and fixed point methods.

Now, first we will recall the fundamental results in fixed point theory.

Theorem 1.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T: X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \leq Ld(x,y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $\lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x,x^*) \leq \frac{1}{1-L}\ d(x,x^*), \forall\ x \in X$.

Theorem 1.2. *[34] Suppose that for a complete generalized metric space (Ω,δ) and a strictly contractive mapping $T: \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y,Ty)$ for all $y \in \Delta$.

In Section 2 the generalized Ulam - Hyers stability of (1.1) and (1.2) are respectively proved via Quasi- beta Banach space using direct and fixed point methods.

In Section 3 the generalized Ulam - Hyers stability of (1.1) and (1.2) are respectively given via Intuitionistic fuzzy Banach space using direct and fixed point methods.

Throughout this paper, let us take the following: Define a constant J_i such that

$$J_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 1. \end{cases} \tag{1.4}$$

2. Stability Results In Quasi Beta Banach Space

2.1 Definitions and Notations On Quasi Beta Banach space

In this section, we present some basic facts concerning quasi-β-Normed spaces and some preliminary results.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:*

$(Q1)$ $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.

$(Q2)$ $\| \lambda x \| = | \lambda |^\beta . \| x \|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

$(Q3)$ There is a constant $K \geq 1$ such that $\| x+y \| \leq K(\| x \| + \| y \|)$ for all $x,y \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. *A quasi-β-Banach space is a complete quasi-β-normed space.*

Definition 2.3. *A qusi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0 < p \leq 1)$ if*

$$\| x+y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x,y \in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space.

In this section, the generalized Ulam - Hyers stability of the functional equations (1.1) and (1.2) are respectively provided using direct and fixed point methods.. Also throughout this section, let us consider $\mathscr{T}_1$ and $\mathscr{T}_2$ to be a Linear Space over $\mathbb{R}$ and quasi - beta Banach space with $||\cdot||_{\mathscr{T}_2}$. respectively.

2.2 Stability Results of (1.1): Direct Method

Theorem 2.4. *Let $j \in \{-1,1\}$ and $\Delta_{AQ}: \mathscr{T}_1 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\Delta_{AQ}\left(2^{nj}x\right)}{2^{nj}} = 0 \tag{2.1}$$

for all $x \in \mathscr{T}_1$. Let $f_a: \mathscr{T}_1 \to \mathscr{T}_2$ be an odd function satisfying the inequality

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\|_{\mathscr{T}_2} \leq \Delta_{AQ}(x) \tag{2.2}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive mapping $A: \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.1) such that

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq \frac{K^{n-1}}{2^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Delta_{AQ}(2^{kj}x)}{2^{kj}} \tag{2.3}$$

for all $x \in \mathscr{T}_1$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^{nj}x)}{2^{nj}} \tag{2.4}$$

for all $x \in \mathscr{T}_1$.

Proof. Assume $j = 1$. Using oddness of f_a in (2.2), it follows that

$$\|f_a(2x) - 2f_a(x)\|_{\mathscr{T}_2} \le \Delta_{AQ}(x)$$
$$\Longrightarrow \left\| \frac{f_a(2x)}{2} - f_a(x) \right\|_{\mathscr{T}_2} \le \frac{\Delta_{AQ}(x)}{2^{\beta}} \tag{2.5}$$

for all $x \in \mathscr{T}_1$. Now replacing x by $2x$ and dividing by 2 in (2.5), we get

$$\left\| \frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2} \right\|_{\mathscr{T}_2} \le \frac{\Delta_{AQ}(2x)}{2^{\beta} \cdot 2} \tag{2.6}$$

for all $x \in \mathscr{T}_1$. From (2.5) and (2.6), we obtain

$$\begin{aligned} &\left\| \frac{f_a(2^2x)}{2^2} - f_a(x) \right\|_{\mathscr{T}_2} \\ &\le \left\| \frac{f_a(2x)}{2} - f_a(x) \right\|_{\mathscr{T}_2} + \left\| \frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2} \right\|_{\mathscr{T}_2} \\ &\le \frac{K}{2^{\beta}} \left[\Delta_{AQ}(x) + \frac{\Delta_{AQ}(2x)}{2} \right] \end{aligned} \tag{2.7}$$

for all $x \in \mathscr{T}_1$. In general for any positive integer n, we have

$$\left\| \frac{f_a(2^nx)}{2^n} - f_a(x) \right\|_{\mathscr{T}_2} \le \frac{K^{n-1}}{2^{\beta}} \sum_{k=0}^{n-1} \frac{\Delta_{AQ}(2^kx)}{2^k} \tag{2.8}$$

for all $x \in \mathscr{T}_1$. In order to prove the convergence of the sequence

$$\left\{ \frac{f_a(2^nx)}{2^n} \right\},$$

replace x by 2^mx and dividing by 2^m in (2.8), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f_a(2^{n+m}x)}{2^{(n+m)}} - \frac{f_a(2^mx)}{2^m} \right\|_{\mathscr{T}_2} &\le \frac{K^{n-1}}{2^{\beta}} \sum_{k=0}^{n-1} \frac{\Delta_{AQ}(2^{k+m}x)}{2^{k+m\beta}} \\ &\to 0 \ \ as\ m \to \infty \end{aligned}$$

for all $x \in \mathscr{T}_1$. Hence the sequence $\left\{ \frac{f_a(2^nx)}{2^n} \right\}$ is a Cauchy sequence. Since $\mathscr{T}_2$ is complete, there exists a mapping $A : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^nx)}{2^n}, \ \forall\, x \in \mathscr{T}_1.$$

Letting $n \to \infty$ in (2.8), we see that (2.3) holds for all $x \in \mathscr{T}_1$. To prove that A satisfies (1.1), replacing x by 2^nx and dividing by 2^n in (2.2), we obtain

$$\frac{1}{2^n} \left\| f_a(2^n \cdot 2x) - 3f_a(2^nx) - f_a(-2^nx) \right\| \le \frac{1}{2^n} \Delta_{AQ}(2^nx)$$

for all $x \in \mathscr{T}_1$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$ and (2.1), we see that

$$A(2x) = 3A(x) + A(-x).$$

Hence A satisfies (1.1) for all $x \in \mathscr{T}_1$. To prove that A is unique, let $B(x)$ be another additive mapping satisfying (1.1) and (2.3), then

$$\begin{aligned} &\|A(x) - B(x)\|_{\mathscr{T}_2} \\ &= \frac{1}{2^n} \|A(2^nx) - B(2^nx)\|_{\mathscr{T}_2} \\ &\le \frac{K}{2^n} \left\{ \|A(2^nx) - f_a(2^nx)\|_{\mathscr{T}_2} + \|f_a(2^nx) - B(2^nx)\|_{\mathscr{T}_2} \right\} \\ &\le \frac{2K^n}{2^{\beta}} \sum_{k=0}^{\infty} \frac{\Delta_{AQ}(2^{k+n}x)}{2^{(k+n)}} \\ &\to 0 \ \ as\ n \to \infty \end{aligned}$$

for all $x \in \mathscr{T}_1$. Hence A is unique. Thus the theorem holds for $j = 1$.

Replacing x by $\frac{x}{2}$ in (2.5), we arrive

$$\left\| f_a(x) - 2f_a\left(\frac{x}{2}\right) \right\|_{\mathscr{T}_2} \le \frac{\Delta_{AQ}\left(\frac{x}{2}\right)}{2^{\beta}} \tag{2.9}$$

for all $x \in \mathscr{T}_1$. The rest of the proof is similar to that of case $j = 1$. Thus, for $j = -1$ also the theorem is true. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 2.4 concerning the stability of (1.1).

Corollary 2.5. *Let λ and r be nonnegative real numbers. Let an odd function $f_a : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\|_{\mathscr{T}_2} \le \begin{cases} \lambda, \\ \lambda ||x||^r, & r \neq 1; \end{cases} \tag{2.10}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive function $A : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \le \begin{cases} \dfrac{K^{n-1}2|\lambda|}{2^{\beta}}, \\ \dfrac{K^{n-1}2|\lambda|||x||^r}{|2 - 2^{r\beta}|}, \end{cases} \tag{2.11}$$

for all $x \in \mathscr{T}_1$.

Theorem 2.6. *Let $j \in \{-1, 1\}$ and $\Delta_{AQ} : \mathscr{T}_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\Delta_{AQ}\left(2^{nj}x\right)}{4^{nj}} = 0 \tag{2.12}$$

for all $x \in \mathscr{T}_1$. Let $f_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even function satisfying the inequality

$$\left\| f_q(2x) - 3f_q(x) - f_q(-x) \right\|_{\mathscr{T}_2} \le \Delta_{AQ}(x) \tag{2.13}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quadratic mapping $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.1) such that

$$\left\| f_q(x) - Q_2(x) \right\|_{\mathscr{T}_2} \le \frac{K^{n-1}}{4^{\beta}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Delta_{AQ}(2^{kj}x)}{4^{kj}} \tag{2.14}$$

for all $x \in \mathscr{T}_1$. The mapping $Q_2(x)$ is defined by

$$Q_2(x) = \lim_{n\to\infty} \frac{f_q(2^{nj}x)}{4^{nj}} \tag{2.15}$$

for all $x \in \mathscr{T}_1$.

Proof. Assume $j = 1$. Using evenness of f_q in (2.13), it follows that

$$\begin{aligned} &\left\| f_q(2x) - 4f_q(x) \right\|_{\mathscr{T}_2} \le \Delta_{AQ}(x) \\ &\Longrightarrow \left\| f_q(x) - \frac{f_q(2x)}{4} \right\|_{\mathscr{T}_2} \le \frac{\Delta_{AQ}(x)}{4^\beta} \end{aligned} \tag{2.16}$$

for all $x \in \mathscr{T}_1$. The rest of the proof is similar to that of Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 2.6 concerning the stability of (1.1).

Corollary 2.7. *Let λ and r be nonnegative real numbers. Let an even function $f_q : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\left\| f_q(2x) - 3f_q(x) - f_q(-x) \right\|_{\mathscr{T}_2} \le \begin{cases} \lambda, \\ \lambda ||x||^r, \quad r \neq 2; \end{cases} \tag{2.17}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quadratic function $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\left\| f_q(x) - Q_2(x) \right\|_{\mathscr{T}_2} \le \begin{cases} \dfrac{K^{n-1} 4|\lambda|}{3\cdot 4^\beta}, \\ \dfrac{K^{n-1} 4|\lambda| ||x||^r}{|4-2^{r\beta}|}, \end{cases} \tag{2.18}$$

for all $x \in \mathscr{T}_1$.

Theorem 2.8. *Let $j \in \{-1, 1\}$ and $\Delta_{AQ} : \mathscr{T}_1 \to [0, \infty)$ be a function with conditions (2.1) and (2.12) for all $x \in \mathscr{T}_1$. Let $f : \mathscr{T}_1 \to \mathscr{T}_2$ be a function satisfying the inequality*

$$\| f(2x) - 3f(x) - f(-x) \|_{\mathscr{T}_2} \le \Delta_{AQ}(x) \tag{2.19}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive mapping $A : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quadratic mapping $Q : \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.1) such that

$$\begin{aligned} &\| f(x) - A(x) - Q_2(x) \|_{\mathscr{T}_2} \\ &\le \frac{K^{n+1}}{2^\beta} \left[\frac{1}{2^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{2^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{2^{kj}} \right) \right. \\ &\qquad \left. + \frac{1}{4^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{4^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{4^{kj}} \right) \right] \end{aligned} \tag{2.20}$$

for all $x \in \mathscr{T}_1$. The mapping $A(x)$ and $Q_2(x)$ are defined in (2.4) and (2.15) respectively for all $x \in \mathscr{T}_1$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in \mathscr{T}_1$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in \mathscr{T}_1$. Hence

$$\| f_o(2x) - 3f_o(x) - f_o(-x) \|_{\mathscr{T}_2} \le \frac{K}{2^\beta} \{ \Delta_{AQ}(x) + \Delta_{AQ}(-x) \} \tag{2.21}$$

for all $x \in \mathscr{T}_1$. By Theorem 2.4, we have

$$\| f_o(x) - A(x) \|_{\mathscr{T}_2} \le \frac{K^n}{4^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{2^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{2^{kj}} \right) \tag{2.22}$$

for all $x \in \mathscr{T}_1$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in \mathscr{T}_1$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathscr{T}_1$. Hence

$$\| f_e(2x) - 3f_e(x) - f_e(-x) \|_{\mathscr{T}_2} \le \frac{K}{2^\beta} \{ \Delta_{AQ}(x) + \Delta_{AQ}(-x) \} \tag{2.23}$$

for all $x \in \mathscr{T}_1$. By Theorem 2.6, we have

$$\| f_e(x) - Q_2(x) \|_{\mathscr{T}_2} \le \frac{K^n}{8^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{4^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{4^{kj}} \right) \tag{2.24}$$

for all $x \in \mathscr{T}_1$. Define

$$f(x) = f_e(x) + f_o(x) \tag{2.25}$$

for all $x \in \mathscr{T}_1$. From (2.22),(2.24) and (2.25), we arrive

$$\begin{aligned} &\| f(x) - A(x) - Q_2(x) \|_{\mathscr{T}_2} \\ &= \| f_e(x) + f_o(x) - A(x) - Q_2(x) \|_{\mathscr{T}_2} \\ &\le K \left\{ \| f_o(x) - A(x) \|_{\mathscr{T}_2} + \| f_e(x) - Q_2(x) \|_{\mathscr{T}_2} \right\} \\ &\le \frac{K}{2^\beta} \left[\frac{K^n}{2^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{2^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{2^{kj}} \right) \right. \\ &\qquad \left. + \frac{K^n}{4^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\Delta_{AQ}(2^{kj}x)}{4^{kj}} + \frac{\Delta_{AQ}(-2^{kj}x)}{4^{kj}} \right) \right] \end{aligned}$$

for all $x \in \mathscr{T}_1$. Hence the theorem is proved. □

Using Corollaries 2.5 and 2.7, we have the following corollary concerning the stability of (1.1).

Corollary 2.9. *Let λ and r be nonnegative real numbers. Let a function $f : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\| f(2x) - 3f(x) - f(-x) \|_{\mathscr{T}_2} \le \begin{cases} \lambda, \\ \lambda ||x||^r, \quad r \neq 1, 2; \end{cases} \tag{2.26}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive function $A : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quadratic function $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|f(x) - A(x) - Q_2(x)\|_{\mathscr{T}_2} \leq \begin{cases} \dfrac{K^{n+1}}{2^\beta}\left\{\dfrac{2|\lambda|}{2^\beta} + \dfrac{4|\lambda|}{3\cdot 4^\beta}\right\}, \\ \dfrac{K^{n+1}}{2^\beta}\left\{\dfrac{2\lambda||x||^r}{|2-2^{r\beta}|} + \dfrac{4\lambda||x||^r}{|4-2^{r\beta}|}\right\}, \end{cases} \tag{2.27}$$

for all $x \in \mathscr{T}_1$.

2.3 Stability Results of (1.2): Direct Method

Theorem 2.10. *Let $j \in \{-1, 1\}$ and $\Delta_{CQ} : \mathscr{T}_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\Delta_{CQ}\left(2^{nj}x\right)}{8^{nj}} = 0 \tag{2.28}$$

for all $x \in \mathscr{T}_1$. Let $g_c : \mathscr{T}_1 \to \mathscr{T}_2$ be an odd function satisfying the inequality

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \tag{2.29}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic mapping $C : \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.2) such that

$$\|g_c(x) - C(x)\|_{\mathscr{T}_2} \leq \frac{K^{n-1}}{8^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Delta_{CQ}(2^{kj}x)}{8^{kj}} \tag{2.30}$$

for all $x \in \mathscr{T}_1$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty} \frac{g_c(2^{nj}x)}{8^{nj}} \tag{2.31}$$

for all $x \in \mathscr{T}_1$.

Proof. Assume $j = 1$. Using oddness of g_c in (2.29), it follows that

$$\begin{aligned} &\|g_c(2x) - 8g_c(x)\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \\ \Longrightarrow &\left\|\frac{g_c(2x)}{8} - g_c(x)\right\|_{\mathscr{T}_2} \leq \frac{\Delta_{CQ}(x)}{8^\beta} \end{aligned} \tag{2.32}$$

for all $x \in \mathscr{T}_1$. The rest of the proof is similar to that of Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 2.10 concerning the stability of (1.2).

Corollary 2.11. *Let μ and r be nonnegative real numbers. Let an odd function $g_c : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\|_{\mathscr{T}_2} \leq \begin{cases} \mu, \\ \mu||x||^r, \quad r \neq 3; \end{cases} \tag{2.33}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic function $C : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|g_c(x) - C(x)\|_{\mathscr{T}_2} \leq \begin{cases} \dfrac{K^{n-1}8|\mu|}{7\cdot 8^\beta}, \\ \dfrac{K^{n-1}8\mu||x||^r}{8^\beta|8-2^{r\beta}|}, \end{cases} \tag{2.34}$$

for all $x \in \mathscr{T}_1$.

Theorem 2.12. *Let $j \in \{-1, 1\}$ and $\Delta_{CQ} : \mathscr{T}_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\Delta_{CQ}\left(2^{nj}x\right)}{16^{nj}} = 0 \tag{2.35}$$

for all $x \in \mathscr{T}_1$. Let $g_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even function satisfying the inequality

$$\left\|g_q(2x) - 12g_q(x) - 4g_q(-x)\right\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \tag{2.36}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quartic mapping $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.2) such that

$$\left\|g_q(x) - Q_4(x)\right\|_{\mathscr{T}_2} \leq \frac{K^{n-1}}{16^\beta} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Delta_{CQ}(2^{kj}x)}{16^{kj}} \tag{2.37}$$

for all $x \in \mathscr{T}_1$. The mapping $Q_4(x)$ is defined by

$$Q_4(x) = \lim_{n\to\infty} \frac{g_q(2^{nj}x)}{16^{nj}} \tag{2.38}$$

for all $x \in \mathscr{T}_1$.

Proof. Assume $j = 1$. Using evenness of g_q in (2.36), it follows that

$$\begin{aligned} &\left\|g_q(2x) - 16g_q(x)\right\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \\ \Longrightarrow &\left\|\frac{g_q(2x)}{16} - g_q(x)\right\|_{\mathscr{T}_2} \leq \frac{\Delta_{CQ}(x)}{16^\beta} \end{aligned} \tag{2.39}$$

for all $x \in \mathscr{T}_1$. The rest of the proof similar to the Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 2.12 concerning the stability of (1.2).

Corollary 2.13. *Let μ and r be nonnegative real numbers. Let an even function $g_q : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\left\|g_q(2x) - 12g_q(x) - 4g_q(-x)\right\|_{\mathscr{T}_2} \leq \begin{cases} \mu, \\ \mu||x||^r, \quad r \neq 4; \end{cases} \tag{2.40}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quartic function $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\left\|g_q(x) - Q_4(x)\right\|_{\mathscr{T}_2} \leq \begin{cases} \dfrac{K^{n-1}16|\mu|}{15\cdot 16^\beta}, \\ \dfrac{K^{n-1}16\mu||x||^r}{16^\beta|16-2^{r\beta}|}, \end{cases} \tag{2.41}$$

for all $x \in \mathscr{T}_1$.

Theorem 2.14. *Let $j \in \{-1,1\}$ and $\Delta_{CQ} : \mathscr{T}_1 \to [0,\infty)$ be a function with conditions (2.28) and (2.35) for all $x \in \mathscr{T}_1$. Let $g : \mathscr{T}_1 \to \mathscr{T}_2$ be a function satisfying the inequality*

$$\|g(2x) - 12g(x) - 4g(-x)\|_{\mathscr{T}_2} \le \Delta_{CQ}(x) \tag{2.42}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic mapping $C : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quartic mapping $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ which satisfying (1.2) such that

$$\begin{aligned}&\|g(x) - C(x) - Q_4(x)\|_{\mathscr{T}_2}\\ &\le \frac{K^{n+1}}{2^\beta}\left[\frac{1}{8^\beta}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Delta_{CQ}(2^{kj}x)}{8^{kj}} + \frac{\Delta_{CQ}(-2^{kj}x)}{8^{kj}}\right)\right.\\ &\quad\left.+\frac{1}{16^\beta}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Delta_{CQ}(2^{kj}x)}{16^{kj}} + \frac{\Delta_{CQ}(-2^{kj}x)}{16^{kj}}\right)\right]\end{aligned} \tag{2.43}$$

for all $x \in \mathscr{T}_1$. The mapping $C(x)$ and $Q_4(x)$ are defined in (2.31) and (2.38) respectively for all $x \in \mathscr{T}_1$.

Proof. The proof of the Theorem is similar to the Theorem 2.8. □

Using Corollaries 2.11 and 2.13, we have the following corollary concerning the stability of (1.2).

Corollary 2.15. *Let μ and r be nonnegative real numbers. Let a function $g : \mathscr{T}_1 \to \mathscr{T}_2$ satisfies the inequality*

$$\|g(2x) - 12g(x) - 4g(-x)\|_{\mathscr{T}_2} \le \begin{cases} \mu, & \\ \mu||x||^r, & r \ne 3,4; \end{cases} \tag{2.44}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic function $C : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quartic function $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\begin{aligned}&\|g(x) - C(x) - Q_4(x)\|_{\mathscr{T}_2}\\ &\le \begin{cases} \dfrac{K^{n+1}}{2^\beta}\left\{\dfrac{8|\lambda|}{7\cdot 8^\beta} + \dfrac{16|\lambda|}{15\cdot 16^\beta}\right\}, \\ \dfrac{K^{n+1}}{2^\beta}\left\{\dfrac{8\lambda||x||^r}{|8-2^{r\beta}|} + \dfrac{16\lambda||x||^r}{|16-2^{r\beta}|}\right\},\end{cases}\end{aligned} \tag{2.45}$$

for all $x \in \mathscr{T}_1$.

2.4 Stability Results of (1.1): Fixed Point Method

Theorem 2.16. *Let $f_a : \mathscr{T}_1 \to \mathscr{T}_2$ be an odd mapping for which there exist a function $\Delta_{AQ} : \mathscr{T}_1 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{J_i^k}\Delta_{AQ}(J_i^k x) = 0 \tag{2.46}$$

where J_i is defined in (1.4) such that the functional inequality

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\|_{\mathscr{T}_2} \le \Delta_{AQ}(x) \tag{2.47}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right),$$

has the property

$$\frac{1}{J_i}\Delta_{CQ}^{AQ}(J_i x) = L\Delta_{CQ}^{AQ}(x). \tag{2.48}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive mapping $A : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.1) and

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \le \frac{L^{1-i}}{1-L}\Delta_{CQ}^{AQ}(x) \tag{2.49}$$

for all $x \in \mathscr{T}_1$.

Proof. Consider the set

$$\mathscr{I} = \{p/p : \mathscr{T}_1 \to \mathscr{T}_2,\ p(0) = 0\}.$$

Introduce the generalized metric on $\mathscr{I}$ as

$$d(p,q) = \inf\{M \in (0,\infty) : \| p(x) - q(x) \|_{\mathscr{T}_2} \le M\Delta_{CQ}^{AQ}(x), x \in \mathscr{T}_1\}.$$

It is easy to see that $(\mathscr{I}, d)$ is complete. Define $\Gamma : \mathscr{I} \to \mathscr{I}$ by

$$\Gamma p(x) = \frac{1}{J_i}p(J_i x),$$

for all $x \in \mathscr{T}_1$. Now $p,q \in \mathscr{I}$,

$$\begin{aligned}d(p,q) \le K &\Rightarrow \| p(x) - q(x) \|_{\mathscr{T}_2} \le M\Delta_{CQ}^{AQ}(x), x \in \mathscr{T}_1,\\ &\Rightarrow \left\|\frac{1}{J_i}p(J_i x) - \frac{1}{J_i}q(J_i x)\right\|_{\mathscr{T}_2} \le \frac{1}{J_i}M\Delta_{CQ}^{AQ}(J_i x), x \in \mathscr{T}_1,\\ &\Rightarrow \left\|\frac{1}{J_i}p(J_i x) - \frac{1}{J_i}q(J_i x)\right\|_{\mathscr{T}_2} \le LM\Delta_{CQ}^{AQ}(x), x \in \mathscr{T}_1,\\ &\Rightarrow \| \Gamma p(x) - \Gamma q(x) \|_{\mathscr{T}_2} \le LM\Delta_{CQ}^{AQ}(x), x \in \mathscr{T}_1,\\ &\Rightarrow d(p,q) \le LM.\end{aligned}$$

This implies $d(\Gamma p, \Gamma q) \le Ld(p,q)$, for all $p,q \in \mathscr{I}$. i.e., Γ is a strictly contractive mapping on $\mathscr{I}$ with Lipschitz constant L.

Using oddness of f_a in (2.47), we arrive

$$\|f_a(2x) - 2f(x)\|_{\mathscr{T}_2} \le \Delta_{AQ}(x) \tag{2.50}$$

for all $x \in \mathscr{T}_1$. It follows from (2.50) that

$$\left\|\frac{f_a(2x)}{2} - f_a(x)\right\|_{\mathscr{T}_2} \le \frac{\Delta_{AQ}(x)}{2^\beta} \tag{2.51}$$

for all $x \in \mathscr{T}_1$. Using (2.48) for the case $i = 0$ it reduces to

$$\left\|\frac{f_a(2x)}{2} - f_a(x)\right\|_{\mathscr{T}_2} \le L\ \Delta_{CQ}^{AQ}(x)$$

for all $x \in \mathscr{T}_1$,

$$\text{i.e.,}\quad d(\Gamma f_a, f_a) \le L \Rightarrow d(\Gamma f_a, f_a) \le L = L^1 < \infty. \tag{2.52}$$

Again replacing $x = \frac{x}{2}$ in (2.50), we get

$$\left\| f_a(x) - 2f_a\left(\frac{x}{2}\right)\right\|_{\mathscr{T}_2} \leq \Delta_{AQ}\left(\frac{x}{2}\right) \tag{2.53}$$

for all $x \in \mathscr{T}_1$. Using (2.48) for the case $i = 1$ it reduces to

$$\left\| f_a(x) - 2f_a\left(\frac{x}{2}\right)\right\|_{\mathscr{T}_2} \leq \Delta^{AQ}_{CQ}(x)$$

for all $x \in \mathscr{T}_1$,

$$\text{i.e.,}\quad d(f_a, \Gamma f_a) \leq 1 \Rightarrow d(f_a, \Gamma f_a) \leq 1 = L^0 < \infty. \tag{2.54}$$

From (2.52) and (2.54), we arrive

$$d(f_a, \Gamma f_a) \leq L^{1-i}.$$

Therefore (FP1) holds. By (FP2), it follows that there exists a fixed point A of Γ in $\mathscr{I}$ such that

$$A(x) = \lim_{k\to\infty} \frac{f_a(J_i^k x)}{J_i^k}, \qquad \forall\ x \in \mathscr{T}_1. \tag{2.55}$$

To order to prove $A : \mathscr{T}_1 \to \mathscr{T}_2$ is additive. Replacing x by $J_i^k x$ in (2.47) and dividing by J_i^k, it follows from (2.46) that

$$\frac{1}{J_i^k}\left\| f_a(J_i^k 2x) - 3f_a(J_i^k x) - f_a(-J_i^k x)\right\|_{\mathscr{T}_2} \leq \frac{1}{J_i^k}\Delta_{AQ}(J_i^k x)$$

for all $x \in \mathscr{T}_1$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$A(2x) = 3A(x) + A(-x)$$

i.e., A satisfies the functional equation (1.1) for all $x \in \mathscr{T}_1$.

By (FP3), A is the unique fixed point of Γ in the set

$$\Delta = \{A \in \mathscr{I} : d(f_a, A) < \infty\},$$

such that

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq K\Delta^{AQ}_{CQ}(x)$$

for all $x \in \mathscr{T}_1$ and $K > 0$. Finally by (FP4), we obtain

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq \frac{L^{1-i}}{1-L}\Delta^{AQ}_{CQ}(x)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.16 concerning the stability of (1.1).

Corollary 2.17. *Let $f_a : \mathscr{T}_1 \to \mathscr{T}_2$ be an odd mapping and there exists real numbers λ and r such that*

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda\|x\|^r, \quad r \neq 1; \end{cases} \tag{2.56}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive function $A : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & |\lambda|, \\ (ii) & \dfrac{\lambda\|x\|^r}{|2 - 2^{r\beta}|}, \end{cases} \tag{2.57}$$

for all $x \in \mathscr{T}_1$.

Proof. Setting

$$\Delta_{AQ}(x) = \begin{cases} \lambda, \\ \lambda\|x\|^r, \end{cases}$$

for all $x \in \mathscr{T}_1$. Now,

$$\frac{1}{J_i^k}\Delta_{AQ}(J_i^k x) = \begin{cases} \dfrac{\lambda}{J_i^k}, \\ \dfrac{\lambda}{J_i^k}\|J_i^k x_i\|^r, \end{cases} = \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (2.46) is holds.

But, we have $\Delta^{AQ}_{CQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right)$ has the property $L\Delta^{AQ}_{CQ}(x) = \frac{1}{J_i}\Delta^{AQ}_{CQ}(J_i x)$ for all $x \in \mathscr{T}_1$. Hence

$$\Delta^{AQ}_{CQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right) = \begin{cases} \lambda \\ \dfrac{\lambda}{2^{r\beta}}\|x\|^r. \end{cases}$$

Now,

$$\frac{1}{J_i}\Delta^{AQ}_{CQ}(J_i x) = \begin{cases} \dfrac{\lambda}{J_i}, \\ \dfrac{\lambda}{J_i}\|J_i x\|^r, \end{cases} = \begin{cases} \dfrac{\lambda}{J_i}, \\ \dfrac{\lambda}{J_i}J_i^{r\beta}\|x\|^r, \end{cases} = \begin{cases} J_i^{-1}\lambda, \\ J_i^{r\beta-1}\lambda\|x\|^r, \end{cases} = \begin{cases} J_i^{-1}\Delta^{AQ}_{CQ}(x), \\ J_i^{r\beta-1}\Delta^{AQ}_{CQ}(x). \end{cases}$$

Hence the inequality (2.48) holds either, $L = 2^{-1}$ if $i = 0$ and $L = \frac{1}{2^{-1}}$ if $i = 1$. Now from (2.49), we prove the following cases for condition (i).

Case:1 $L = 2^{-1}$ if $i = 0$

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq \frac{\left(2^{-1}\right)^{1-0}}{1 - 2^{-1}}\Delta^{AQ}_{CQ}(x) = \lambda.$$

Case:2 $L = \frac{1}{2^{-1}}$ if $i = 1$

$$\|f_a(x) - A(x)\|_{\mathscr{T}_2} \leq \frac{\left(\frac{1}{2^{-1}}\right)^{1-1}}{1 - \frac{1}{2^{-1}}}\Delta^{AQ}_{CQ}(x) = -\lambda.$$

Also the inequality (2.48) holds either, $L = 2^{r\beta-1}$ for $r < 1$ if $i = 0$ and $L = \frac{1}{2^{r\beta-1}}$ for $r > 1$ if $i = 1$. Now from (2.49), we prove the following cases for condition (ii).

Case:3 $L = 2^{r\beta-1}$ for $r < 1$ if $i = 0$

$$\begin{aligned}\|f_a(x) - A(x)\|_{\mathscr{T}_2} &\leq \frac{\left(2^{(r\beta-1)}\right)^{1-0}}{1-2^{(r\beta-1)}}\Delta_{CQ}^{AQ}(x)\\ &= \frac{2^{r\beta}}{2-2^{r\beta}}\frac{\lambda}{2^{r\beta}}||x||^r\\ &= \frac{\lambda||x||^r}{2-2^{r\beta}}.\end{aligned}$$

Case:4 $L = \frac{1}{2^{r\beta-1}}$ for $r > 1$ if $i = 1$

$$\begin{aligned}\|f_a(x) - A(x)\|_{\mathscr{T}_2} &\leq \frac{\left(\frac{1}{2^{(r\beta-1)}}\right)^{1-1}}{1-\frac{1}{2^{(r\beta-1)}}}\Delta_{CQ}^{AQ}(x)\\ &= \frac{2^{r\beta}}{2^{r\beta}-2}\frac{\lambda}{2^{r\beta}}||x||^r\\ &= \frac{\lambda||x||^r}{2^{r\beta}-2}.\end{aligned}$$

Hence the proof is complete. □

Theorem 2.18. *Let $f_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even mapping for which there exist a function $\Delta_{AQ} : \mathscr{T}_1 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{J_i^{2k}}\Delta_{AQ}(J_i^k x) = 0 \tag{2.58}$$

where J_i is defined in (1.4) such that the functional inequality

$$\left\|f_q(2x) - 3f_q(x) - f_q(-x)\right\|_{\mathscr{T}_2} \leq \Delta_{AQ}(x) \tag{2.59}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right),$$

has the property

$$L\,\Delta_{CQ}^{AQ}(x) = \frac{1}{J_i^2}\,\Delta_{CQ}^{AQ}(J_i x). \tag{2.60}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quadratic mapping $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.1) and

$$\left\|f_q(x) - Q_2(x)\right\|_{\mathscr{T}_2} \leq \frac{L^{1-i}}{1-L}\Delta_{CQ}^{AQ}(x) \tag{2.61}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the theorem is similar ideas given in Theorem 2.16 by defining a mapping $\Gamma : \mathscr{I} \to \mathscr{I}$ by

$$\Gamma p(x) = \frac{1}{J_i^2}p(J_i x),$$

for all $x \in \mathscr{T}_1$. □

The following corollary is an immediate consequence of Theorem 2.16 concerning the stability of (1.1).

Corollary 2.19. *Let $f_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even mapping and there exists real numbers λ and r such that*

$$\|f(2x) - 3f(x) - f(-x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda||x||^r, \quad r \neq 2; \end{cases} \tag{2.62}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quadratic function $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\left\|f_q(x) - Q_2(x)\right\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \dfrac{\lambda}{|3|}, \\ (ii) & \dfrac{\lambda||x||^r}{|4-2^{r\beta}|}, \end{cases} \tag{2.63}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the corollary is similar lines to the of Corollary 2.17. □

Theorem 2.20. *Let $f : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping for which there exist a function $\Delta_{AQ} : E \to [0,\infty)$ with the conditions (2.46) and (2.58) where J_i is defined (1.4) such that the functional inequality*

$$\|f(2x) - 3f(x) - f(-x)\|_{\mathscr{T}_2} \leq \Delta_{AQ}(x) \tag{2.64}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right),$$

with the properties (2.48) and (2.60) for all $x \in \mathscr{T}_1$. Then there exists a unique additive mapping $A : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation and a unique quadratic mapping $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.1) and

$$\|f(x) - A(x) - Q_2(x)\|_{\mathscr{T}_2} \leq \frac{L^{1-i}}{1-L}(\Delta_{CQ}^{AQ}(x) + \Delta_{CQ}^{AQ}(-x)) \tag{2.65}$$

for all $x \in \mathscr{T}_1$.

Proof. Using definition of f_o and Theorem 2.16, we have

$$\|f_o(x) - A(x)\|_{\mathscr{T}_2} \leq \frac{K}{2^\beta}\frac{L^{1-i}}{1-L}\left(\Delta_{CQ}^{AQ}(x) + \Delta_{CQ}^{AQ}(-x)\right) \tag{2.66}$$

for all $x \in \mathscr{T}_1$. Also, using definition of f_e and Theorem 2.18, we have

$$\|f_e(x) - Q(x)\|_{\mathscr{T}_2} \leq \frac{K}{2^\beta}\frac{L^{1-i}}{1-L}\left(\Delta_{CQ}^{AQ}(x) + \Delta_{CQ}^{AQ}(-x)\right) \tag{2.67}$$

for all $x \in \mathscr{T}_1$. Define

$$f(x) = f_e(x) + f_o(x) \tag{2.68}$$

for all $x \in \mathscr{T}_1$. From (2.66),(2.67) and (2.68), we arrive

$$\begin{aligned} &\|f(x)-A(x)-Q(x)\|_{\mathscr{T}_2} \\ &= \|f_e(x)+f_o(x)-A(x)-Q(x)\|_{\mathscr{T}_2} \\ &\leq K\left\{\|f_o(x)-A(x)\|_{\mathscr{T}_2}+\|f_e(x)-Q(x)\|_{\mathscr{T}_2}\right\} \\ &\leq \frac{K^2}{2^\beta}\frac{L^{1-i}}{1-L}\left[\left(\Delta_{CQ}^{AQ}(x)+\Delta_{CQ}^{AQ}(-x)\right)\right. \\ &\qquad\qquad \left.+\left(\Delta_{CQ}^{AQ}(x)+\Delta_{CQ}^{AQ}(-x)\right)\right] \\ &\leq \frac{2K^2}{2^\beta}\cdot\frac{L^{1-i}}{1-L}\left\{\left(\Delta_{CQ}^{AQ}(x)+\Delta_{CQ}^{AQ}(-x)\right)\right\} \end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

The following corollary is an immediate consequence of Theorem 2.20, using Corollaries 2.17 and 2.19 concerning the stability of (1.1).

Corollary 2.21. *Let $f : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping and there exists real numbers λ and r such that*

$$\|f(2x)-3f(x)-f(-x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda||x||^r, \quad r \neq 1,2; \end{cases} \tag{2.69}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique additive function $A : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quadratic function $Q_2 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|f(x)-A(x)-Q_2(x)\|_{\mathscr{T}_2} \leq \begin{cases} \dfrac{2K^2|\lambda|}{2^\beta}\left(1+\dfrac{1}{3}\right), \\ \dfrac{2K^2\lambda||x||^r}{2^\beta}\left(\dfrac{1}{|2-2^{r\beta}|}+\dfrac{1}{|4-2^{r\beta}|}\right), \end{cases} \tag{2.70}$$

for all $x \in \mathscr{T}_1$.

2.5 Stability Results of (1.2): Fixed Point Method

Theorem 2.22. *Let $g_c : \mathscr{T}_1 \to \mathscr{T}_2$ be an odd mapping for which there exist a function $\Delta_{CQ} : \mathscr{T}_1 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{J_i^{3k}}\Delta_{CQ}(J_i^k x) = 0 \tag{2.71}$$

where J_i is defined in (1.4) such that the functional inequality

$$\|g_c(2x)-12g_c(x)-4g_c(-x)\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \tag{2.72}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{CQ}\left(\frac{x}{2}\right),$$

has the property

$$L\Delta_{CQ}^{AQ}(x) = \frac{1}{J_i^3}\,\Delta_{CQ}^{AQ}(J_i x). \tag{2.73}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic mapping $C : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.2) and

$$\|g_c(x)-C(x)\|_{\mathscr{T}_2} \leq \frac{L^{1-i}}{1-L}\Delta_{CQ}^{AQ}(x) \tag{2.74}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the theorem is similar ideas given in Theorem 2.16 by defining a mapping $\Gamma : \mathscr{I} \to \mathscr{I}$ by

$$\Gamma p(x) = \frac{1}{J_i^3}p(J_i x),$$

for all $x \in \mathscr{T}_1$. □

The following corollary is an immediate consequence of Theorem 2.22 concerning the stability of (1.2).

Corollary 2.23. *Let $g_c : \mathscr{T}_1 \to \mathscr{T}_2$ be an odd mapping and there exists real numbers μ and r such that*

$$\|g_c(2x)-12g_c(x)-4g_c(-x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \mu, \\ (ii) & \mu||x||^r, \quad r \neq 3; \end{cases} \tag{2.75}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic function $C : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\|g_c(x)-C(x)\|_{\mathscr{T}_2} \leq \begin{cases} (i) & \dfrac{\mu}{|7|}, \\ (ii) & \dfrac{\mu||x||^r}{|8-2^{r\beta}|}, \end{cases} \tag{2.76}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the corollary is similar lines to the of Corollary 2.17. □

Theorem 2.24. *Let $g_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even mapping for which there exist a function $\Delta_{CQ} : \mathscr{T}_1 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty}\frac{1}{J_i^{4k}}\Delta_{CQ}(J_i^k x) = 0 \tag{2.77}$$

where J_i is defined in (1.4) such that the functional inequality

$$\left\|g_q(2x)-12g_q(x)-4g_q(-x)\right\|_{\mathscr{T}_2} \leq \Delta_{CQ}(x) \tag{2.78}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{CQ}\left(\frac{x}{2}\right),$$

has the property

$$L\Delta_{CQ}^{AQ}(x) = \frac{1}{J_i^4}\,\Delta_{CQ}^{AQ}(J_i x). \tag{2.79}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quartic mapping $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.2) and

$$\left\|g_q(x)-Q_4(x)\right\|_{\mathscr{T}_2} \leq \frac{L^{1-i}}{1-L}\Delta_{CQ}^{AQ}(x) \tag{2.80}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the theorem is similar ideas given in Theorem 2.16 by defining a mapping $\Gamma : \mathscr{I} \to \mathscr{I}$ by

$$\Gamma p(x) = \frac{1}{J_i^4} p(J_i x),$$

for all $x \in \mathscr{T}_1$. □

The following corollary is an immediate consequence of Theorem 2.24 concerning the stability of (1.2).

Corollary 2.25. *Let $g_q : \mathscr{T}_1 \to \mathscr{T}_2$ be an even mapping and there exists real numbers μ and r such that*

$$\left\| g_q(2x) - 12 g_q(x) - 4 g_q f(-x) \right\|_{\mathscr{T}_2} \le \begin{cases} (i) & \mu, \\ (ii) & \mu ||x||^r, \quad r \neq 4; \end{cases} \tag{2.81}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique quartic function $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\left\| g_q(x) - Q_2(x) \right\|_{\mathscr{T}_2} \le \begin{cases} (i) & \dfrac{\mu}{|15|}, \\ (ii) & \dfrac{\mu ||x||^r}{|16 - 2^{r\beta}|}, \end{cases} \tag{2.82}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the corollary is similar lines to the of Corollary 2.17. □

Theorem 2.26. *Let $g : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping for which there exist a function $\Delta_{CQ} : \mathscr{T}_1 \to [0, \infty)$ with the conditions (2.71) and (2.77) where J_i is defined (1.4) such that the functional inequality*

$$\| g(2x) - 12 g(x) - 4 g(-x) \|_{\mathscr{T}_2} \le \Delta_{CQ}(x) \tag{2.83}$$

for all $x \in \mathscr{T}_1$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Delta_{CQ}^{AQ}(x) = \Delta_{CQ}\left(\frac{x}{2}\right),$$

with the properties (2.73) and (2.79) for all $x \in \mathscr{T}_1$. Then there exists a unique cubic mapping $C : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation and a unique quartic mapping $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ satisfying the functional equation (1.2) and

$$\| g(x) - C(x) - Q_4(x) \|_{\mathscr{T}_2} \le \frac{2K^2}{2^\beta} \frac{L^{1-i}}{1-L} (\Delta_{CQ}^{AQ}(x) + \Delta_{CQ}^{AQ}(-x)) \tag{2.84}$$

for all $x \in \mathscr{T}_1$.

Proof. The proof of the Theorem is similar to the Theorem 2.20. □

The following Corollary is an immediate consequence of Theorem 2.26, using Corollaries 2.23 and 2.25 concerning the stability of (1.2).

Corollary 2.27. *Let $g : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping and there exists real numbers μ and r such that*

$$\| g(2x) - 12 g(x) - 4 g(-x) \|_{\mathscr{T}_2} \le \begin{cases} (i) & \mu, \\ (ii) & \mu ||x||^r, \quad r \neq 2, 4; \end{cases} \tag{2.85}$$

for all $x \in \mathscr{T}_1$. Then there exists a unique cubic function $C : \mathscr{T}_1 \to \mathscr{T}_2$ and a unique quartic function $Q_4 : \mathscr{T}_1 \to \mathscr{T}_2$ such that

$$\begin{aligned} &\| g(x) - C(x) - Q_4(x) \|_{\mathscr{T}_2} \\ &\le \begin{cases} \dfrac{2K^2 |\mu|}{2^\beta} \left(\dfrac{1}{7} + \dfrac{1}{15} \right), \\ \dfrac{2K^2 \mu ||x||^r}{2^\beta} \left(\dfrac{1}{|8 - 2^{r\beta}|} + \dfrac{1}{|16 - 2^{r\beta}|} \right), \end{cases} \end{aligned} \tag{2.86}$$

for all $x \in \mathscr{T}_1$.

3. Stability Results In Intuitionistic Fuzzy Banach Space

3.1 Definitions and Notations of Intuitionistic Fuzzy Banach Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space.

Definition 3.1. *A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:*

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 3.2. *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:*

(1') $\diamond$ is commutative and associative;

(2') $\diamond$ is continuous;

(3') $a \diamond 0 = a$ for all $a \in [0,1]$;

(4') $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [50] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 3.3. *The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*1$ is a continuous t-norm, $\diamond$ is a continuous $t-$conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$*

(IFN1) $\mu(x,t)+\nu(x,t)\leq 1$,

(IFN2) $\mu(x,t)>0$,

(IFN3) $\mu(x,t)=1$, *if and only if* $x=0$.

(IFN4) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{\alpha}\right)$ *for each* $\alpha\neq 0$,

(IFN5) $\mu(x,t)*\mu(y,s)\leq\mu(x+y,t+s)$,

(IFN6) $\mu(x,\cdot):(0,\infty)\to[0,1]$ *is continuous,*

(IFN7) $\lim_{t\to\infty}\mu(x,t)=1$ *and* $\lim_{t\to 0}\mu(x,t)=0$,

(IFN8) $\nu(x,t)<1$,

(IFN9) $\nu(x,t)=0$, *if and only if* $x=0$.

(IFN10) $\nu(\alpha x,t)=\nu\left(x,\frac{t}{\alpha}\right)$ *for each* $\alpha\neq 0$,

(IFN11) $\nu(x,t)\diamond\nu(y,s)\geq\nu(x+y,t+s)$,

(IFN12) $\nu(x,\cdot):(0,\infty)\to[0,1]$ *is continuous,*

(IFN13) $\lim_{t\to\infty}\nu(x,t)=0$ *and* $\lim_{t\to 0}\nu(x,t)=1$.

In this case, (μ,ν) *is called an intuitionistic fuzzy norm.*

Example 3.4. *Let* $(X,\|\cdot\|)$ *be a normed space. Let* $a*b=ab$ *and* $a\diamond b=\min\{a+b,1\}$ *for all* $a,b\in[0,1]$. *For all* $x\in X$ *and every* $t>0$, *consider*

$$\mu(x,t)=\begin{cases}\frac{t}{t+\|x\|} & if \quad t>0;\\ 0 & if \quad t\leq 0;\end{cases}$$

and

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|} & if \quad t>0;\\ 0 & if \quad t\leq 0.\end{cases}$$

Then $(X,\mu,\nu,*,\diamond)$ *is an IFN-space.*

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are investigated in [50].

Definition 3.5. *Let* $(X,\mu,\nu,*,\diamond)$ *be an IFNS. Then, a sequence* $x=\{x_k\}$ *is said to be intuitionistic fuzzy convergent to a point* $L\in X$ *if*

$$\lim \quad \mu(x_k-L,t)=1 \quad and \quad \lim \quad \nu(x_k-L,t)=0$$

for all $t>0$. *In this case, we write*

$$x_k \xrightarrow{IF} L \qquad as \qquad k\to\infty$$

Definition 3.6. *Let* $(X,\mu,\nu,*,\diamond)$ *be an IFN-space. Then,* $x=\{x_k\}$ *is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\left(x_{k+p}-x_k,t\right)=1 \qquad and \qquad \nu\left(x_{k+p}-x_k,t\right)=0$$

for all $t>0$, *and* $p=1,2\cdots$.

Definition 3.7. *Let* $(X,\mu,\nu,*,\diamond)$ *be an IFN-space. Then* $(X,\mu,\nu,*,\diamond)$ *is said to be complete if every intuitionistic fuzzy Cauchy sequence in* $(X,\mu,\nu,*,\diamond)$ *is intuitionistic fuzzy convergent* $(X,\mu,\nu,*,\diamond)$.

In this section, the generalized Ulam - Hyers stability of the functional equations (1.1) and (1.2) are respectively provided with the help of direct and fixed point methods. Here and subsequently, assume that X is a linear space, (Z,μ',ν') is an intuitionistic fuzzy normed space and (Y,μ,ν) an intuitionistic fuzzy Banach space.

3.2 Stability Results of (1.1): Direct Method

Theorem 3.8. *Let* $j\in\{1,-1\}$. *Let* $\Delta_{AQ}:X\longrightarrow Z$ *be a function such that for some* $0<\left(\frac{p}{2}\right)^j<1$,

$$\left.\begin{aligned}\mu'\left(\Delta_{AQ}\left(2^{nj}x\right),t\right)\geq\mu'\left(p^{nj}\Delta_{AQ}(x),t\right)\\ \nu'\left(\Delta_{AQ}\left(2^{nj}x\right),t\right)\leq\nu'\left(p^{nj}\Delta_{AQ}(x),t\right)\end{aligned}\right\} \tag{3.1}$$

for all $x\in X$ *and all* $t>0$ *and*

$$\left.\begin{aligned}\lim_{n\to\infty}\mu'\left(\Delta_{AQ}\left(2^{jn}x\right),2^{jn}t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Delta_{AQ}\left(2^{jn}x\right),2^{jn}t\right)=0\end{aligned}\right\} \tag{3.2}$$

for all $x\in X$ *and all* $t>0$. *Let* $f_a:X\to Y$ *be an odd function satisfying the inequality*

$$\left.\begin{aligned}\mu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\leq\nu'\left(\Delta_{AQ}(x),t\right)\end{aligned}\right\} \tag{3.3}$$

for all $x\in X$ *and all* $t>0$. *Then there exists a unique additive mapping* $\mathscr{A}:X\longrightarrow Y$ *satisfying (1.1) and*

$$\left.\begin{aligned}\mu\left(f_a(x)-\mathscr{A}(x),t\right)\geq\mu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ \nu\left(f_a(x)-\mathscr{A}(x),t\right)\leq\nu'\left(\Delta_{AQ}(x),2|2-p|t\right)\end{aligned}\right\} \tag{3.4}$$

for all $x\in X$ *and all* $t>0$.

Proof. **Case (i):** Let $j=1$. Using oddness of f in in (3.3), we obtain

$$\left.\begin{aligned}\mu\left(f_a(2x)-2f(x),t\right)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_a(2x)-2f(x),t\right)\leq\nu'\left(\Delta_{AQ}(x),t\right)\end{aligned}\right\} \tag{3.5}$$

for all $x\in X$ and all $t>0$. Using (IFN4) and (IFN10) in (3.5), we arrive

$$\left.\begin{aligned}\mu\left(\frac{f_a(3x)}{2}-f_a(x),\frac{t}{2}\right)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(\frac{f_a(3x)}{2}-f_a(x),\frac{t}{2}\right)\leq\nu'\left(\Delta_{AQ}(x),t\right)\end{aligned}\right\} \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Substituting x by $2^n x$ in (3.6), we have

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+1}x)}{2} - f_a(2^n x), \frac{t}{2}\right) \geq \mu'\left(\Delta_{AQ}(2^n x), t\right) \\ &\nu\left(\frac{f_a(2^{n+1}x)}{2} - f_a(2^n x), \frac{t}{2}\right) \leq \nu'\left(\Delta_{AQ}(2^n x), t\right) \end{aligned}\right\} \tag{3.7}$$

for all $x \in X$ and all $t > 0$. It is easy to verify from (3.7) and using (3.1), (IFN4), (IFN10) that

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+1}x)}{2^{(n+1)}} - \frac{f_a(2^n x)}{2^n}, \frac{t}{2 \cdot 2^n}\right) \geq \mu'\left(\Delta_{AQ}(x), \frac{t}{p^n}\right) \\ &\nu\left(\frac{f_a(2^{n+1}x)}{2^{(n+1)}} - \frac{f_a(2^n x)}{2^n}, \frac{t}{2 \cdot 2^n}\right) \leq \nu'\left(\Delta_{AQ}(x), \frac{t}{p^n}\right) \end{aligned}\right\} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. Interchanging t into $p^n t$ in (3.8), we have

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+1}x)}{2^{(n+1)}} - \frac{f_a(2^n x)}{2^n}, \frac{t \cdot p^n}{2 \cdot 2^n}\right) \geq \mu'\left(\Delta_{AQ}(x), t\right) \\ &\nu\left(\frac{f_a(2^{n+1}x)}{2^{(n+1)}} - \frac{f_a(2^n x)}{2^n}, \frac{t \cdot p^n}{2 \cdot 2^n}\right) \leq \nu'\left(\Delta_{AQ}(x), t\right) \end{aligned}\right\} \tag{3.9}$$

for all $x \in X$ and all $t > 0$. It is easy to see that

$$\frac{f_a(2^n x)}{2^n} - f_a(x) = \sum_{i=0}^{n-1} \frac{f_a(2^{i+1}x)}{2^{(i+1)}} - \frac{f_a(2^i x)}{2^i} \tag{3.10}$$

for all $x \in X$. It follows from (3.9) and (3.10), we get

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \\ &= \mu\left(\sum_{i=0}^{n-1} \frac{f_a(2^{i+1}x)}{2^{(i+1)}} - \frac{f_a(2^i x)}{2^i}, \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \\ &\nu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \\ &= \nu\left(\sum_{i=0}^{n-1} \frac{f_a(2^{i+1}x)}{2^{(i+1)}} - \frac{f_a(2^i x)}{2^i}, \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \end{aligned}\right\} \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Using (IFNS5) and (IFNA11) in (3.11), we have

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i t}{2 \cdot 2^i}\right) \\ &\geq \textstyle\prod_{i=0}^{n-1} \mu\left(\frac{f_a(2^{i+1}x)}{2^{(i+1)}} - \frac{f_a(2^i x)}{2^i}, \frac{p^i\, tr}{2 \cdot 2^i}\right) \\ &\nu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i t}{2 \cdot 2^i}\right) \\ &\leq \textstyle\prod_{i=0}^{n-1} \nu\left(\frac{f_a(2^{i+1}x)}{2^{(i+1)}} - \frac{f_a(2^i x)}{2^i}, \frac{p^i\, t}{2 \cdot 2^i}\right) \end{aligned}\right\} \tag{3.12}$$

where

$$\prod_{i=0}^{n-1} c_j = c_1 * c_2 * \cdots * c_n$$

and

$$\prod_{i=0}^{n-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_n$$

for all $x \in X$ and all $t > 0$. Hence

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \\ &\geq \textstyle\prod_{i=0}^{n-1} \mu'\left(\Delta_{AQ}(x), t\right) = \mu'\left(\Delta_{AQ}(x), t\right) \\ &\nu\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^i}\right) \\ &\leq \textstyle\prod_{i=0}^{n-1} \nu'\left(\Delta_{AQ}(x), t\right) = \nu'\left(\Delta_{AQ}(x), t\right) \end{aligned}\right\} \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^m x$ in (3.13) and using (3.2), (IFN4), (IFN10), we obtain

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^{(i+m)}}\right) \\ &\geq \mu'\left(\Delta_{AQ}(2^m x), t\right) = \mu'\left(\Delta_{AQ}(x), \tfrac{t}{p^m}\right) \\ &\nu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^i\, t}{2 \cdot 2^{(i+m)}}\right) \\ &\leq \nu'\left(\Delta_{AQ}(2^m x), t\right) = \nu'\left(\Delta_{AQ}(x), \tfrac{t}{p^m}\right) \end{aligned}\right\} \tag{3.14}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Replacing t by $p^m t$ in (3.14), we get

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\, t}{2 \cdot 2^{(i+m)}}\right) \geq \mu'\left(\Delta_{AQ}(x), t\right) \\ &\nu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\, t}{2 \cdot 2^{(i+m)}}\right) \leq \nu'\left(\Delta_{AQ}(x), t\right) \end{aligned}\right\} \tag{3.15}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. The relation (3.14) implies that

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}}-\frac{f_a(2^m x)}{2^m},t\right) \\ &\geq \mu'\left(\Delta_{AQ}(x),\frac{t}{\sum_{i=m}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \\ &\nu\left(\frac{f_a(2^{n+m}x)}{2^{(n+m)}}-\frac{f_a(2^m x)}{2^m},t\right) \\ &\leq \nu'\left(\Delta_{AQ}(x),\frac{t}{\sum_{i=m}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \end{aligned}\right\} \tag{3.16}$$

holds for all $x\in X$ and all $t>0$ and all $m,n\geq 0$. Since $0<p<2$ and $\sum_{i=0}^{n}\left(\frac{p}{2}\right)^i<\infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{f_a(2^n x)}{2^n}\right\}$ is Cauchy in (Y,μ,ν). Since (Y,μ,ν) is a complete IFN-space this sequence converges to some point $\mathscr{A}(x)\in Y$. So, one can define the mapping $\mathscr{A}:X\longrightarrow Y$ by

$$\lim_{n\to\infty}\mu\left(\frac{f_a(2^n x)}{2^n}-\mathscr{A}(x),t\right)=1,$$

$$\lim_{n\to\infty}\nu\left(\frac{f_a(2^n x)}{2^n}-\mathscr{A}(x),t\right)=0$$

for all $x\in X$ and all $t>0$. Hence

$$\frac{f_a(2^n x)}{2^n}\xrightarrow{IF}\mathscr{A}(x),\quad as\quad n\to\infty.$$

Letting $m=0$ in (3.15), we arrive

$$\left.\begin{aligned} &\mu\left(\frac{f_a(2^n x)}{2^n}-f_a(x),t\right)\geq\mu'\left(\Delta_{AQ}(x),\frac{t}{\sum_{i=0}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \\ &\nu\left(\frac{f_a(2^n x)}{2^n}-f_a(x),t\right)\leq\nu'\left(\Delta_{AQ}(x),\frac{t}{\sum_{i=0}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \end{aligned}\right\} \tag{3.17}$$

for all $x\in X$ and all $t>0$. Letting $n\to\infty$ in (3.17), we arrive

$$\left.\begin{aligned} &\mu\left(\mathscr{A}(x)-f_a(x),t\right)\geq\mu'\left(\Delta_{AQ}(x),2\,t|2-p|\right) \\ &\nu\left(\mathscr{A}(x)-f_a(x),t\right)\leq\nu'\left(\Delta_{AQ}(x),2\,t|2-p|\right) \end{aligned}\right\} \tag{3.18}$$

for all $x\in X$ and all $t>0$. To prove $\mathscr{A}$ satisfies (1.1), replacing x by $2^n x$ in (3.3) respectively, we obtain

$$\left.\begin{aligned} &\mu\left(\tfrac{1}{2^n}\left[f_a(2\cdot 2^n x)-3f_a(2^n x)-f_a(-2^n x)\right],t\right) \\ &\geq\mu'\left(\Delta_{AQ}(2^n x),2^n t\right) \\ &\nu\left(\tfrac{1}{2^n}\left[f_a(2\cdot 2^n x)-3f_a(2^n x)-f_a(-2^n x)\right],t\right) \\ &\geq\nu'\left(\Delta_{AQ}(2^n x),2^n t\right) \end{aligned}\right\} \tag{3.19}$$

for all $x\in X$ and all $t>0$. Now,

$$\begin{aligned} &\mu\Big(\mathscr{A}(2x)-3\mathscr{A}(x)-\mathscr{A}(-x),t\Big) \\ &\geq\mu\Big(\mathscr{A}(2x)-\frac{1}{2^n}f_a(2x),\frac{t}{4}\Big) \\ &\quad*\mu\Big(-3\mathscr{A}(x)+3\frac{1}{2^n}f_a(x),\frac{t}{4}\Big) \\ &\quad*\mu\Big(-\mathscr{A}(-x)+\frac{1}{2^n}f_a(-x),\frac{t}{4}\Big) \\ &\quad*\mu\Big(\frac{1}{2^n}f_a(2x)-3\frac{1}{2^n}f_a(x)-\frac{1}{2^n}f_a(-x),\frac{t}{4}\Big) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\nu\Big(\mathscr{A}(2x)-3\mathscr{A}(x)-\mathscr{A}(-x),t\Big) \\ &\geq\nu\Big(\mathscr{A}(2x)-\frac{1}{2^n}f_a(2x),\frac{t}{4}\Big) \\ &\quad\diamond\nu\Big(-3\mathscr{A}(x)+3\frac{1}{2^n}f_a(x),\frac{t}{4}\Big) \\ &\quad\diamond\nu\Big(-\mathscr{A}(-x)+\frac{1}{2^n}f_a(-x),\frac{t}{4}\Big) \\ &\quad\diamond\nu\Big(\frac{1}{2^n}f_a(2x)-3\frac{1}{2^n}f_a(x)-\frac{1}{2^n}f_a(-x),\frac{t}{4}\Big) \end{aligned} \tag{3.21}$$

for all $x\in X$ and all $t>0$. Also,

$$\left.\begin{aligned} &\lim_{n\to\infty}\mu\left(\tfrac{1}{2^n}\left[f_a(2\cdot 2^n x)-3f_a(2^n x)-f_a(-2^n x)\right],\tfrac{t}{4}\right)=1 \\ &\lim_{n\to\infty}\nu\left(\tfrac{1}{2^n}\left[f_a(2\cdot 2^n x)-3f_a(2^n x)-f_a(-2^n x)\right],\tfrac{t}{4}\right)=0 \end{aligned}\right\} \tag{3.22}$$

for all $x\in X$ and all $t>0$. Letting $n\to\infty$ in (3.20), (3.21) and using (3.22), we find that $\mathscr{A}$ fulfills (1.1). Therefore, $\mathscr{A}$ is a additive mapping. In order to prove $\mathscr{A}(x)$ is unique, let $\mathscr{A}'(x)$ be another additive functional equation satisfying (1.1) and (3.4). Hence,

$$\begin{aligned} &\mu(\mathscr{A}(x)-\mathscr{A}'(x),t) \\ &\geq\mu\left(\mathscr{A}(2^n x)-f_a(2^n x),\frac{t.2^n}{2}\right)*\mu\left(f_a(2^n x)-\mathscr{A}'(2^n x),\frac{t.2^n}{2}\right) \\ &\geq\mu'\left(\Delta_{AQ}(2^n x),\frac{2t\ 2^n|2-p|}{2}\right)\geq\mu'\left(\Delta_{AQ}(x),\frac{2t\ 2^n|2-p|}{2\cdot p^n}\right) \\ &\nu(\mathscr{A}(x)-\mathscr{A}'(x),t) \\ &\leq\nu\left(\mathscr{A}(2^n x)-f_a(2^n x),\frac{t.2^n}{2}\right)\diamond\nu\left(f_a(2^n x)-\mathscr{A}'(2^n x),\frac{t.2^n}{2}\right) \\ &\leq\nu'\left(\Delta_{AQ}(2^n x),\frac{2t\ 2^n|2-p|}{2}\right)\leq\nu'\left(\Delta_{AQ}(x),\frac{2t\ 2^n|2-p|}{2\cdot p^n}\right) \end{aligned}$$

for all $x\in X$ and all $t>0$. Since $\lim_{n\to\infty}\frac{2t\ 2^n|2-p|}{2\ p^n}=\infty$, we obtain

$$\left.\begin{aligned} &\lim_{n\to\infty}\mu'\left(\Delta_{AQ}(x),\tfrac{2t\ 2^n|2-p|}{2\cdot p^n}\right)=1 \\ &\lim_{n\to\infty}\nu'\left(\Delta_{AQ}(x),\tfrac{2t\ 2^n|2-p|}{2\cdot p^n}\right)=0 \end{aligned}\right\}$$

for all $x \in X$ and all $t > 0$. Thus

$$\left.\begin{array}{l}\mu(\mathscr{A}(x)-\mathscr{A}'(x),t)=1\\ \nu(\mathscr{A}(x)-\mathscr{A}'(x),t)=0\end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Hence, $\mathscr{A}(x) = \mathscr{A}'(x)$. Therefore, $\mathscr{A}(x)$ is unique.

Case 2: For $j = -1$. Putting x by $\frac{x}{2}$ in (3.5), we get

$$\left.\begin{array}{l}\mu\left(f_a(x)-2f\left(\frac{x}{2}\right),t\right)\geq \mu'\left(\Delta_{AQ}\left(\frac{x}{2}\right),t\right)\\ \nu\left(f_a(x)-2f\left(\frac{x}{2}\right),t\right)\leq \nu'\left(\Delta_{AQ}\left(\frac{x}{2}\right),t\right)\end{array}\right\} \tag{3.23}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Case 1. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.8, regarding the stability of (1.1)

Corollary 3.9. *Suppose that an odd function $f_a : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\\ \geq\left\{\begin{array}{l}\mu'(\lambda,t),\\ \mu'(\lambda(||x||^r),t),\end{array}\right.\\ \nu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\\ \leq\left\{\begin{array}{l}\nu'(\lambda,t),\\ \nu'(\lambda(||x||^r),t),\end{array}\right.\end{array}\right\} \tag{3.24}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 2$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ such that

$$\left.\begin{array}{l}\mu\left(f_a(x)-\mathscr{A}(x),t\right)\geq\left\{\begin{array}{l}\mu'(\lambda,|2|t),\\ \mu'(\lambda||x||^r,2|2-2^r|t),\end{array}\right.\\ \nu\left(f_a(x)-\mathscr{A}(x),t\right)\leq\left\{\begin{array}{l}\nu'(\lambda,|2|t),\\ \mu'(\lambda||x||^r,2|2-2^r|t),\end{array}\right.\end{array}\right\} \tag{3.25}$$

for all $x \in X$ and all $t > 0$.

Theorem 3.10. *Let $j \in \{1, -1\}$. Let $\Delta_{AQ} : X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{4}\right)^j < 1$,*

$$\left.\begin{array}{l}\mu'\left(\Delta_{AQ}\left(2^{nj}x\right),t\right)\geq \mu'\left(p^{nj}\Delta_{AQ}(x),t\right)\\ \nu'\left(\Delta_{AQ}\left(2^{nj}x\right),t\right)\leq \nu'\left(p^{nj}\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.26}$$

for all $x \in X$ and all $t > 0$ and

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}\mu'\left(\Delta_{AQ}\left(2^{jn}x\right),4^{jn}t\right)=1\\ \lim\limits_{n\to\infty}\nu'\left(\Delta_{AQ}\left(2^{jn}x\right),4^{jn}t\right)=0\end{array}\right\} \tag{3.27}$$

for all $x \in X$ and all $t > 0$. Let $f_q : X \to Y$ be an even function satisfying the inequality

$$\left.\begin{array}{l}\mu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)\geq \mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)\leq \nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.28}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : X \longrightarrow Y$ satisfying (1.1) and

$$\left.\begin{array}{l}\mu\left(f_q(x)-\mathscr{Q}_2(x),t\right)\geq \mu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ \nu\left(f_q(x)-\mathscr{Q}_2(x),t\right)\leq \nu'\left(\Delta_{AQ}(x),4|4-p|t\right)\end{array}\right\} \tag{3.29}$$

for all $x \in X$ and all $t > 0$.

Proof. **Case (i):** Let $j = 1$. Using evenness of f in in (3.28), we obtain

$$\left.\begin{array}{l}\mu\left(f_q(2x)-4f(x),t\right)\geq \mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_q(2x)-4f(x),t\right)\leq \nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.30}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Theorem 3.8. □

The following corollary is an immediate consequence of Theorem 3.10, regarding the stability of (1.1)

Corollary 3.11. *Suppose that an even function $f : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)\geq\left\{\begin{array}{l}\mu'(\lambda,t),\\ \mu'(\lambda(||x||^r),t),\end{array}\right.\\ \nu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)\leq\left\{\begin{array}{l}\nu'(\lambda,t),\\ \nu'(\lambda(||x||^r),t),\end{array}\right.\end{array}\right\} \tag{3.31}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 2$. Then there exists a unique quadratic mapping $\mathscr{Q}_2 : X \longrightarrow Y$ such that

$$\left.\begin{array}{l}\mu\left(f_q(x)-\mathscr{Q}_2(x),t\right)\geq\left\{\begin{array}{l}\mu'(\lambda,4|3|t),\\ \mu'(\lambda||x||^r,4|4-2^r|t),\end{array}\right.\\ \nu\left(f_q(x)-\mathscr{Q}_2(x),t\right)\leq\left\{\begin{array}{l}\nu'(\lambda,4|3|t),\\ \mu'(\lambda||x||^r,4|4-2^r|t),\end{array}\right.\end{array}\right\} \tag{3.32}$$

for all $x \in X$ and all $t > 0$.

Theorem 3.12. *Let $j \in \{1, -1\}$. Let $\Delta_{AQ} : X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{2}\right)^j, 0 < \left(\frac{p}{4}\right)^j < 1$, with conditions (3.1), (3.26), (3.2) and (3.27) for all $x \in X$ and all $t > 0$. Let $f : X \to Y$ be a function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(f(2x)-3f(x)-f(-x),t\right)\geq \mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f(2x)-3f(x)-f(-x),t\right)\leq \nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.33}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ and a unique quadratic mapping $\mathscr{Q}_2$:

$X \longrightarrow Y$ satisfying (1.1) and

$$\left.\begin{array}{l}\mu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right)\\ \geq \mu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ \quad *\mu'\left(\Delta_{AQ}(-x),2|2-p|t\right)\\ \quad *\mu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ \quad *\mu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\\ \\ \nu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right)\\ \leq \nu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{AQ}(-x),2|2-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\end{array}\right\} \tag{3.34}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $f_o(x) = \dfrac{f_a(x)-f_a(-x)}{2}$ for all $x \in \mathscr{T}_1$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence by Theorem 3.8, we have

$$\left.\begin{array}{l}\mu\left(f_o(x)-\mathscr{A}(x),t\right) \geq \mu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ \qquad\qquad *\mu'\left(\Delta_{AQ}(-x),2|2-p|t\right)\\ \\ \nu\left(f_o(x)-\mathscr{A}(x),t\right) \leq \nu'\left(\Delta_{AQ}(x),2|2-p|tt\right)\\ \qquad\qquad \diamond\nu'\left(\Delta_{AQ}(-x),2|2-p|tt\right)\end{array}\right\} \tag{3.35}$$

for all $x \in X$ and all $t > 0$. Also, let $f_e(x) = \dfrac{f_q(x)+f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathscr{T}_1$. Hence by Theorem 3.10, we have

$$\left.\begin{array}{l}\mu\left(f_e(x)-\mathscr{Q}_2(x),t\right) \geq \mu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ \qquad\qquad *\mu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\\ \\ \nu\left(f_e(x)-\mathscr{Q}_2(x),t\right) \leq \nu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ \qquad\qquad \diamond\nu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\end{array}\right\} \tag{3.36}$$

for all $x \in X$ and all $t > 0$. Define

$$f(x) = f_o(x) + f_e(x) \tag{3.37}$$

for all $x \in X$. From (3.35),(3.36) and (3.37), we arrive

$$\begin{aligned}&\mu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),2t\right)\\ &= \mu\left(f_o(x)+f_e(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),2t\right)\\ &\geq \mu\left(f_o(x)-\mathscr{A}(x),t\right)*\mu\left(f_e(x)-\mathscr{Q}_2(x),t\right)\\ &\geq \mu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ &\quad *\mu'\left(\Delta_{AQ}(-x),2|2-p|t\right)\\ &\quad *\mu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ &\quad *\mu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\end{aligned}$$

and

$$\begin{aligned}&\nu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),2t\right)\\ &= \nu\left(f_o(x)+f_e(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),2t\right)\\ &\leq \nu\left(f_o(x)-\mathscr{A}(x),t\right)*\nu\left(f_e(x)-\mathscr{Q}_2(x),t\right)\\ &\leq \nu'\left(\Delta_{AQ}(x),2|2-p|t\right)\\ &\quad \diamond\nu'\left(\Delta_{AQ}(-x),2|2-p|t\right)\\ &\quad \diamond\nu'\left(\Delta_{AQ}(x),4|4-p|t\right)\\ &\quad \diamond\nu'\left(\Delta_{AQ}(-x),4|4-p|t\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$. □

The following corollary is an immediate consequence of Theorem 3.12, regarding the stability of (1.1)

Corollary 3.13. *Suppose that a function $f : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(f(2x)-3f(x)-f(-x),t\right) \geq \begin{cases}\mu'(\lambda,t),\\ \mu'\left(\lambda\left(||x||^r\right),t\right),\end{cases}\\ \nu\left(f(2x)-3f(x)-f(-x),t\right) \leq \begin{cases}\nu'(\lambda,t),\\ \nu'\left(\lambda\left(||x||^r\right),t\right),\end{cases}\end{array}\right\} \tag{3.38}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 1,2$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ and a unique quadratic mapping $\mathscr{Q}_2 : X \longrightarrow Y$ such that

$$\left.\begin{array}{l}\mu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right)\\ \geq \begin{cases}\mu'(4\lambda,|2|t)*\mu'(4\lambda,4|3|t),\\ \mu'\left(4\lambda||x||^r,2|2-2^r|t\right)*\mu'\left(4\lambda||x||^r,4|4-2^r|t\right),\end{cases}\\ \\ \nu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right)\\ \leq \begin{cases}\nu'(4\lambda,|2|t)\diamond\nu'(4\lambda,4|3|t),\\ \nu'\left(4\lambda||x||^r,2|2-2^r|t\right)\diamond\nu'\left(4\lambda||x||^r,4|4-2^r|t\right),\end{cases}\end{array}\right\} \tag{3.39}$$

for all $x \in X$ and all $t > 0$.

3.3 Stability Results of (1.2): Direct Method

Theorem 3.14. *Let $j \in \{1,-1\}$. Let $\Delta_{CQ} : X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{8}\right)^j < 1$,*

$$\left.\begin{array}{l}\mu'\left(\Delta_{CQ}\left(2^{nj}x\right),t\right) \geq \mu'\left(p^{nj}\Delta_{CQ}(x),t\right)\\ \\ \nu'\left(\Delta_{CQ}\left(2^{nj}x\right),t\right) \leq \nu'\left(p^{nj}\Delta_{CQ}(x),t\right)\end{array}\right\} \tag{3.40}$$

for all $x \in X$ and all $t > 0$ and

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}\mu'\left(\Delta_{CQ}\left(2^{jn}x\right),8^{jn}t\right) = 1\\ \\ \lim\limits_{n\to\infty}\nu'\left(\Delta_{CQ}\left(2^{jn}x\right),8^{jn}t\right) = 0\end{array}\right\} \tag{3.41}$$

for all $x \in X$ and all $t > 0$. Let $g_c : X \to Y$ be an odd function satisfying the inequality

$$\left.\begin{aligned} \mu\left(g_c(2x) - 12g_c(x) - 4g_c(-x), t\right) \geq \mu'\left(\Delta_{CQ}(x), t\right) \\ \nu\left(g_c(2x) - 12g_c(x) - 4g_c(-x), t\right) \leq \nu'\left(\Delta_{CQ}(x), t\right) \end{aligned}\right\} \tag{3.42}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique cubic mapping $\mathscr{C} : X \longrightarrow Y$ satisfying (1.2) and

$$\left.\begin{aligned} \mu\left(g_c(x) - \mathscr{C}(x), t\right) \geq \mu'\left(\Delta_{CQ}(x), 8|8-p|t\right) \\ \nu\left(g_c(x) - \mathscr{C}(x), t\right) \leq \nu'\left(\Delta_{CQ}(x), 8|8-p|t\right) \end{aligned}\right\} \tag{3.43}$$

for all $x \in X$ and all $t > 0$.

Proof. **Case (i):** Let $j = 1$. Using oddness of g_c in in (3.42), we obtain

$$\left.\begin{aligned} \mu\left(g_c(2x) - 8f(x), t\right) \geq \mu'\left(\Delta_{CQ}(x), t\right) \\ \nu\left(g_c(2x) - 8f(x), t\right) \leq \nu'\left(\Delta_{CQ}(x), t\right) \end{aligned}\right\} \tag{3.44}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Theorem 3.8. □

The following corollary is an immediate consequence of Theorem 3.14, regarding the stability of (1.2)

Corollary 3.15. *Suppose that an odd function $g_c : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{aligned} &\mu\left(g_c(2x) - 12g_c(x) - 4g_c(-x), t\right) \\ &\geq \begin{cases} \mu'(\lambda, t), \\ \mu'\left(\lambda\left(||x||^r\right), t\right), \end{cases} \\ &\nu\left(g_c(2x) - 12g_c(x) - 4g_c(-x), t\right) \\ &\leq \begin{cases} \nu'(\lambda, t), \\ \nu'\left(\lambda\left(||x||^r\right), t\right), \end{cases} \end{aligned}\right\} \tag{3.45}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 3$. Then there exists a unique cubic mapping $\mathscr{C} : X \longrightarrow Y$ such that

$$\left.\begin{aligned} \mu\left(g_c(x) - \mathscr{C}(x), t\right) \geq \begin{cases} \mu'(\lambda, 8|7|t), \\ \mu'\left(\lambda||x||^r, 8|8-2^r|t\right), \end{cases} \\ \nu\left(g_c(x) - \mathscr{C}(x), t\right) \leq \begin{cases} \nu'(\lambda, 8|7|t), \\ \mu'\left(\lambda||x||^r, 8|8-2^r|t\right), \end{cases} \end{aligned}\right\} \tag{3.46}$$

for all $x \in X$ and all $t > 0$.

Theorem 3.16. *Let $j \in \{1, -1\}$. Let $\Delta_{CQ} : X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{16}\right)^j < 1$,*

$$\left.\begin{aligned} \mu'\left(\Delta_{CQ}\left(2^{nj}x\right), t\right) \geq \mu'\left(p^{nj}\Delta_{CQ}(x), t\right) \\ \nu'\left(\Delta_{CQ}\left(2^{nj}x\right), t\right) \leq \nu'\left(p^{nj}\Delta_{CQ}(x), t\right) \end{aligned}\right\} \tag{3.47}$$

for all $x \in X$ and all $t > 0$ and

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'\left(\Delta_{CQ}\left(2^{jn}x\right), 16^{jn}t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Delta_{CQ}\left(2^{jn}x\right), 16^{jn}t\right) = 0 \end{aligned}\right\} \tag{3.48}$$

for all $x \in X$ and all $t > 0$. Let $g_q : X \to Y$ be an even function satisfying the inequality

$$\left.\begin{aligned} \mu\left(g_q(2x) - 12g_q(x) - 4g_q(-x), t\right) \geq \mu'\left(\Delta_{CQ}(x), t\right) \\ \nu\left(g_q(2x) - 12g_q(x) - 4g_q(-x), t\right) \leq \nu'\left(\Delta_{CQ}(x), t\right) \end{aligned}\right\} \tag{3.49}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique quartic mapping $\mathscr{Q}_4 : X \longrightarrow Y$ satisfying (1.2) and

$$\left.\begin{aligned} \mu\left(g_q(x) - \mathscr{Q}_4(x), t\right) \geq \mu'\left(\Delta_{CQ}(x), 16|16-p|t\right) \\ \nu\left(g_q(x) - \mathscr{Q}_4(x), t\right) \leq \nu'\left(\Delta_{CQ}(x), 16|16-p|t\right) \end{aligned}\right\} \tag{3.50}$$

for all $x \in X$ and all $t > 0$.

Proof. **Case (i):** Let $j = 1$. Using evenness of g_q in in (3.49), we obtain

$$\left.\begin{aligned} \mu\left(g_q(2x) - 16f(x), t\right) \geq \mu'\left(\Delta_{CQ}(x), t\right) \\ \nu\left(g_q(2x) - 16f(x), t\right) \leq \nu'\left(\Delta_{CQ}(x), t\right) \end{aligned}\right\} \tag{3.51}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Theorem 3.8. □

The following corollary is an immediate consequence of Theorem 3.16, regarding the stability of (1.2)

Corollary 3.17. *Suppose that an even function $f : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{aligned} \mu\left(g_q(2x) - 12g_q(x) - 4g_q(-x), t\right) \geq \begin{cases} \mu'(\lambda, t), \\ \mu'\left(\lambda\left(||x||^r\right), t\right), \end{cases} \\ \nu\left(g_q(2x) - 12g_q(x) - 4g_q(-x), t\right) \leq \begin{cases} \nu'(\lambda, t), \\ \nu'\left(\lambda\left(||x||^r\right), t\right), \end{cases} \end{aligned}\right\} \tag{3.52}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 4$. Then there exists a unique quartic mapping $\mathscr{Q}_4 : X \longrightarrow Y$ such that

$$\left.\begin{aligned} \mu\left(g_q(x) - \mathscr{Q}_4(x), t\right) \geq \begin{cases} \mu'(\lambda, 16|15|t), \\ \mu'\left(\lambda||x||^r, 16|16-2^r|t\right), \end{cases} \\ \nu\left(g_q(x) - \mathscr{Q}_4(x), t\right) \leq \begin{cases} \nu'(\lambda, 16|15|t), \\ \mu'\left(\lambda||x||^r, 16|16-2^r|t\right), \end{cases} \end{aligned}\right\} \tag{3.53}$$

for all $x \in X$ and all $t > 0$.

Theorem 3.18. *Let $j \in \{1,-1\}$. Let $\Delta_{CQ} : X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{8}\right)^j, 0 < \left(\frac{p}{16}\right)^j < 1$, with conditions (3.40), (3.47), (3.41) and (3.48) for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(g(2x)-12(x)-4(-x),t\right) \geq \mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(g(2x)-12(x)-4(-x),t\right) \leq \nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.54}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique cubic mapping $\mathscr{C} : X \longrightarrow Y$ and a unique quartic mapping $\mathscr{Q}_4 : X \longrightarrow Y$ satisfying (1.2) and

$$\left.\begin{array}{l}\mu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ \geq \mu'\left(\Delta_{CQ}(x),8|8-p|t\right)\\ \quad *\mu'\left(\Delta_{CQ}(-x),8|8-p|t\right)\\ \quad *\mu'\left(\Delta_{CQ}(x),16|16-p|t\right)\\ \quad *\mu'\left(\Delta_{CQ}(-x),16|16-p|t\right)\\ \nu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ \leq \nu'\left(\Delta_{CQ}(x),8|8-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{CQ}(-x),8|8-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{CQ}(x),16|16-p|t\right)\\ \quad \diamond\nu'\left(\Delta_{CQ}(-x),16|16-p|t\right)\end{array}\right\} \tag{3.55}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof of the Theorem is similar to the Theorem 3.12 □

The following corollary is an immediate consequence of Theorem 3.18, regarding the stability of (1.2)

Corollary 3.19. *Suppose that a function $g : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(g(2x)-12g(x)-4g(-x),t\right)\\ \geq \begin{cases}\mu'(\lambda,t),\\ \mu'\left(\lambda\left(||x||^r\right),t\right),\end{cases}\\ \nu\left(g(2x)-12g(x)-4g(-x),t\right)\\ \leq \begin{cases}\nu'(\lambda,t),\\ \nu'\left(\lambda\left(||x||^r\right),t\right),\end{cases}\end{array}\right\} \tag{3.56}$$

for all $x \in X$ and all $t > 0$, where λ, r are constants with $\lambda > 0$ and $r \neq 3,2$. Then there exists a unique cubic mapping $\mathscr{C} : X \longrightarrow Y$ and a unique quartic mapping $\mathscr{Q}_4 : X \longrightarrow Y$ such that

$$\left.\begin{array}{l}\mu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ \geq \begin{cases}\mu'\left(4\lambda,8|7|t\right)*\mu'\left(4\lambda,16|15|t\right),\\ \mu'\left(4\lambda||x||^r,8|8-2^r|t\right)\\ \qquad *\mu'\left(4\lambda||x||^r,16|16-2^r|t\right),\end{cases}\\ \nu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ \leq \begin{cases}\nu'\left(4\lambda,8|7|t\right)\diamond\nu'\left(4\lambda,16|15|t\right),\\ \nu'\left(4\lambda||x||^r,8|8-2^r|t\right)\\ \qquad \diamond\nu'\left(4\lambda||x||^r,16|16-2^r|t\right),\end{cases}\end{array}\right\} \tag{3.57}$$

for all $x \in X$ and all $t > 0$.

3.4 Stability Results of (1.1): Fixed Point Method

Theorem 3.20. *Let $f_a : X \longrightarrow Y$ be an odd mapping for which there exists a function $\Delta_{AQ} : X \longrightarrow Z$ with the double condition*

$$\left.\begin{array}{l}\lim_{n\to\infty}\mu'\left(\Delta_{AQ}\left(J_i^n x\right),J_i^n t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Delta_{AQ}\left(J_i^n x\right),J_i^n t\right)=0\end{array}\right\} \tag{3.58}$$

for all $x,y \in X$ and all $t > 0$ where J_i is defined in (1.4) and satisfying the double functional inequality

$$\left.\begin{array}{l}\mu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right) \geq \mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right) \leq \nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\} \tag{3.59}$$

for all $x \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\Delta_{AQ}(x) = \Delta_{AQ}\left(\frac{x}{2}\right), \tag{3.60}$$

has the property

$$\left.\begin{array}{l}\mu'\left(J_i\Delta_{AQ}(J_i x),t\right)=\mu'\left(\Delta_{AQ}(x),Lt\right)\\ \nu'\left(J_i\Delta_{AQ}(J_i x),t\right)=\nu'\left(\Delta_{AQ}(x),Lt\right)\end{array}\right\} \tag{3.61}$$

for all $x \in X$ and all $t > 0$, then there exists a unique additive function $\mathscr{A} : X \longrightarrow Y$ satisfying the functional equation (1.1) and

$$\left.\begin{array}{l}\mu\left(f_a(x)-\mathscr{A}(x),t\right) \geq \mu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right)\\ \nu\left(f_a(x)-\mathscr{A}(x),t\right) \leq \nu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right)\end{array}\right\} \tag{3.62}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set

$$\Lambda = \{h | h : X \longrightarrow Y,\ h(0) = 0\}$$

and introduce the generalized metric on Λ, as

$$d(h,f) = \inf\left\{L \in (0,\infty) : \left\{\begin{array}{l}\mu(h(x)-f(x),t)\\ \geq \mu'(\Delta_{AQ}(x),Lt),\\ \nu(h(x)-f(x),t)\\ \leq \nu'(\Delta_{AQ}(x),Lt),\end{array}\right\}\right\} \tag{3.63}$$

for all $x \in X$ and all $t > 0$. It is easy to see that (3.63) is complete with respect to the defined metric. Define $\Gamma : \Lambda \longrightarrow \Lambda$ by

$$\Gamma h(x) = \frac{1}{J_i}h(J_i x),$$

for all $x \in \mathscr{X}$. Now, from (3.63) and $h, f \in \Lambda$

$$\begin{cases}\mu(h(x)-f(x),t) \geq \mu'(\Delta_{AQ}(x),t), x \in X, t > 0\\ \mu(\frac{1}{J_i}h(J_i x)-\frac{1}{J_i}f(J_i x),t) \geq \mu'(\Delta_{AQ}(J_i x),J_i t), x \in X, t > 0\\ \mu(\frac{1}{J_i}h(J_i x)-\frac{1}{J_i}f(J_i x),t) \geq \mu'(\Delta_{AQ}(x),Lt), x \in X, t > 0\\ \mu(\Gamma h(x)-\Gamma f(x),t) \geq \mu'(\Delta_{AQ}(x),Lt), x \in X, t > 0\end{cases}$$

$$\begin{cases}\nu(h(x)-f(x),t) \leq \nu'(\Delta_{AQ}(x),t), x \in X, t > 0\\ \nu(\frac{1}{J_i}h(J_i x)-\frac{1}{J_i}f(J_i x),t) \leq \nu'(\Delta_{AQ}(J_i x),J_i t), x \in X, t > 0\\ \nu(\frac{1}{J_i}h(J_i x)-\frac{1}{J_i}f(J_i x),t) \leq \nu'(\Delta_{AQ}(x),Lt), x \in X, t > 0\\ \nu(\Gamma h(x)-\Gamma f(x),t) \leq \nu'(\Delta_{AQ}(x),Lt), x \in X, t > 0\end{cases}$$

This implies $d(\Gamma h,\Gamma g)\leq Ld(h,g)$. i.e., Γ is a strictly contractive mapping on Λ with Lipschitz constant L.

Using oddness of f in (3.59), we reach

$$\left.\begin{array}{l}\mu\Big(f(2x)-2f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\Big(f(2x)-2f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right\}\tag{3.64}$$

for all $x\in X$ and all $t>0$. Now, from (3.64) and (3.61) for the case $i=0$, we reach

$$\left\{\begin{array}{l}\mu\Big(f(2x)-2f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \mu\Big(\frac{f(2x)}{2}-f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),2t\right)\\ \mu\Big(\Gamma f(x)-f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),Lt\right)\\ \mu\Big(\Gamma f(x)-f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),Lt\right)\\ \mu\Big(\Gamma f(x)-f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),Lt\right)\\ \nu\Big(f(2x)-2f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),t\right)\\ \nu\Big(\frac{f(2x)}{2}-f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),2t\right)\\ \nu\Big(\Gamma f(x)-f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),Lt\right)\\ \nu\Big(\Gamma f(x)-f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),Lt\right)\\ \nu\Big(\Gamma f(x)-f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),Lt\right)\end{array}\right.\tag{3.65}$$

for all $x\in X$ and all $t>0$. Again by interchanging x into $\dfrac{x}{2}$ in (3.64) and (3.61) for the case $i=1$, we get

$$\left\{\begin{array}{l}\mu\Big(f(2x)-2f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(\tfrac{x}{2}),t\right)\\ \mu\Big(f(x)-\Gamma f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \mu\Big(f(x)-\Gamma f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \mu\Big(f(x)-\Gamma f(x),t\Big)\geq\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\Big(f(2x)-2f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(\tfrac{x}{2}),t\right)\\ \nu\Big(f(x)-\Gamma f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),t\right)\\ \nu\Big(f(x)-\Gamma f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),t\right)\\ \nu\Big(f(x)-\Gamma f(x),t\Big)\leq\nu'\left(\Delta_{AQ}(x),t\right)\end{array}\right.\tag{3.66}$$

for all $x\in X$ and all $t>0$. Thus, from (3.64) and (3.66), we arrive

$$\left.\begin{array}{l}\mu(\Gamma f(x)-f(x),t)\geq\mu'(\Delta_{AQ}(x),L^{1-i}t),x\in X\\ \nu(\Gamma f(x)-f(x),t)\leq\nu'(\Delta_{AQ}(x),L^{1-i}t),x\in X\end{array}\right\}\tag{3.67}$$

Hence property (FP1) holds.

By (FP2), it follows that there exists a fixed point $\mathscr{A}$ of J in Λ such that

$$\lim_{n\to\infty}\mu\left(\frac{f(J_i^nx)}{J_i^n}-\mathscr{A}(x),t\right)=1,\lim_{n\to\infty}\nu\left(\frac{f(J_i^nx)}{J_i^n}-\mathscr{A}(x),t\right)=0$$

for all $x\in X$ and all $t>0$. To order to prove $A:X\longrightarrow Y$ is additive, the proof is similar to that of Theorem 3.8

By (FP3), $\mathscr{A}$ is the unique fixed point of Γ in the set $\Delta=\{\mathscr{A}\in\Lambda:d(f,A)<\infty\}$, $\mathscr{A}$ is the unique function such that

$$\left.\begin{array}{l}\mu(f(x)-\mathscr{A}(x),t)\geq\mu'(\Delta_{AQ}(x),L^{1-i}t),x\in X\\ \nu(f(x)-\mathscr{A}(x),t)\leq\nu'(\Delta_{AQ}(x),L^{1-i}t),x\in X\end{array}\right\}$$

for all $x\in X$ and and all $t>0$. Finally by (FP4), we obtain

$$\left.\begin{array}{l}\mu\left(f(x)-\mathscr{A}(x),t\right)\geq\mu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right)\\ \nu\left(f(x)-\mathscr{A}(x),t\right)\leq\nu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right)\end{array}\right\}$$

for all $x\in X$ and all $t>0$. So, the proof is complete. □

The next corollary is a direct consequence of Theorem 3.20 which shows that (1.1) can be stable.

Corollary 3.21. *Suppose that an odd function $f_a:X\longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\geq\left\{\begin{array}{l}\mu'\left(\lambda,t\right),\\ \mu'\left(\lambda||x||^r),t\right),\end{array}\right.\\ \nu\left(f_a(2x)-3f_a(x)-f_a(-x),t\right)\leq\left\{\begin{array}{l}\nu'\left(\lambda,t\right),\\ \nu'\left(\lambda||x||^r,t\right),\end{array}\right.\end{array}\right\}\tag{3.68}$$

for all $x,y\in X$ and all $t>0$, where $\lambda,r\neq 1$ are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{A}:X\longrightarrow Y$ such that the double inequality

$$\left.\begin{array}{l}\mu\left(f_a(x)-\mathscr{A}(x),t\right)\geq\left\{\begin{array}{l}\mu'\left(\lambda,|1|t\right),\\ \mu'\left(\lambda||x||^r,\frac{4^r}{|2-2^r|}\right),\end{array}\right.\\ \nu\left(f_a(x)-\mathscr{A}(x),t\right)\leq\left\{\begin{array}{l}\nu'\left(\lambda,|1|t\right),\\ \nu'\left(\lambda||x||^r,\frac{4^r}{|2-2^r|}\right),\end{array}\right.\end{array}\right\}\tag{3.69}$$

holds for all $x\in X$ and all $t>0$.

Proof. Now,

$$\mu'\left(\Delta_{AQ}(J_i^nx,J_i^ny),J_i^kt\right)=\left\{\begin{array}{l}\mu'\left(\lambda,J_i^kt\right),\\ \mu'\left(\lambda||x||^r,J_i^{k-a}t\right),\end{array}\right.$$
$$=\left\{\begin{array}{l}\to 1\text{ as }k\to\infty\\ \to 1\text{ as }k\to\infty\end{array}\right.$$

$$\nu'\left(\Delta_{AQ}(J_i^nx,J_i^ny),J_i^kt\right)=\left\{\begin{array}{l}\nu'\left(\lambda,J_i^kt\right),\\ \nu'\left(\lambda||x||^r,J_i^{k-a}t\right),\end{array}\right.$$
$$=\left\{\begin{array}{l}\to 0\text{ as }k\to\infty\\ \to 0\text{ as }k\to\infty\end{array}\right.$$

for all $x\in X$ and all $t>0$. Thus, the relation (3.58) holds. It follows from (3.60), (3.61) and (3.68), we arrive

$$\mu'\left(\Delta_{AQ},t\right)=\mu'\left(\Delta_{AQ}\left(\frac{x}{2}\right),t\right)=\left\{\begin{array}{l}\mu'\left(\lambda,t\right)\\ \mu'\left(\frac{\lambda||x||^r}{2^r},t\right)\end{array}\right.$$

$$\nu'\left(\Delta_{AQ},t\right)=\nu'\left(\Delta_{AQ}\left(\frac{x}{2}\right),t\right)=\left\{\begin{array}{l}\nu'\left(\lambda,t\right)\\ \nu'\left(\frac{\lambda||x||^r}{2^r},t\right)\end{array}\right.$$

for all $x,y \in X$ and all $t>0$. Also from (3.61), we have

$$\mu'\left(J_i\Delta_{AQ}(J_ix),t\right)=\begin{cases}\mu'\left(\lambda,J_i^{-1}t\right)\\ \mu'\left(\lambda\|x\|^r,J_i^{r-1}t\right)\end{cases}$$

$$\nu'\left(J_i\Delta_{AQ}(J_ix),t\right)=\begin{cases}\nu'\left(\lambda,J_i^{-1}t\right)\\ \nu'\left(\lambda\|x\|^r,J_i^{r-1}t\right)\end{cases}$$

for all $x \in X$ and all $t>0$.

For the case $L=J_i^{-1}=2^{-1}$ for $i=0$ and $L=J_i^{-1}=\left(\frac{1}{2}\right)^{-1}=2$ for $i=1$ from the inequality (3.62), we arrive

$$\left.\begin{aligned}\mu\left(f(x)-\mathscr{A}(x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),\tfrac{(2^{-1})^{1-0}}{1-2^{-1}}t\right)\\&=\mu'(\lambda,t)\\ \nu\left(f(x)-\mathscr{A}(x),t\right)&\le\nu'\left(\Delta_{AQ}(x),\tfrac{(2^{-1})^{1-0}}{1-2^{-1}}t\right)\\&=\nu'(\lambda,t)\end{aligned}\right\}$$

$$\left.\begin{aligned}\mu\left(f(x)-\mathscr{A}(x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),\tfrac{(2)^{1-1}}{1-2}t\right)\\&=\mu'(\lambda,-t)\\ \nu\left(f(x)-\mathscr{A}(x),t\right)&\le\nu'\left(\Delta_{AQ}(x),\tfrac{(2)^{1-1}}{1-2}t\right)\\&=\nu'(\lambda,-t)\end{aligned}\right\}$$

for all $x \in X$ and all $t>0$.

For the case $L=J_i^{r-1}=2^{r-1}$ for $i=0$ and $L=J_i^{r-1}=\left(\frac{1}{2}\right)^{r-1}=2^{1-r}$ for $i=1$ from the inequality (3.62), we arrive

$$\left.\begin{aligned}\mu\left(f(x)-\mathscr{A}(x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),\tfrac{(2^{r-1})^{1-0}}{1-2^{r-1}}t\right)\\&=\mu'\left(\lambda\|x\|^r,\tfrac{4^r}{2-2^r}t\right)\\ \nu\left(f(x)-\mathscr{A}(x),t\right)&\le\nu'\left(\Delta_{AQ}(x),\tfrac{(2^{r-1})^{1-0}}{1-2^{r-1}}t\right)\\&=\nu'\left(\lambda\|x\|^r,\tfrac{4^r}{2-2^r}t\right)\end{aligned}\right\}$$

$$\left.\begin{aligned}\mu\left(f(x)-\mathscr{A}(x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),\tfrac{(2^{1-r})^{1-1}}{1-2^{1-r}}t\right)\\&=\mu'\left(\lambda\|x\|^r,\tfrac{4^r}{2^r-2}t\right)\\ \nu\left(f(x)-\mathscr{A}(x),t\right)&\le\nu'\left(\Delta_{AQ}(x),\tfrac{(2^{1-r})^{1-1}}{1-2^{1-r}}t\right)\\&=\nu'\left(\lambda\|x\|^r,\tfrac{4^r}{2^r-2}t\right)\end{aligned}\right\}$$

for all $x \in X$ and all $t>0$. This finishes the proof. □

Theorem 3.22. *Let $f_q: X \longrightarrow Y$ be an even mapping for which there exists a function $\Delta_{AQ}: X \longrightarrow Z$ with the double condition*

$$\left.\begin{aligned}\lim_{n\to\infty}\mu'\left(\Delta_{AQ}\left(J_i^nx\right),J_i^{2n}t\right)&=1\\ \lim_{n\to\infty}\nu'\left(\Delta_{AQ}\left(J_i^nx\right),J_i^{2n}t\right)&=0\end{aligned}\right\}\tag{3.70}$$

for all $x,y\in X$ and all $t>0$ where J_i is defined in (1.4) and satisfying the double functional inequality

$$\left.\begin{aligned}\mu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)&\le\nu'\left(\Delta_{AQ}(x),t\right)\end{aligned}\right\}\tag{3.71}$$

for all $x\in X$ and all $t>0$. If there exists $L=L(i)$ such that the function

$$\Delta_{AQ}(x)=\Delta_{AQ}\left(\frac{x}{2}\right),\tag{3.72}$$

has the property

$$\left.\begin{aligned}\mu'\left(J_i\Delta_{AQ}(J_ix),t\right)&=\mu'\left(\Delta_{AQ}(x),Lt\right)\\ \nu'\left(J_i\Delta_{AQ}(J_ix),t\right)&=\nu'\left(\Delta_{AQ}(x),Lt\right)\end{aligned}\right\}\tag{3.73}$$

for all $x\in X$ and all $t>0$, then there exists a unique quadratic function $\mathscr{Q}_2: X\longrightarrow Y$ satisfying the functional equation (1.1) and

$$\left.\begin{aligned}\mu\left(f_q(x)-\mathscr{Q}_2(x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\\ \nu\left(f_q(x)-\mathscr{Q}_2(x),t\right)&\le\nu'\left(\Delta_{AQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\end{aligned}\right\}\tag{3.74}$$

for all $x\in X$ and all $t>0$.

Proof. The proof of the theorem is similar ideas given in Theorem 3.20 by defining a mapping $\Gamma:\Lambda\to\Lambda$ by

$$\Gamma h(x)=\frac{1}{J_i^2}h(J_ix),$$

for all $x\in X$. □

The next corollary is a direct consequence of which shows that (1.1) can be stable.

Corollary 3.23. *Suppose that an even function $f_q: X\longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{aligned}\mu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)&\ge\begin{cases}\mu'(\lambda,t),\\ \mu'(\lambda\|x\|^r),t),\end{cases}\\ \nu\left(f_q(2x)-3f_q(x)-f_q(-x),t\right)&\le\begin{cases}\nu'(\lambda,t),\\ \nu'(\lambda\|x\|^r,t),\end{cases}\end{aligned}\right\}\tag{3.75}$$

for all $x,y\in X$ and all $t>0$, where $\lambda, r\neq 2$ are constants with $\lambda>0$. Then there exists a unique quadratic mapping $\mathscr{Q}_2: X\longrightarrow Y$ such that the double inequality

$$\left.\begin{aligned}\mu\left(f_q(x)-\mathscr{Q}_2(x),t\right)&\ge\begin{cases}\mu'(\lambda,|3|t),\\ \mu'\left(\lambda\|x\|^r,\tfrac{4^r}{|4-2^r|}\right),\end{cases}\\ \nu\left(f_q(x)-\mathscr{Q}_2(x),t\right)&\le\begin{cases}\nu'(\lambda,|3|t),\\ \nu'\left(\lambda\|x\|^r,\tfrac{4^r}{|4-2^r|}\right),\end{cases}\end{aligned}\right\}\tag{3.76}$$

holds for all $x\in X$ and all $t>0$.

Theorem 3.24. *Let $f: X\longrightarrow Y$ be a mapping for which there exists a function $\Delta_{AQ}: X\longrightarrow Z$ with the double conditions (3.58), (3.70) for all $x,y\in X$ and all $t>0$ and satisfying the double functional inequality*

$$\left.\begin{aligned}\mu\left(f(2x)-3f(x)-f(-x),t\right)&\ge\mu'\left(\Delta_{AQ}(x),t\right)\\ \nu\left(f(2x)-3f(x)-f(-x),t\right)&\le\nu'\left(\Delta_{AQ}(x),t\right)\end{aligned}\right\}\tag{3.77}$$

for all $x \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the functions (3.60) and (3.72) has the properties (3.61) and (3.73) for all $x \in X$ and all $t > 0$, then there exists a unique additive function $\mathscr{A} : X \longrightarrow Y$ and a unique quadratic function $\mathscr{Q}_2 : X \longrightarrow Y$ satisfying the functional equation (1.1) and

$$\left.\begin{array}{l} \mu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right) \\ \geq \mu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right) * \mu'\left(\Delta_{AQ}(-x),\frac{L^{1-i}}{1-L}t\right) \\ \quad * \mu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right) * \mu'\left(\Delta_{AQ}(-x),\frac{L^{1-i}}{1-L}t\right) \\ \nu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right) \\ \leq \nu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right) \diamond \nu'\left(\Delta_{AQ}(-x),\frac{L^{1-i}}{1-L}t\right) \\ \quad \diamond \nu'\left(\Delta_{AQ}(x),\frac{L^{1-i}}{1-L}t\right) \diamond \nu'\left(\Delta_{AQ}(-x),\frac{L^{1-i}}{1-L}t\right) \end{array}\right\} \tag{3.78}$$

for all $x \in X$ and all $t > 0$.

The next corollary is a direct consequence of Theorem 3.24 which shows that (1.1) can be stable.

Corollary 3.25. *Suppose that a function $f : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l} \mu\left(f(2x)-3f(x)-f(-x),t\right) \geq \begin{cases} \mu'(\lambda,t), \\ \mu'(\lambda\|x\|^r),t), \end{cases} \\ \nu\left(f(2x)-3f(x)-f(-x),t\right) \leq \begin{cases} \nu'(\lambda,t), \\ \nu'(\lambda\|x\|^r,t), \end{cases} \end{array}\right\} \tag{3.79}$$

for all $x,y \in X$ and all $t > 0$, where $\lambda, r \neq 1,2$ are constants with $\lambda > 0$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ and a unique quadratic function $\mathscr{Q}_2 : X \longrightarrow Y$ such that the double inequality

$$\left.\begin{array}{l} \mu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x),t\right) \\ \geq \begin{cases} \mu'(4\lambda,|1|t) * \mu'(4\lambda,|3|t),, \\ \mu'\left(4\lambda\|x\|^r,\frac{4^r}{|2-2^r|}\right) * \mu'\left(4\lambda\|x\|^r,\frac{4^r}{|4-2^r|}\right), \end{cases} \\ \nu\left(f(x)-\mathscr{A}(x)-\mathscr{Q}_2(x)),t\right) \\ \leq \begin{cases} \nu'(4\lambda,|1|t) \diamond \nu'(4\lambda,|3|t), \\ \nu'\left(4\lambda\|x\|^r,\frac{4^r}{|2-2^r|}\right) \diamond \nu'\left(4\lambda\|x\|^r,\frac{4^r}{|4-2^r|}\right) \end{cases} \end{array}\right\} \tag{3.80}$$

holds for all $x \in X$ and all $t > 0$.

3.5 Stability Results of (1.2): Fixed Point Method

Theorem 3.26. *Let $g_c : X \longrightarrow Y$ be an odd mapping for which there exists a function $\Delta_{CQ} : X \longrightarrow Z$ with the double condition*

$$\left.\begin{array}{l} \lim_{n\to\infty} \mu'\left(\Delta_{CQ}(J_i^n x), J_i^n t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(\Delta_{CQ}(J_i^n x), J_i^n t\right) = 0 \end{array}\right\} \tag{3.81}$$

for all $x,y \in X$ and all $t > 0$ where J_i is defined in (1.4) and satisfying the double functional inequality

$$\left.\begin{array}{l} \mu\left(g_c(2x)-12g_c(x)-4g_c(-x),t\right) \geq \mu'\left(\Delta_{CQ}(x),t\right) \\ \nu\left(g_c(2x)-12g_c(x)-4g_c(-x),t\right) \leq \nu'\left(\Delta_{CQ}(x),t\right) \end{array}\right\} \tag{3.82}$$

for all $x \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\Delta_{CQ}(x) = \Delta_{CQ}\left(\frac{x}{2}\right), \tag{3.83}$$

has the property

$$\left.\begin{array}{l} \mu'\left(J_i\Delta_{CQ}(J_i x),t\right) = \mu'\left(\Delta_{CQ}(x),Lt\right) \\ \nu'\left(J_i\Delta_{CQ}(J_i x),t\right) = \nu'\left(\Delta_{CQ}(x),Lt\right) \end{array}\right\} \tag{3.84}$$

for all $x \in X$ and all $t > 0$, then there exists a unique cubic function $\mathscr{C} : X \longrightarrow Y$ satisfying the functional equation (1.2) and

$$\left.\begin{array}{l} \mu\left(g_c(x)-\mathscr{C}(x),t\right) \geq \mu'\left(\Delta_{CQ}(x),\frac{L^{1-i}}{1-L}t\right) \\ \nu\left(g_c(x)-\mathscr{C}(x),t\right) \leq \nu'\left(\Delta_{CQ}(x),\frac{L^{1-i}}{1-L}t\right) \end{array}\right\} \tag{3.85}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof of the theorem is similar ideas given in Theorem 3.20 by defining a mapping $\Gamma : \Lambda \to \Lambda$ by

$$\Gamma h(x) = \frac{1}{J_i^3} h(J_i x),$$

for all $x \in X$. □

The next corollary is a direct consequence of Theorem 3.26 which shows that (1.1) can be stable.

Corollary 3.27. *Suppose that an odd function $g_c : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{array}{l} \mu\left(g_c(2x)-12g_c(x)-4g_c(-x),t\right) \geq \begin{cases} \mu'(\lambda,t), \\ \mu'(\lambda\|x\|^r),t), \end{cases} \\ \nu\left(g_c(2x)-12g_c(x)-4g_c(-x),t\right) \leq \begin{cases} \nu'(\lambda,t), \\ \nu'(\lambda\|x\|^r,t), \end{cases} \end{array}\right\} \tag{3.86}$$

for all $x,y \in X$ and all $t > 0$, where $\lambda, r \neq 3$ are constants with $\lambda > 0$. Then there exists a unique cubic mapping $\mathscr{C} : X \longrightarrow Y$ such that the double inequality

$$\left.\begin{array}{l} \mu\left(g_c(x)-\mathscr{C}(x),t\right) \geq \begin{cases} \mu'(\lambda,|7|t), \\ \mu'\left(\lambda\|x\|^r,\frac{4^r}{|8-2^r|}\right), \end{cases} \\ \nu\left(g_c(x)-\mathscr{C}(x),t\right) \leq \begin{cases} \nu'(\lambda,|7|t), \\ \nu'\left(\lambda\|x\|^r,\frac{4^r}{|8-2^r|}\right), \end{cases} \end{array}\right\} \tag{3.87}$$

holds for all $x \in X$ and all $t > 0$.

Theorem 3.28. *Let $g_q : X \longrightarrow Y$ be an even mapping for which there exists a function $\Delta_{CQ} : X \longrightarrow Z$ with the double condition*

$$\left.\begin{aligned}\lim_{n\to\infty}\mu'\left(\Delta_{CQ}(J_i^n x), J_i^{2n}t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Delta_{CQ}(J_i^n x), J_i^{2n}t\right)=0\end{aligned}\right\} \tag{3.88}$$

for all $x,y\in X$ and all $t>0$ where J_i is defined in (1.4) and satisfying the double functional inequality

$$\left.\begin{aligned}\mu\left(g_q(2x)-12g_q(x)-4g_q(-x),t\right)\geq\mu'\left(\Delta_{CQ}(x),t\right)\\ \nu\left(g_q(2x)-12g_q(x)-4g_q(-x),t\right)\leq\nu'\left(\Delta_{CQ}(x),t\right)\end{aligned}\right\} \tag{3.89}$$

for all $x\in X$ and all $t>0$. If there exists $L=L(i)$ such that the function

$$\Delta_{CQ}(x)=\Delta_{CQ}\left(\frac{x}{2}\right), \tag{3.90}$$

has the property

$$\left.\begin{aligned}\mu'\left(J_i\Delta_{CQ}(J_i x),t\right)=\mu'\left(\Delta_{CQ}(x),Lt\right)\\ \nu'\left(J_i\Delta_{CQ}(J_i x),t\right)=\nu'\left(\Delta_{CQ}(x),Lt\right)\end{aligned}\right\} \tag{3.91}$$

for all $x\in X$ and all $t>0$, then there exists a unique quartic function $\mathscr{Q}_4 : X \longrightarrow Y$ satisfying the functional equation (1.2) and

$$\left.\begin{aligned}\mu\left(g_q(x)-\mathscr{Q}_4(x),t\right)\geq\mu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\\ \nu\left(g_q(x)-\mathscr{Q}_4(x),t\right)\leq\nu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\end{aligned}\right\} \tag{3.92}$$

for all $x\in X$ and all $t>0$.

Proof. The proof of the theorem is similar ideas given in Theorem 3.20 by defining a mapping $\Gamma:\Lambda\to\Lambda$ by

$$\Gamma h(x)=\frac{1}{J_i^4}h(J_i x),$$

for all $x\in X$. □

The next corollary is a direct consequence of which shows that (1.1) can be stable.

Corollary 3.29. *Suppose that an even function $g_q : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{aligned}&\mu\left(g_q(2x)-12g_q(x)-4g_q(-x),t\right)\\ &\geq\begin{cases}\mu'(\lambda,t),\\ \mu'(\lambda\|x\|^r),t),\end{cases}\\ &\nu\left(g_q(2x)-12g_q(x)-4g_q(-x),t\right)\\ &\leq\begin{cases}\nu'(\lambda,t),\\ \nu'(\lambda\|x\|^r,t),\end{cases}\end{aligned}\right\} \tag{3.93}$$

for all $x,y\in X$ and all $t>0$, where $\lambda, r\neq 4$ are constants with $\lambda>0$. Then there exists a unique quartic mapping $\mathscr{Q}_4 : X \longrightarrow Y$ such that the double inequality

$$\left.\begin{aligned}\mu\left(g_q(x)-\mathscr{Q}_4(x),t\right)\geq\begin{cases}\mu'(\lambda,|15|t),\\ \mu'\left(\lambda\|x\|^r,\frac{4^r}{|16-2^r|}\right),\end{cases}\\ \nu\left(g_q(x)-\mathscr{Q}_4(x),t\right)\leq\begin{cases}\nu'(\lambda,|15|t),\\ \nu'\left(\lambda\|x\|^r,\frac{4^r}{|16-2^r|}\right),\end{cases}\end{aligned}\right\} \tag{3.94}$$

holds for all $x\in X$ and all $t>0$.

Theorem 3.30. *Let $g : X \longrightarrow Y$ be a mapping for which there exists a function $\Delta_{CQ} : X \longrightarrow Z$ with the double conditions (3.81), (3.88) for all $x,y\in X$ and all $t>0$ and satisfying the double functional inequality*

$$\left.\begin{aligned}\mu\left(g(2x)-12g(x)-4g(-x),t\right)\geq\mu'\left(\Delta_{CQ}(x),t\right)\\ \nu\left(g(2x)-12g(x)-4g(-x),t\right)\leq\nu'\left(\Delta_{CQ}(x),t\right)\end{aligned}\right\} \tag{3.95}$$

for all $x\in X$ and all $t>0$. If there exists $L=L(i)$ such that the functions (3.60) and (3.72) has the properties (3.84) and (3.91) for all $x\in X$ and all $t>0$, then there exists a unique cubic function $\mathscr{C} : X \longrightarrow Y$ and a unique quartic function $\mathscr{Q}_4 : X \longrightarrow Y$ satisfying the functional equation (1.2) and

$$\left.\begin{aligned}&\mu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ &\geq\mu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)*\mu'\left(\Delta_{CQ}(-x),\tfrac{L^{1-i}}{1-L}t\right)\\ &\quad*\mu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)*\mu'\left(\Delta_{CQ}(-x),\tfrac{L^{1-i}}{1-L}t\right)\\ &\nu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ &\leq\nu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\diamond\nu'\left(\Delta_{CQ}(-x),\tfrac{L^{1-i}}{1-L}t\right)\\ &\quad\diamond\nu'\left(\Delta_{CQ}(x),\tfrac{L^{1-i}}{1-L}t\right)\diamond\nu'\left(\Delta_{CQ}(-x),\tfrac{L^{1-i}}{1-L}t\right)\end{aligned}\right\} \tag{3.96}$$

for all $x\in X$ and all $t>0$.

The next corollary is a direct consequence of Theorem 3.24 which shows that (1.2) can be stable.

Corollary 3.31. *Suppose that a function $g : X \longrightarrow Y$ satisfies the double inequality*

$$\left.\begin{aligned}\mu\left(g(2x)-12g(x)-4g(-x),t\right)\geq\begin{cases}\mu'(\lambda,t),\\ \mu'(\lambda\|x\|^r),t),\end{cases}\\ \nu\left(g(2x)-12g(x)-4g(-x),t\right)\leq\begin{cases}\nu'(\lambda,t),\\ \nu'(\lambda\|x\|^r,t),\end{cases}\end{aligned}\right\} \tag{3.97}$$

for all $x,y\in X$ and all $t>0$, where $\lambda, r\neq 3,4$ are constants with $\lambda>0$. Then there exists a unique cubic mapping $\mathscr{C}$: $X \longrightarrow Y$ and a unique quartic function $\mathscr{Q}_4 : X \longrightarrow Y$ such *that the double inequality*

$$\left.\begin{array}{l}\mu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x),t\right)\\ \geq \begin{cases}\mu'\left(4\lambda,|7|t\right)*\mu'\left(4\lambda,|15|t\right),,\\ \mu'\left(4\lambda||x||^r,\frac{4^r}{|8-2^r|}\right)*\mu'\left(4\lambda||x||^r,\frac{4^r}{|16-2^r|}\right),\end{cases}\\ \\ \nu\left(g(x)-\mathscr{C}(x)-\mathscr{Q}_4(x)),t\right)\\ \leq \begin{cases}\nu'\left(4\lambda,|7|t\right)\diamond\nu'\left(4\lambda,|15|t\right),\\ \nu'\left(4\lambda||x||^r,\frac{4^r}{|8-2^r|}\right)\diamond\nu'\left(4\lambda||x||^r,\frac{4^r}{|16-2^r|}\right)\end{cases}\end{array}\right\} \quad (3.98)$$

holds for all $x \in X$ and all $t > 0$.

References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables,* Cambridge Univ, Press, 1989.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64-66.

[3] M. Arunkumar, S. Hema Latha, *Orthogonal Stability Of 2 Dimensional Mixed Type Additive And Quartic Functional Equation*, International Journal of pure and Applied Mathematics, 63 No.4, (2010), 461-470.

[4] M. Arunkumar, John M. Rassias, *On the generalized Ulam-Hyers stability of an AQ-mixed type functional equation with counter examples*, Far East Journal of Applied Mathematics, Volume 71, No. 2, (2012), 279-305.

[5] M. Arunkumar, Matina J. Rassias, Yanhui Zhang, *Ulam - Hyers stability of a 2- variable AC - mixed type functional equation: direct and fixed point methods*, Journal of Modern Mathematics Frontier (JMMF), 1 (3), 2012, 10-26.

[6] M. Arunkumar, *Solution and stability of modified additive and quadratic functional equation in generalized 2-normed spaces*, International Journal Mathematical Sciences and Engineering Applications, Vol. 7 No. I (January, 2013), 383-391.

[7] M. Arunkumar, *Generalized Ulam - Hyers stability of derivations of a AQ - functional equation*, Cubo A Mathematical Journal dedicated to Professor Gaston M. N'Guerekata on the occasion of his 60th Birthday Vol.15, No 01, (159-169), March 2013.

[8] M. Arunkumar, P. Agilan, *Additive Quadratic functional equation are Stable in Banach space: A Fixed Point Approach*, International Journal of pure and Applied Mathematics, 86 No.6, (2013), 951-963, .

[9] M. Arunkumar, P. Agilan, *Additive Quadratic functional equation are Stable in Banach space: A Direct Method*, Far East Journal of Applied Mathematics, Volume 80, No. 1, (2013), 105 - 121.

[10] M. Arunkumar, P. Agilan, S. Ramamoorthi, *Perturbation of AC-mixed type functional equation*, Proceedings of National conference on Recent Trends in Mathematics and Computing (NCRTMC-2013), 7-14, ISBN 978-93-82338-68-0.

[11] M. Arunkumar, *Perturbation of n Dimensional AQ - mixed type Functional Equation via Banach Spaces and Banach Algebra: Hyers Direct and Alternative Fixed Point Methods*, International Journal of Advanced Mathematical Sciences (IJAMS), Vol. 2 (1), 2014, 34-56.

[12] M. Arunkumar, P. Agilan, C. Devi Shyamala Mary, *Permanence of A Generalized AQ Functional Equation In Quasi-Beta Normed Spaces, A Fixed Point Approach*, Proceedings of the International Conference on Mathematical Methods and Computation, Jamal Academic Research Journal an Interdisciplinary, (February 2014), 315-324.

[13] M. Arunkumar, C. Devi Shyamala Mary, G. Shobana, *Simple AQ And Simple CQ Functional Equations*, Journal Of Concrete And Applicable Mathematics (JCAAM), Vol 13, Issue 1/2 , Jan - Apr 2015, 120 - 151.

[14] M. Arunkumar, P. Agilan, C. Devi Shyamala Mary, *Permanence of A Generalized AQ Functional Equation In Quasi-Beta Normed Spaces*, International Journal of Pure and Applied Mathematics, Vol. 101, No. 6 (2015), 1013-1025.

[15] M. Arunkumar, G.Shobana, S. Hemalatha, *Ulam - Hyers, Ulam - Trassias, Ulam-Grassias, Ulam - Jrassias Stabilities of A Additive - Quadratic Mixed Type Functional Equation In Banach Spaces*, International Journal of pure and Applied Mathematics, Vol. 101, No. 6 (2015), 1027-1040.

[16] M. Arunkumar, T. Namachivayam, *Intuitionistic fuzzy stability of a n-dimensional cubic functional equation: Direct and fixed point methods*, Intern. J. Fuzzy Mathematical Archive, Vol. 7(1)(2015), 1-11 .

[17] M. Arunkumar, E. Sathya, P. Narasimman, N. Mahesh kumar, *3 Dimensional Additive Quadratic Functional Equations*, Malaya Journal of Matematik, 5(1) (2017), 72-103.

[18] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy sets and Systems. 20 (1986), No. 1, 87–96.

[19] C.Borelli, G.L.Forti, *On a general Hyers-Ulam stability*, Internat J.Math.Math.Sci, 18 (1995), 229-236.

[20] P.W.Cholewa, *Remarks on the stability of functional equations ,* Aequationes Math., 27 (1984), 76-86.

[21] S.Czerwik, *On the stability of the quadratic mappings in normed spaces,* Abh.Math.Sem.Univ Hamburg., 62 (1992), 59-64.

[22] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[23] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings ,* J. Math. Anal. Appl., 184 (1994), 431-436.

[24] M. Eshaghi Gordji, H. Khodaie, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.

[25] M. Eshaghi Gordji, N.Ghobadipour, J. M. Rassias, *Fuzzy stability of additive-quadratic functional Equations*,

arXiv:0903.0842v1 [math.FA] 4 Mar 2009.

[26] M. Eshaghi Gordji, M. Bavand Savadkouhi, and Choonkil Park, *Quadratic-Quartic Functional Equations in RN-Spaces*, Journal of Inequalities and Applications, Vol. 2009, Article ID 868423, 14 pages, doi:10.1155/2009/868423

[27] M. Eshaghi Gordji, H. Khodaei, J.M. Rassias, *Fixed point methods for the stability of general quadratic functional equation*, Fixed Point Theory 12 (2011), no. 1, 71-82.

[28] D.H. Hyers, *On the stability of the linear functional equation,* Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.

[29] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*,Birkhauser, Basel, 1998.

[30] Sun Sook Jin, Yang-Hi Lee, *A Fixed Point Approach to the Stability of the Cauchy Additive and Quadratic Type Functional Equation*, Journal of Applied Mathematics, doi:10.1155/2011/817079, 16 pages.

[31] K.W. Jun, H.M. Kim, *On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation*, Bull. Korean Math. Soc. 42(1) (2005), 133-148.

[32] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[33] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[34] B.Margoils, J.B.Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull.Amer. Math. Soc. 126 74 (1968), 305-309.

[35] S. A. Mohiuddine, Q. M. Danish Lohani, *On generalized statistical convergence in intuitionistic fuzzy normed space*, Chaos, Solitons Fract., 42 (1), (2009), 731–1737.

[36] S. Murthy, M. Arunkumar, G. Ganapathy, P. Rajarethinam, *Stability of mixed type additive quadratic functional equation in Random Normed space*, International Journal of Applied Mathematics (IJAM), Vol. 26. No. 2 (2013), 123-136.

[37] S. Murthy, M. Arunkumar, G. Ganapathy, *Perturbation of n- dimensional quadratic functional equation: A fixed point approach*, International Journal of Advanced Computer Research (IJACR), Volume-3, Number-3, Issue-11 September-2013, 271-276.

[38] A. Najati, M.B. Moghimi, *On the stability of a quadratic and additive functional equation*, J. Math. Anal. Appl. 337 (2008), 399-415.

[39] Choonkil Park, *Orthogonal Stability of an Additive-Quadratic Functional Equation*, Fixed Point Theory and Applications 2011 2011:66.

[40] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, 22 (2004), 1039–1046.

[41] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings,* J. Funct. Anal. USA, 46, (1982) 126-130.

[42] J.M. Rassias, K.Ravi, M.Arunkumar and B.V.Senthil Kumar, *Ulam Stability of Mixed type Cubic and Additive functional equation*, Functional Ulam Notions (F.U.N) Nova Science Publishers, 2010, Chapter 13, 149 - 175.

[43] M.J. Rassias, M. Arunkumar, S. Ramamoorthi, *Stability of the Leibniz additive-quadratic functional equation in Quasi-Beta normed space: Direct and fixed point methods,* Journal Of Concrete And Applicable Mathematics, Vol. 14 No. 1-2, (2014), 22 - 46.

[44] J. M. Rassias, M. Arunkumar, E.sathya, N. Mahesh Kumar, *Generalized Ulam - Hyers Stability Of A (AQQ): Additive - Quadratic - Quartic Functional Equation*, Malaya Journal of Matematik, 5(1) (2017), 122-142.

[45] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc.Amer.Math. Soc., 72 (1978), 297-300.

[46] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[47] K.Ravi and M.Arunkumar, *On a n- dimensional additive Functional Equation with fixed point Alternative*, Proceedings of International Conference on Mathematical Sciences 2007, Malaysia.

[48] K. Ravi, M. Arunkumar and J.M. Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[49] K. Ravi, J.M. Rassias, M. Arunkumar, R. Kodandan, *Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation*, J. Inequal. Pure Appl. Math. 10 (2009), no. 4, Article 114, 29 pp.

[50] R. Saadati, J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons and Fractals. 27 (2006), 331–344.

[51] R. Saadati, J. H. Park, *Intuitionstic fuzzy Euclidean normed spaces*, Commun. Math. Anal., 1 (2006), 85–90.

[52] R. Saadati, S. Sedghi and N. Shobe, *Modified intuitionistic fuzzy metric spaces and some fixed point theorems*, Chaos, Solitons and Fractals, 38 (2008), 36–47.

[53] S.M. Ulam, *Problems in Modern Mathematics,* Science Editions, Wiley, NewYork, 1964.

[54] T.Z. Xu, J.M. Rassias, W.X Xu, *Generalized Ulam-Hyers stability of a general mixed AQCQ-functional equation in multi-Banach spaces: a fixed point approach*, Eur. J. Pure Appl. Math. 3 (2010), no. 6, 1032-1047.

[55] T.Z. Xu, J.M. Rassias, M.J. Rassias, W.X. Xu, *A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces*, J. Inequal. Appl. 2010, Art. ID 423231, 23 pp.

[56] T.Z. Xu, J.M Rassias, W.X. Xu, *A fixed point approach to the stability of a general mixed AQCQ-functional equation in non-Archimedean normed spaces*, Discrete Dyn. Nat. Soc. 2010, Art. ID 812545, 24 pp.

[57] L. A. Zadeh, *Fuzzy sets*, Inform. Control, 8 (1965), 338–353.

[58] G. Zamani Eskandani, Hamid Vaezi, Y. N. Dehghan, *Stability Of A Mixed Additive And Quadratic Functional Equation In Non-Archimedean Banach Modules*, Taiwanese Journal Of Mathematics, Vol. 14, No. 4, (2010), 1309-1324.

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